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Boundedness for multilinear commutator of Marcinkiewicz operator on Hardy and Herz-Hardy spaces

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1. INTRODUCTION AND DEFINITION

Let T be the Calderón-Zygmund operator and $b \in BMO(R^n)$. The commutator $[b, T]$ generated by T and b is defined by

$$[b, T](f)(x) = b(x)T(f)(x) - T(bf)(x).$$

A classical result of Coifman, Rochberg and Weiss (see [2][3]) proved that the commutator $[b, T]$ is bounded on $L^p(R^n)$ $(1 < p < \infty)$. However, it was observed that the $[b, T]$ is not bounded, in general, from $H^p(R^n)$ to $L^p(R^n)$. But if $H^p(R^n)$ is replaced by a suitable atomic space $H^P_b(R^n)$, then $[b, T]$ maps continuously $H^P_b(R^n)$ into $L^p(R^n)$(see [1]). In addition we easily known that $H^p_b(R^n) \subset H^p(R^n)$. In recent years, the theory of Herz type Hardy spaces have been developed(see [4][7][8][9]). The main purpose of this paper is to consider the continuity of the multilinear commutators related to the Marcinkiewicz operators and $BMO(R^n)$ functions on certain Hardy and Herz-Hardy spaces. Let us first introduce some definitions(see [1][4-14]).

Given a positive integer m and $1 \le j \le m$, we denote by C^m_j the family of all finite subsets $\sigma = \{\sigma(1), \cdots, \sigma(j)\}$ of $\{1, \cdots, m\}$ of j different elements. For $\sigma \in C^m_j$, set $\sigma^c = \{1, \cdots, m\} \setminus \sigma$. For $\vec{b} = (b_1, \cdots, b_m)$ and $\sigma = \{\sigma(1), \cdots, \sigma(j)\} \in C^m_j$, set $\vec{b}_\sigma = (b_{\sigma(1)}, \cdots, b_{\sigma(j)})$, $b_\sigma = b_{\sigma(1)} \cdots b_{\sigma(j)}$ and $||\vec{b}_\sigma||_{BMO} = ||b_{\sigma(1)}||_{BMO} \cdots ||b_{\sigma(j)}||_{BMO}$.

Definition 1. Let b_i $(i = 1, \cdots, m)$ be a locally integrable function and $0 < p \le 1$. A bounded measurable function a on R^n is said a $(p, \vec{b})$ atom, if

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(1) $suppa \subset B = B(x_0, r)$,
(2) $||a||_{L^\infty} \leq |B|^{-1/p}$,
(3) $\int_B a(y)dy = \int_B a(y) \prod_{l \in \sigma} b_l(y)dy = 0$ for any $\sigma \in C_j^m$, $1 \leq j \leq m$.

A temperate distribution f is said to belong to $H^p_{\vec{b}}(R^n)$, if, in the Schwartz distribution sense, it can be written as

$$f(x) = \sum_{j=1}^{\infty} \lambda_j a_j(x),$$

where every a_j is $(p, \vec{b})$ atom, $\lambda \in C$ and $\sum_{j=1}^{\infty} |\lambda|^p < \infty$. Moreover, $||f||_{H^p_{\vec{b}}(R^n)} \approx (\sum_{j=1}^{\infty} |\lambda_j|^p)^{1/p}$.

Given a set $E \subset R^n$, the characteristic function of E is defined by χ_E. Let $B_k = \{x \in R^n : |x| \leq 2^k\}$ and $C_k = B_k \setminus B_{k-1}$and $\chi_k = \chi_{B_k}, k \in Z$.

Definition 2. Let $0 < p, q < \infty,\ \alpha \in R$. For $k \in Z$, set $B_k = \{x \in R^n : |x| \leq 2^k\}$ and $C_k = B_k \setminus B_{k-1}$. Denote by χ_k the characteristic function of C_k and χ_0 the characteristic function of B_0.

(1) The homogeneous Herz space is defined by

$$\dot{K}_q^{\alpha,p}(R^n) = \left\{ f \in L^q_{loc}(R^n \setminus \{0\}) : ||f||_{\dot{K}_q^{\alpha,p}} < \infty \right\},$$

where

$$||f||_{\dot{K}_q^{\alpha,p}} = \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} ||f\chi_k||^p_{L^q} \right]^{1/p}.$$

(2) The nonhomogeneous Herz space is defined by

$$K_q^{\alpha,p}(R^n) = \left\{ f \in L^q_{loc}(R^n) : ||f||_{K_q^{\alpha,p}} < \infty \right\},$$

where

$$||f||_{K_q^{\alpha,p}} = \left[\sum_{k=1}^{\infty} 2^{k\alpha p} ||f\chi_k||^p_{L^q} + ||f\chi_0||^p_{L^q} \right]^{1/p}.$$

Definition 3. Let $\alpha \in R,\ 1 < q < \infty,\ 0 < \alpha < n(1 - 1/q),\ \ b_i \in BMO(R^n),\ 1 \leq i \leq m$. A function $a(x)$ on R^n is called a central $(\alpha, q, \vec{b})$ -atom (or a central $(\alpha, q, \vec{b})$-atom of restrict type), if

(1) $suppa \subset B = B(0, r)$(or for some $r \geq 1$),
(2) $||a||_{L^q} \leq |B|^{-\alpha/n}$,

(3) $\int_B a(x)dx = \int_B a(x)\prod_{l\in\sigma} b_l(x)dx = 0$ for any $\sigma \in C_j^m$, $1 \le j \le m$.
A temperate distribution f is said to belong to $H\dot{K}^{\alpha,p}_{q,\vec{b}}(R^n)$ (or $HK^{\alpha,p}_{q,\vec{b}}(R^n)$), if it can be written as $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$ (or $f = \sum_{j=0}^{\infty} \lambda_j a_j$), in the Schwartz distribution sense, where a_j is a central $(\alpha, q, \vec{b})$-atom(or a central $(\alpha, q, \vec{b})$-atom of restrict type) supported on $B(0, 2^j)$ and $\sum_{-\infty}^{\infty} |\lambda_j|^p < \infty$(or $\sum_{j=0}^{\infty} |\lambda_j|^p < \infty$). Moreover, $||f||_{H\dot{K}^{\alpha,p}_{q,\vec{b}}}$(or $||f||_{HK^{\alpha,p}_{q,\vec{b}}}$) $\approx (\sum_j |\lambda_j|^p)^{1/p}$.

Definition 4. Let $0 < \delta < n$, $0 < \gamma \le 1$ and Ω be homogeneous of degree zero on R^n such that $\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0$. Assume that $\Omega \in Lip_\gamma(S^{n-1})$, that is there exists a constant $M > 0$ such that for any $x, y \in S^{n-1}$, $|\Omega(x) - \Omega(y)| \le M|x - y|^\gamma$. The Marcinkiewicz multilinear commutator is defined by

$$\mu_\delta^{\tilde{b}}(f)(x) = \left(\int_0^\infty |F_t^{\tilde{b}}(f)(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

where

$$F_t^{\tilde{b}}(f)(x) = \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1-\delta}} \left[\prod_{j=1}^m (b_j(x) - b_j(y))\right] f(y)dy.$$

Set

$$F_t(f)(x) = \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1-\delta}} f(y)dy,$$

we also define that

$$\mu_\delta(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

which is the Marcinkiewicz operator(see [6][13][14]).

2. THEOREMS AND PROOFS

We begin with two preliminary lemmas.

Lemma 1. (see[12]) Let $1 < r < \infty$, $b_j \in BMO$ for $j = 1, \cdots, k$ and $k \in N$. Then, we have

$$\frac{1}{|Q|}\int_Q \prod_{j=1}^k |b_j(y) - (b_j)_Q| dy \le C \prod_{j=1}^k ||b_j||_{BMO}$$

and

$$\left(\frac{1}{|Q|}\int_Q\prod_{j=1}^{k}|b_j(y)-(b_j)_Q|^r dy\right)^{1/r}\leq C\prod_{j=1}^{k}||b_j||_{BMO}.$$

Lemma 2. (see[14]) Let $0<\delta<n$, $1<s<n/\delta$ and $1/r=1/s-\delta/n$. Then $\mu_\delta^{\vec{b}}$ is bounded from $L^s(R^n)$ to $L^r(R^n)$.

Theorem 1. Let $0<\delta<n$, $\max\left(n/(n+\gamma-\delta),n/(n+1/2-\delta)\right)<p\leq 1, 1/q=1/p-\delta/n$, $\vec{b}=(b_1,\cdots,b_m)$, $b_i\in BMO, 1\leq i\leq m$. Then $\mu_\delta^{\vec{b}}$ is bounded from $H_{\vec{b}}^p(R^n)$ to $L^q(R^n)$.

Proof It suffices to show that there exist a constant $C>0$, such that for every $(p,\vec{b})$ atom a,

$$||\mu_\delta^{\vec{b}}(a)||_{L^q}\leq C.$$

Let a be a $(p,\vec{b})$ atom supported on a ball $B=B(x_0,2d)$.

$$\int_{R^n}|\mu_\delta^{\vec{b}}(a)(x)|^q dx=\int_{|x-x_0|\leq 2d}|\mu_\delta^{\vec{b}}(a)(x)|^q dx+$$

$$+\int_{|x-x_0|>2d}|\mu_\delta^{\vec{b}}(a)(x)|^q dx=I+II.$$

For I, taking $r,s>1$ with $q<s<n/\delta$ and $1/r=1/s-\delta/n$, by Hölder's inequality and the (L^s,L^r)-boundedness of $\mu_\delta^{\vec{b}}$, we get

$$I\leq C||\mu_\delta^{\vec{b}}(a)||_{L^r}^q|B(x_0,2d)|^{1-q/r}\leq$$

$$\leq C||a||_{L^s}^q|B|^{1-q/r}\leq C|B|^{-q/p+q/s+1-q/r}\leq C.$$

For II, denoting $\lambda=(\lambda_1,\cdots,\lambda_m)$ with $\lambda_i=(b_i)_B,\ 1\leq i\leq m$, where $(b_i)_B=|B(x_0,2d)|^{-1}\int_{B(x_0,2d)}b_i(x)dx$, by Hölder's inequality and the vanishing moment of a, we get

$$\begin{aligned}II &\leq \left[\int_0^{|x-x_0|+2d}\left|\int_{|x-y|<t}\prod_{j=1}^{m}(b_j(x)-b_j(y))a(y)\frac{\Omega(x-y)}{|x-y|^{n-1-\delta}}dy\right|^2\frac{dt}{t^3}\right]^{1/2}\\ &+\left[\int_{|x-x_0|+2d}^{\infty}\left|\int_{|x-y|<t}\prod_{j=1}^{m}(b_j(x)-b_j(y))a(y)\frac{\Omega(x-y)}{|x-y|^{n-1-\delta}}dy\right|^2\frac{dt}{t^3}\right]^{1/2}\\ &= II_1+II_2.\end{aligned}$$

Note that $|x-y| \sim |x-x_0| \sim |x-x_0|+2d$ for $|x-x_0|>2d$, $y\in B$. Then we have

$$II_1 \le C\int_{R^n}\left(\int_{|x-y|}^{|x-x_0|+2d}\frac{dt}{t^3}\right)^{1/2}\prod_{j=1}^{m}|b_j(x)-b_j(y)||a(y)|\frac{\Omega(x-y)}{|x-y|^{n-1-\delta}}dy$$

$$\le C\int_{R^n}\left|\frac{1}{|x-y|^2}-\frac{1}{(|x-x_0|+2d)^2}\right|^{1/2}\prod_{j=1}^{m}|b_j(x)-b_j(y)||a(y)|\frac{\Omega(x-y)}{|x-y|^{n-1-\delta}}dy$$

$$\le C\int_{R^n}\prod_{j=1}^{m}|b_j(x)-b_j(y)||a(y)|\frac{|\Omega(x-y)|}{|x-x_0|^{n-1-\delta}}\frac{|y-x_0|^{1/2}}{|x-x_0|^{3/2}}dy$$

$$\le C\int_{R^n}\prod_{j=1}^{m}|b_j(x)-b_j(y)||a(y)|\frac{|\Omega(x-y)|}{|x-x_0|^{n+1/2-\delta}}|y-x_0|^{1/2}dy$$

$$\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{|x-x_0|^{n+1/2-\delta}}\left(\int_B|(\vec{b}(y)-\lambda)_{\sigma^c}||a(y)||y-x_0|^{1/2}dy\right)\cdot$$

$$\cdot|(\vec{b}(x)-\lambda)_\sigma| \le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{d^{n(1-1/p+1/2n)}}{|x-x_0|^{n+1/2-\delta}}||\vec{b}_{\sigma^c}||_{BMO}\cdot|(\vec{b}(x)-\lambda)_\sigma|.$$

For II_2, using the following inequality (see [14])

$$\left|\frac{\Omega(x-y)}{|x-y|^{n-1-\delta}}-\frac{\Omega(x-x_0)}{|x-x_0|^{n-1-\delta}}\right| \le C\left(\frac{|y-x_0|}{|x-x_0|^{n-\delta}}+\frac{|y-x_0|^\gamma}{|x-x_0|^{n-1+\gamma-\delta}}\right),$$

we can obtain

$$II_2 = \int_{R^n}\left(\int_{|x-x_0|+2d}^{\infty}\frac{dt}{t^3}\right)^{1/2}\left|\prod_{j=1}^{m}(b_j(x)-b_j(y))a(y)\frac{\Omega(x-y)}{|x-y|^{n-1-\delta}}dy\right| \le$$

$$\le C\left|\int_{R^n}\prod_{j=1}^{m}(b_j(x)-b_j(y))a(y)\left(\frac{\Omega(x-y)}{|x-y|^{n-1-\delta}}-\frac{\Omega(x-x_0)}{|x-x_0|^{n-1-\delta}}\right)dy\right|\frac{1}{|x-x_0|+2d} \le$$

$$\leq C\int_{R^n}\prod_{j=1}^{m}|b_j(x)-b_j(y)||a(y)|\left(\frac{|y-x_0|}{|x-x_0|^{n+1-\delta}}+\frac{|y-x_0|^{\gamma}}{|x-x_0|^{n+\gamma-\delta}}\right)dy\leq$$

$$\leq C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{|(\vec{b}(x)-\lambda)_\sigma|}{|x-x_0|^{n+1-\delta}}\int_B|(\vec{b}(y)-\lambda)_{\sigma^c}||a(y)||y-x_0|dy+$$

$$+C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{|(\vec{b}(x)-\lambda)_\sigma|}{|x-x_0|^{n+\gamma-\delta}}\int_B|(\vec{b}(y)-\lambda)_{\sigma^c}||a(y)||y-x_0|^{\gamma}dy\leq$$

$$\leq C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||\vec{b}_{\sigma^c}||_{BMO}\left(\frac{d^{n(1-1/p+1/n)}}{|x-x_0|^{n+1-\delta}}+\frac{d^{n(1-1/p+\gamma/n)}}{|x-x_0|^{n+\gamma-\delta}}\right)|(\vec{b}(x)-\lambda)_\sigma|.$$

Thus, we have

$$II\leq\sum_{k=1}^{\infty}\int_{2^{k+1}dr\geq|x-x_0|>2^kd}|\mu_\delta^{\vec{b}}(a)(x)|^qdx\leq C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||\vec{b}_{\sigma^c}||_{BMO}^q\sum_{k=1}^{\infty}\int_{C_k}\cdot$$

$$\cdot\left(\frac{d^{n(1+1/2n-1/p)}}{|x-y|^{n+1/2-\delta}}+\frac{d^{n(1+1/n-1/p)}}{|x-x_0|^{n+1-\delta}}+\frac{d^{n(1+\gamma/n-1/p)}}{|x-x_0|^{n+\gamma-\delta}}\right)^q\left|(\vec{b}(x)-\lambda)_\sigma\right|^qdx\leq$$

$$\leq C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||\vec{b}_{\sigma^c}||_{BMO}^q\sum_{k=1}^{\infty}\left(\frac{d^{qn(1+1/2n-1/q-\delta/n)}}{|2^kd|^{q(n+1/2-\delta)}}+\frac{d^{qn(1+1/n-1/q-\delta/n)}}{|2^kd|^{q(n+1-\delta)}}+\right.$$

$$\left.\frac{d^{qn(1+\gamma/n-1/q-\delta/n)}}{|2^kd|^{q(n+\gamma-\delta)}}\right)(2^kd)^n\times\left(\frac{1}{|2^kB|}\int_{2^kB}|(\vec{b}(x)-\lambda)_\sigma|dx\right)^q\leq$$

$$\leq C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}k^q\left(2^{-k(q(n+1/2-\delta)-n)}+2^{-k(q(n+1-\delta)-n)}+2^{-k(q(n+\gamma-\delta)-n)}\right)\cdot$$

$$\cdot||\vec{b}||_{BMO}^q\leq C||\vec{b}||_{BMO}^p.$$

This finishes the proof of Theorem 1.

Theorem 2. Let $0<\delta<n$, $0<p<\infty$, $1<q_1,q_2<\infty$,$1/q_1-1/q_2=\delta/n$, $n(1-1/q_1)+1/2+\delta\le\alpha<\min((n(1-1/q_1)+\gamma+\delta, n(1-1/q_1)+1+\delta)$ and $b_i\in BMO(R^n), 1\le i\le m$, $\vec{b}=(b_1,\cdots,b_m)$. Then $\mu_\delta^{\vec{b}}$ is bounded from $H\dot{K}^{\alpha,p}_{q_1,\vec{b}}(R^n)$ to $\dot{K}^{\alpha,p}_{q_2}(R^n)$.

Proof. Let $f\in H\dot{K}^{\alpha,p}_{q,\vec{b}}(R^n)$ and $f(x)=\sum_{j=-\infty}^{\infty}\lambda_j a_j(x)$ be the atomic decomposition for f as in Definition 3, we write

$$||\mu_\delta^{\vec{b}}(f)(x)||^p_{\dot{K}^{\alpha,p}_{q_2}}\le\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=-\infty}^{k-3}|\lambda_j|||\mu_\delta^{\vec{b}}(a_j)(x)\chi_k||_{L^{q_2}}\right)^p+$$

$$+\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=k-2}^{\infty}|\lambda_j|||\mu_\delta^{\vec{b}}(a_j)(x)\chi_k||_{L^{q_2}}\right)^p=J+JJ.$$

For JJ, by the (L^{q_1},L^{q_2})-boundedness of $\mu_\delta^{\vec{b}}$, we get

$$JJ\le C\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=k-2}^{\infty}|\lambda_j|||a_j||_{L^{q_1}}\right)^p\le C\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=k-2}^{\infty}|\lambda_j|2^{-j\alpha}\right)^p\le$$

$$\le\begin{cases}C\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=k-2}^{\infty}|\lambda_j|^p2^{-j\alpha p}\right), & 0<p\le 1\\ C\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=k-2}^{\infty}|\lambda_j|^p2^{-j\alpha p/2}\right)\left(\sum_{j=k-2}^{\infty}2^{-j\alpha p'/2}\right)^{p/p'}, & p>1\end{cases}\le$$

$$\le\begin{cases}C\sum_{j=-\infty}^{\infty}|\lambda_j|^p\left(\sum_{k=-\infty}^{j+2}2^{(k-j)\alpha p}\right), & 0<p\le 1\\ C\sum_{j=-\infty}^{\infty}|\lambda_j|^p\left(\sum_{k=-\infty}^{j+2}2^{(k-j)\alpha p/2}\right)\left(\sum_{k=-\infty}^{j+2}2^{(k-j)\alpha p'/2}\right)^{p/p'}, & p>1\end{cases}\le$$

$$\le C\sum_{j=-\infty}^{\infty}|\lambda_j|^p\le C||f||^p_{H\dot{K}^{\alpha,p}_{q_1,\vec{b}}}.$$

For J, let $x\in B_k\setminus B_{k-1}$, $b_j^i=|B_j|^{-1}\int_{B_j}b_i(x)dx$, $1\le i\le m$, $\vec{b'}=(b_j^1,...,b_j^m)$, we have

$$\mu_\delta^{\vec{b}}(a_j)(x)=\left(\int_0^\infty\left|\int_{|x-y|<t}\prod_{i=1}^m(b_i(x)-b_i(y))\frac{\Omega(x-y)}{|x-y|^{n-1}}a_j(y)dy\right|^2\frac{dt}{t^3}\right)^{1/2}=$$

$$=\left(\int_0^{|x|+2^j}\left|\int_{|x-y|<t}\prod_{i=1}^m(b_i(x)-b_i(y))\frac{\Omega(x-y)}{|x-y|^{n-1}}a_j(y)dy\right|^2\frac{dt}{t^3}\right)^{1/2}+$$

$$+\left(\int_{|x|+2^j}^\infty\left|\int_{|x-y|<t}\prod_{i=1}^m(b_i(x)-b_i(y))\frac{\Omega(x-y)}{|x-y|^{n-1}}a_j(y)dy\right|^2\frac{dt}{t^3}\right)^{1/2}=G+H.$$

For G, noting that $y\in B_j,\ x\in B(0,2^k)\setminus B(0,2^{k-1})$, $j\le k-3$, we know $|x-y|\sim|x|\sim|x|+2^j$. Then, similar to the proof of Theorem 1, we obtain

$$G\le C\int_{B_j}\left|\int_{|x-y|}^{|x|+2^j}\frac{dt}{t^3}\right|^{1/2}\prod_{i=1}^m|b_i(x)-b_i(y)|\frac{|a_j(y)|}{|x-y|^{n-1-\delta}}dy\le$$

$$\le\int_{B_j}\left|\frac{1}{|x-y|^2}-\frac{1}{(|x|+2^j)^2}\right|^{1/2}\prod_{i=1}^m|b_i(x)-b_i(y)|\frac{|a_j(y)|}{|x-y|^{n-1-\delta}}dy\le$$

$$\le C2^{j(1/2+\delta)}\int_{B_j}\frac{1}{|x|^{n+1/2}}\prod_{i=1}^m|b_i(x)-b_i(y)||a_j(y)|dy\le$$

$$\le C\frac{2^{j(1/2+\delta)}}{|x|^{n+1/2}}\sum_{i=0}^m\sum_{\sigma\in C_i^m}|(\vec{b}(x)-\vec{b'})_\sigma|\int_{B_j}|a_j(y)||(\vec{b}(y)-\vec{b})_{\sigma^c}|dy\le$$

$$\le C\frac{2^{j(1/2+\delta+n(1-1/q_1)-\alpha)}}{|x|^{n+1/2}}\sum_{i=0}^m\sum_{\sigma\in C_i^m}||\vec{b}_{\sigma^c}||_{BMO}|(\vec{b}(x)-\vec{b'})_\sigma|,$$

$$H\le\left(\int_{|x|+2^j}^\infty\left|\int_{|x-y|<t}\prod_{i=1}^m(b_i(x)-b_i(y))\left(\frac{\Omega(x-y)}{|x-y|^{n-1-\delta}}-\frac{\Omega(x)}{|x|^{n-1-\delta}}\right)a_j(y)dy\right|^2\frac{dt}{t^3}\right)^{1/2}\le$$

$$\le\int_{B_j}\frac{1}{|x|+2^j}\left(\frac{2^j}{|x|^{n-\delta}}+\frac{(2^j)^\gamma}{|x|^{n-1+\gamma-\delta}}\right)\prod_{i=1}^m|b_i(x)-b_i(y)||a_j(y)|dy\le$$

$$\leq C\left(\frac{2^{j(1+\delta)}}{|x|^{n+1}}+\frac{2^{j(\gamma+\delta)}}{|x|^{n+\gamma}}\right)\sum_{i=0}^{m}\sum_{\sigma\in C_i^m}|(\vec{b}(x)-\vec{b'})_\sigma|\int_{B_j}|a_j(y)||(\vec{b}(y)-\vec{b})_{\sigma^c}|dy\leq$$

$$\leq C\left(\frac{2^{j(1+\delta+n(1-1/q_1)-\alpha)}}{|x|^{n+1}}+\frac{2^{j(\gamma+\delta+n(1-1/q_1)-\alpha)}}{|x|^{n+\gamma}}\right)\sum_{i=0}^{m}\sum_{\sigma\in C_i^m}||\vec{b}_{\sigma^c}||_{BMO}|(\vec{b}(x)-\vec{b'})_\sigma|,$$

thus

$$||\mu_\Omega^{\vec{b}}(a_j)\chi_k||_{L^{q_2}}\leq C2^{j(1/2+\delta+n(1-1/q_1)-\alpha)}\sum_{i=0}^{m}\sum_{\sigma\in C_i^m}||\vec{b}_{\sigma^c}||_{BMO}\cdot$$

$$\cdot\left[\int_{C_k}\left(|x|^{-(n+1/2)}|(\vec{b}(x)-\vec{b'})_\sigma|\right)^{q_2}\right]^{1/q_2}+C2^{j(1+\delta+n(1-1/q_1)-\alpha)}.$$

$$\cdot\sum_{i=0}^{m}\sum_{\sigma\in C_i^m}||\vec{b}_{\sigma^c}||_{BMO}\left[\int_{C_k}\left(|x|^{-(n+1)}|(\vec{b}(x)-\vec{b'})_\sigma|\right)^{q_2}\right]^{1/q_2}+C2^{j(\gamma+\delta+n(1-1/q_1)-\alpha)}.$$

$$\cdot\sum_{i=0}^{m}\sum_{\sigma\in C_i^m}||\vec{b}_{\sigma^c}||_{BMO}\left[\int_{C_k}\left(|x|^{-(n+\gamma)}|(\vec{b}(x)-\vec{b'})_\sigma|\right)^{q_2}\right]^{1/q_2}\leq$$

$$\leq C2^{j(1/2+\delta+n(1-1/q_1)-\alpha)-(k-1)(n+1/2)+kn(1/q_1-\delta/n)}||\vec{b}||_{BMO}+$$

$$+C2^{j(1+\delta+n(1-1/q_1)-\alpha)-(k-1)(n+1)+kn(1/q_1-\delta/n)}||\vec{b}||_{BMO}+$$

$$+C2^{j(\gamma+\delta+n(1-1/q_1)-\alpha)-(k-1)(n+\gamma)+kn(1/q_1-\delta/n)}||\vec{b}||_{BMO},$$

to be simply, we denote

$$\begin{aligned} & W(j,k)= \\ = \ & 2^{j(1/2+\delta+n(1-1/q_1)-\alpha)-(k-1)(n+1/2)+kn(1/q_1-\delta/n)} \\ + \ & 2^{j(1+\delta+n(1-1/q_1)-\alpha)-(k-1)(n+1)+kn(1/q_1-\delta/n)} \\ + \ & 2^{j(\gamma+\delta+n(1-1/q_1)-\alpha)-(k-1)(n+\gamma)+kn(1/q_1-\delta/n)}, \end{aligned}$$

then

$$||\mu_\Omega^{\vec{b}}(a_j)\chi_k||_{L^{q_2}}\leq C||\vec{b}||_{BMO}W(j,k),$$

we obtain

$$
\begin{aligned}
J &\le C||\vec{b}||_{BMO}^{p}\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=-\infty}^{k-3}|\lambda_j|W(j,k)\right)^{p}\\
&\le \begin{cases} C||\vec{b}||_{BMO}^{p}\sum_{j=-\infty}^{\infty}|\lambda_j|^{p}\sum_{k=j+3}^{\infty}W(j,k)^{p}, & 0<p\le 1\\ C||\vec{b}||_{BMO}^{p}\sum_{j=-\infty}^{\infty}|\lambda_j|^{p}\left[\sum_{k=j+3}^{\infty}W(j,k)^{p/2}\right]\left[\sum_{k=j+3}^{\infty}W(j,k)^{p'/2}\right]^{p/p'}, & p>1\end{cases}\\
&\le C||\vec{b}||_{BMO}^{p}\sum_{j=-\infty}^{\infty}|\lambda_j|^{p}\le C||\vec{b}||_{BMO}^{p}||f||_{H\dot{K}_{q_1,\vec{b}}^{\alpha,p}}^{p}.
\end{aligned}
$$

This completes the proof of the Theorem 2.

Remark. *Theorem 2 also hold for nonhomogeneous Herz-type spaces.*

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Some fixed point theorems involving weak contraction conditions of the derivative type

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ABSTRACT. In this paper, we shall establish two fixed point theorems by using two contractive definitions involving the derivative of a certain function. Our contractive conditions are independent of that of Branciari [10].

Our results are generalizations of the classical Banach's fixed point theorem [1, 3, 5, 8, 25] as well as some of the results of Berinde [4, 5, 8].

1. INTRODUCTION

Let (X, d) be a complete metric space and $f : X \to X$ a selfmap of X. Suppose that $F_f = \{x \in X \mid f(x) = x\}$ is the set of fixed points of f. The classical Banach's fixed point theorem is established in Banach [3] by using the following contractive definition: there exists $c \in [0, 1)$ (fixed) such that $\forall\ x,\ y \in X$, we have

$$d(f(x), f(y)) \leq c\ d(x, y). \tag{1}$$

In generalizing the Banach's fixed point theorem, Kannan [17] used the following contractive definition: there exists $a \in [0, \frac{1}{2})$ such that

$$d(f(x), f(y)) \leq a[d(x, f(x)) + d(y, f(y))], \quad \in x,\ y \in X. \tag{2}$$

Chatterjea [11] employed a dual contractive condition: that is, there exists $a \in [0, \frac{1}{2})$ such that

$$d(f(x), f(y)) \leq a[d(x, f(y)) + d(y, f(x))], \quad \in x,\ y \in X. \tag{3}$$

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In 1972, Zamfirescu [24] used a combination of the conditions (1), (2) and (3) to establish a nice generalization of Banach's fixed point theorem.

Literature abounds with several generalizations of the classical Banach's fixed point theorem since 1922. For some of these generalizations of the classical Banach's fixed point theorem and various contractive definitions that have been employed, we refer the readers to [1, 2, 4, 5, 7, 8, 9, 12, 13, 21, 24] and other references listed in the reference section of this paper. Imoru and Olatinwo [15] and Olatinwo [19] have also investigated stability of iteration processes with some other contractive-type conditions.

In this paper, we shall establish two fixed point theorems by using two contractive definitions involving the derivative of a certain function.

Our results are generalizations of the classical Banach's fixed point theorem [1, 3, 5, 8, 25] as well as some of the results of Berinde [4, 5, 8].
We shall require the following definition in the sequel:

Definition 1.1. [Berinde [5,8]]: A function $\psi R_+ \to R_+$ is called a *comparison function* if:
(i) ψ is monotone increasing;
(ii) $\lim\limits_{n\to\infty} \psi^n(t) = 0, \forall\ t \geq 0.$

Remark 1.1. *Every comparison function satisfies the condition* $\psi(0) = 0$. *Also, both conditions (i) and (ii) of Definition 1.1 imply that* $\psi(t) < t,\ \forall\ t > 0.$

Definition 1.2. X is said to be *d−bounded* if
$\delta_d(X) = sup\,\{d(x, y) \mid x,\ y \in X\} < \infty.$
We shall employ the following contractive conditions:
(a) there exists $k \in [0, 1)$ such that $\forall\ x,\ y \in X$, we have

$$\frac{d\varphi}{dt}|_{t=d(f(x),f(y))} \leq k\frac{d\varphi}{dt}|_{t=d(x,y)}, \tag{4}$$

where $\varphi R^+ \to R^+$ is a function such that $\frac{d\varphi}{dt}|_{t=\epsilon} > 0$ for each $\epsilon > 0$;
(b) and also there exists a comparison function $\psi R^+ \to R^+$ such that

$\forall\ x,\ y \in X$, we have

$$\frac{d\varphi}{dt}|_{t=d(f(x),f(y))} \leq \psi\left(\frac{d\varphi}{dt}|_{t=d(x,y)}\right), \tag{5}$$

where $\varphi R^+ \to R^+$ is as in (4).

Remark 1.2. *(i) In (4), if we put* $\frac{d\varphi}{dt}|_{t=d(f(x),f(y))} = t = \frac{d\varphi}{dt}|_{t=d(x,y)},\ \forall\ t \in R^+$, *then we obtain the Banach's contraction condition which is defined by eqn. (1).*
(ii) Also, if in (5), $\psi(u) = ku,\ \forall\ u \in R^+,\ k \in [0,1)$, *then we have condition (4).*
(iii) If in (5), $\frac{d\varphi}{dt}|_{t=d(f(x),f(y))} = t = \frac{d\varphi}{dt}|_{t=d(x,y)},\ \forall\ t \in R^+$, *then we obtain an extension of Banach's fixed point theorem in Berinde [4,5,8].*
(iv) Our contractive conditions are independent of those in (2) and (3) as well as that of Branciari [10].

2. THE MAIN RESULTS

Theorem 2.1. Let (X,d) be a complete metric space and $f : X \to X$ a mapping satisfying (4). Suppose that X is $d-$bounded. Let $\varphi R^+ \to R^+$ be a function such that $\frac{d\varphi}{dt}|_{t=\epsilon} > 0$ for each $\epsilon > 0$.
Then, f has a unique fixed point $z \in X$ such that for each $x \in X,\ \lim_{n\to\infty} f^n(x) = z$.
Proof. Let $x_0 \in X$ and let $\{x_n\}_{n=0}^{\infty}$ defined by $x_n = f(x_{n-1}) = f^n x_0,\ n = 1, 2, \cdots$, be the Picard iteration associated to f. From (4), we have that

$$\begin{aligned}\frac{d\varphi}{dt}|_{t=d(x_n,x_{n+m})} = \frac{d\varphi}{dt}|_{t=d(f(x_{n-1}),f(x_{n+m-1}))} &\leq k\frac{d\varphi}{dt}|_{t=d(x_{n-1},x_{n+m-1})}\\ &\leq k^2\frac{d\varphi}{dt}|_{t=d(x_{n-2},x_{n+m-2})}\\ &\leq \cdots \leq k^n\frac{d\varphi}{dt}|_{t=d(x_0,x_m)}\\ &\leq k^n\frac{d\varphi}{dt}|_{t=\delta_d(X)},\end{aligned}$$

from which we obtain that

$$\frac{d\varphi}{dt}|_{t=d(x_n,x_{n+m})} \leq k^n\frac{d\varphi}{dt}|_{t=\delta_d(X)}, \tag{6}$$

where $d(x_0,x_m) \leq \delta_d(X)$ and $\delta_d(X)$ is as stated in Definition 1.2. Therefore, we have from (6) that $k^n\frac{d\varphi}{dt}|_{t=\delta_d(X)} \to 0$ as $n \to \infty$ since k $\in [0,1)$. It follows that $\frac{d\varphi}{dt}|_{t=d(x_n,x_{n+m})} \to 0$ as $n \to \infty$. By the condition on φ, then we have that $d(x_n, x_{n+m}) \to 0$ as $n \to \infty$. Hence, $\{x_n\}$ is a Cauchy sequence and so convergent. Since (X, d) is a complete metric space, $\{x_n\}$ converges to some $z \in X$, that is, $\lim\limits_{n\to\infty} x_n = z$. Again, from (4), we have that

$$\tfrac{d\varphi}{dt}|_{t=d(x_{n+1},f(z))} \quad = \tfrac{d\varphi}{dt}|_{t=d(f(x_n),f(z))} \leq k\tfrac{d\varphi}{dt}|_{t=d(x_n,z)}. \tag{7}$$

By taking the limits in (7) as $n \to \infty$, then we get

$$\frac{d\varphi}{dt}|_{t=d(z,f(z))} \leq k\frac{d\varphi}{dt}|_{t=0}. \tag{8}$$

The condition on φ gives $\frac{d\varphi}{dt}|_{t=0} = 0$ so that from (8) $\frac{d\varphi}{dt}|_{t=d(z,f(z))} \leq 0$, which is a contradiction. Therefore, by the condition on φ again, we obtain that $\frac{d\varphi}{dt}|_{t=d(z,f(z))} = 0$, thus leading to $d(z, f(z)) = 0$, or $z = f(z)$.
We now prove that f has a unique fixed point: Suppose this is not true. Then, there exist $z_1,\ z_2 \in F_f,\ z_1 \neq z_2,\ d(z_1, z_2) > 0$. Therefore, we obtain by using (4) again that

$$\tfrac{d\varphi}{dt}|_{t=d(z_1,z_2)} \quad = \tfrac{d\varphi}{dt}|_{t=d(f(z_1),f(z_2))} \leq k\tfrac{d\varphi}{dt}|_{t=d(z_1,z_2)},$$

leading to $(1-k)\frac{d\varphi}{dt}|_{t=d(z_1,z_2)} \leq 0$, from which it follows that $1-k > 0$, but $\frac{d\varphi}{dt}|_{t=d(z_1,z_2)} \leq 0$. Therefore, by the condition on φ again, we get $\frac{d\varphi}{dt}|_{t=d(z_1,z_2)} = 0$ which leads to $d(z_1, z_2) = 0$, or $z_1 = z_2$. Hence, f has a unique fixed point.

Theorem 2.2. Let (X, d) be a complete metric space and $f : X \to X$ a mapping satisfying (5). Suppose that X is $d-$bounded. Let $\psi R^+ \to R^+$ be a continuous comparison function and $\varphi R^+ \to R^+$ a function such that $\frac{d\varphi}{dt}|_{t=\epsilon} > 0$ for each $\epsilon > 0$. Then, f has a unique fixed point $z \in X$ such that for each $x \in X$, $\lim\limits_{n\to\infty} f^n(x) = z$.

Proof. Let $x_0 \in X$ and let $\{x_n\}_{n=0}^{\infty}$ defined by $x_n = f(x_{n-1}) = f^n x_0,\ n = 1, 2, \cdots$, be the Picard iteration associated to f. From (5), we have that

$$\begin{aligned}\tfrac{d\varphi}{dt}|_{t=d(x_n,x_{n+m})} = \tfrac{d\varphi}{dt}|_{t=d(f(x_{n-1}),f(x_{n+m-1}))} \quad &\leq \psi\left(\tfrac{d\varphi}{dt}|_{t=d(x_{n-1},x_{n+m-1})}\right)\\ &\leq \psi^2\left(\tfrac{d\varphi}{dt}|_{t=d(x_{n-2},x_{n+m-2})}\right)\\ &\leq \cdots \leq \psi^n\left(\tfrac{d\varphi}{dt}|_{t=d(x_0,x_m)}\right)\\ &\leq \psi^n\left(\tfrac{d\varphi}{dt}|_{t=\delta_d(X)}\right),\end{aligned}$$

from which we obtain that

$$\frac{d\varphi}{dt}|_{t=d(x_n,x_{n+m})} \leq \psi^n \left(\frac{d\varphi}{dt}|_{t=\delta_d(X)} \right). \tag{9}$$

Therefore, we have from (9) that $\psi^n \left(\frac{d\varphi}{dt}|_{t=\delta_d(X)} \right) \to 0$ as $n \to \infty$ since ψ is a comparison function. It follows that $\frac{d\varphi}{dt}|_{t=d(x_n,x_{n+m})} \to 0$ as $n \to \infty$.

Hence, by the condition on φ, we have that $\{x_n\}$ is a Cauchy sequence and so convergent. Since (X, d) is a complete metric space, $\{x_n\}$ converges to some $z \in X$. Again, from (5), we have that

$$\tfrac{d\varphi}{dt}|_{t=d(x_{n+1},f(z))} = \tfrac{d\varphi}{dt}|_{t=d(f(x_n),f(z))} \leq \psi \left(\tfrac{d\varphi}{dt}|_{t=d(x_n,z)} \right). \tag{10}$$

By taking the limits in (10) as $n \to \infty$, then we get

$$\frac{d\varphi}{dt}|_{t=d(z,f(z))} \leq \psi \left(\frac{d\varphi}{dt}|_{t=0} \right). \tag{11}$$

The condition on φ gives $\psi \left(\frac{d\varphi}{dt}|_{t=0} \right) = \psi(0) = 0$ so that from (11) $\frac{d\varphi}{dt}|_{t=d(z,f(z))} \leq 0$, which is a contradiction. Therefore, by the condition on φ again, we have that $\frac{d\varphi}{dt}|_{t=d(z,f(z))} = 0$, thus leading to $d(z, f(z)) = 0$, or z = f(z).
We now prove that f has a unique fixed point: Suppose this is not true. Then, there exist $z_1,\ z_2 \in F_f,\ z_1 \neq z_2,\ d(z_1, z_2) > 0$. Therefore, we obtain by using (5) again that

$$\begin{aligned} \tfrac{d\varphi}{dt}|_{t=d(z_1,z_2)} &= \tfrac{d\varphi}{dt}|_{t=d(f(z_1),f(z_2))} \\ &\leq \psi \left(\tfrac{d\varphi}{dt}|_{t=d(z_1,z_2)} \right) < \tfrac{d\varphi}{dt}|_{t=d(z_1,z_2)} \text{ (by Remark 1.1)}, \end{aligned}$$

leading to a contradiction. Therefore $\frac{d\varphi}{dt}|_{t=d(z_1,z_2)} = 0$, from which it follows that $d(z_1, z_2) = 0$, or z$_1$ = z$_2$. Hence, f has a unique fixed point.

Remark 2.1. *Theorem 2.1 is a generalization and extension of the celebrated Banach's fixed point [1, 2, 3, 4, 5, 8, 25] while Theorem 2.2 generalizes and extends Theorem 2 of Berinde [4] (which is Theorem 2.8 of Berinde [5,8]).*

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Some cyclical inequalities in inner product spaces

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ABSTRACT. In this paper elementary numerical inequalities are used to derive some elementary cyclical inequalities in real or complex inner product spaces.

1. INTRODUCTION

Inequalities in Inner Product Spaces have important applications in several topics in Contemporary Mathematics.

They are applied in Probability and Statistics, Nonlinear Analysis, Approximation Theory, Optimization and Numerical Analysis among other fields. No doubt that the most famous inequality in inner product spaces is the well-known inequality involving the norms and the inner product of two vectors and published by Schwarz in 1885 [1].

The study of mathematical inequalities in general has drawn the attention of many mathematicians and a lot of papers devoted to the subject have appeared in the scientific literature given simpler and shorter proof of classical results or obtaining refinements or generalizations of them. They have been extensively documented in the works of Hardy, Littlewood and Pólya [2], Beckenbach and Bellman [3] and mainly in Mitrinovic [4].

Inequalities in inner product spaces have been studied mainly by Dragomir among others and they have been documented in a lot of papers and books published in the last decades (see [5], [6], [7]).

Our goal in this paper is to describe an elementary method to obtain cyclical inequalities, in general inner product spaces.

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2. THE INEQUALITIES

In what follows, elementary inequalities are used to obtain new cyclical inequalities in inner product spaces.
We begin with the following

Theorem 1. Let $(X, < \cdot, \cdot >)$ be a real or complex inner product space and let x, y, z be nonzero vectors in X. Then, the following inequality

$$\frac{||x||}{||y||\,||z||}\ |< y, z >|^2 + \frac{||y||}{||z||\,||x||}\ |< z, x >|^2 + \frac{||z||}{||x||\,||y||}\ |< x, y >|^2 \geq$$

$$\geq \left|\frac{< z, x >< x, y >}{||x||}\right| + \left|\frac{< x, y >< y, z >}{||y||}\right| + \left|\frac{< y, z >< z, x >}{||z||}\right|$$

holds.
Proof. Setting $a = \left|\frac{< x, y >}{||x||\,||y||}\right|$, $b = \left|\frac{< y, z >}{||y||\,||z||}\right|$ and $c = \left|\frac{< z, x >}{||z||\,||x||}\right|$ into the well known elementary inequality $a^2 + b^2 + c^2 \geq ab + bc + ca$, we get

$$\left|\frac{< x, y >}{||x||\,||y||}\right|^2 + \left|\frac{< y, z >}{||y||\,||z||}\right|^2 + \left|\frac{< z, x >}{||z||\,||x||}\right|^2 \geq \left|\frac{< x, y >< y, z >}{||x||\,||y||^2\,||z||}\right| +$$

$$+ \left|\frac{< y, z >< z, x >}{||x||\,||y||\,||z||^2}\right| + \left|\frac{< z, x >< x, y >}{||x||^2\,||y||\,||z||}\right|$$

Multiplying up both members of the preceding inequality by $||x||\,||y||\,||z||$ and rearranging terms yields the inequality claimed. Equality holds when x, y, z are collinear and when x, y, z are orthogonal.
This completes the proof.

Theorem 2. Let $(X, < \cdot, \cdot >)$ be a real or complex inner product space and let x, y, z be nonzero vectors in X. Then, the following inequality holds:

$$\frac{|< x, y >| + |< y, z >|}{||y||\,|< z, x >|} + \frac{|< y, z >| + |< z, x >|}{||z||\,|< x, y >|} +$$

$$+ \frac{|< z, x >| + |< x, y >|}{||x||\,|< y, z >|} \geq 6$$

Proof. Setting $a = \left|\frac{< x, y >}{||x||\,||y||}\right|$, $b = \left|\frac{< y, z >}{||y||\,||z||}\right|$ and $c = \left|\frac{< z, x >}{||z||\,||x||}\right|$ into the well known elementary inequality

$$\frac{a + b}{c} + \frac{b + c}{a} + \frac{c + a}{b} \geq 6,$$

we have

$$\sum_{cyclic}\left[\left(\left|\frac{< x,y >}{||x||\,||y||}\right| + \left|\frac{< y,z >}{||y||\,||z||}\right|\right) \Big/ \left|\frac{< z,x >}{||z||\,||x||}\right|\right] =$$

$$= \sum_{cyclic} \frac{|< x,y >| + |< y,z >|}{||y||\,|< z,x >|} \geq 6$$

Theorem 3. Let $(X, < \cdot, \cdot >)$ be a real or complex inner product space and let x, y, z be nonzero vectors in X. Then, the following inequality holds:

$$\frac{|< x,y >|\,||z|| + |< y,z >|\,||x||}{|< z,x >|\,||y||} + \frac{|< y,z >|\,||x|| + |< z,x >|\,||y||}{|< x,y >|\,||z||}$$

$$+ \frac{|< z,x >|\,||y|| + |< x,y >|\,||z||}{|< y,z >|\,||x||} \geq 6$$

Proof. Setting $a = \left|\frac{< x,y >}{||x||\,||y||}\right|$, $b = \left|\frac{< y,z >}{||y||\,||z||}\right|$ and $c = \left|\frac{< z,x >}{||z||\,||x||}\right|$ into the well known elementary inequality

$$\frac{a+b}{c} + \frac{b+c}{a} + \frac{c+a}{b} \geq 6,$$

we have

$$\sum_{cyclic}\left[\left(\left|\frac{< x,y >}{||x||\,||y||}\right| + \left|\frac{< y,z >}{||y||\,||z||}\right|\right) \Big/ \left|\frac{< z,x >}{||x||\,||z||}\right|\right] =$$

$$= \sum_{cyclic} \frac{|< x,y >|\,||z|| + |< y,z >|\,||x||}{|< z,x >|\,||y||} \geq 6$$

Equality holds when $x = y = z$ and the proof is complete.

Theorem 4. Let $(X, < \cdot, \cdot >)$ be a real or complex inner product space and let x, y, z be nonzero vectors in X. Then, holds:

$$\left|\frac{< x,y >}{< y,z >}\right| \frac{||z||}{||x||} + \left|\frac{< y,z >}{< z,x >}\right| \frac{||x||}{||y||} + \left|\frac{< z,x >}{< x,y >}\right| \frac{||y||}{||z||} \geq 3$$

Proof. Applying AM-GM inequality to the positive numbers a, b, c immediately follows

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3$$

Now, setting $a = \left|\dfrac{< x,y >}{||x||\,||y||}\right|$, $b = \left|\dfrac{< y,z >}{||y||\,||z||}\right|$ and $c = \left|\dfrac{< z,x >}{||z||\,||x||}\right|$ into the preceding inequality and rearranging terms we get the inequality claimed.

Theorem 5. Let $(X, < \cdot, \cdot >)$ be a real or complex inner product space and let x, y, z be nonzero vectors in X. Then,

$$\left|\frac{< x,y >}{||z||}\right|^3 + \left|\frac{< y,z >}{||x||}\right|^3 + \left|\frac{< z,x >}{||y||}\right|^3 \geq 3\left|\frac{< x,y >< y,z >< z,x >}{||x||\,||y||\,||z||}\right|$$

Proof. Setting $a = \left|\dfrac{< x,y >}{||z||}\right|$, $b = \left|\dfrac{< y,z >}{||x||}\right|$, and $c = \left|\dfrac{< z,x >}{||y||}\right|$ into the well known elementary inequality $a^3 + b^3 + c^3 \geq 3abc$, we get

$$\left|\frac{< x,y >}{||z||}\right|^3 + \left|\frac{< y,z >}{||x||}\right|^3 + \left|\frac{< z,x >}{||y||}\right|^3 \geq 3\left|\frac{< x,y >< y,z >< z,x >}{||x||\,||y||\,||z||}\right|$$

Multiplying up both members of the preceding inequality by $||x||\,||y||\,||z||$

$$\frac{||x||\,||y||}{||z||^2}| < x,y > |^3 + \frac{||y||\,||z||}{||x||^2}| < y,z > |^3 + \frac{||z||\,||x||}{||y||^2}| < z,x > |^3$$

$$\geq 3| < x,y >< y,z >< z,x > |$$

$$\frac{1}{3}\sum_{cyclic}\frac{||x||\,||y||}{||z||^2}| < x,y > |^3 \geq | < x,y >< y,z >< z,x > |$$

Equality holds when x, y, z are orthogonal. This completes the proof.

Theorem 6. Let $(X, < \cdot, \cdot >)$ be a real or complex inner product space and let x, y, z be nonzero vectors in X. Then,

$$\frac{1}{3}\sum_{cyclic}\frac{||x||^2}{||y||\,||z||}| < y,z > |^3 \geq | < x,y >< y,z >< z,x > |$$

Proof. Setting $a = \left|\dfrac{< x,y >}{||x||\,||y||}\right|$, $b = \left|\dfrac{< y,z >}{||y||\,||z||}\right|$, and $c = \left|\dfrac{< z,x >}{||z||\,||x||}\right|$ into the well known elementary inequality $a^3 + b^3 + c^3 \geq 3abc$, we get

$$\left|\frac{< x,y >}{||x||\,||y||}\right|^3 + \left|\frac{< y,z >}{||y||\,||z||}\right|^3 + \left|\frac{< z,x >}{||z||\,||x||}\right|^3 \geq 3\left|\frac{< x,y >< y,z >< z,x >}{||x||^2\,||y||^2\,||z||^2}\right|$$

Multiplying up both members of the preceding inequality by $||x||^2\,||y||^2\,||z||^2$

$$\frac{||z||^2}{||x||\,||y||}|<x,y>|^3+\frac{||x||^2}{||y||\,||z||}|<y,z>|^3+\frac{||y||^2}{||z||\,||x||}|<z,x>|^3$$

$$\geq 3|<x,y><y,z><z,x>|$$

Taking into account the well-known Cesaro's inequality. Namely, $(a+b)(b+c)(c+a)\geq 8abc$ valid for all triples of positive real numbers, we derive two new inequalities.

Theorem 7. Let $(X,<\cdot,\cdot>)$ be a real or complex inner product space and let x,y,z be nonzero vectors in X. Then, holds

$$\frac{1}{8}\prod_{cyclic}(||x||\,|<y,z>|+||z||\,|<x,y>|)\geq\left|\frac{<x,y><y,z><z,x>}{||x||\,||y||\,||z||}\right| \qquad (a)$$

$$\frac{1}{8}\prod_{cyclic}\left(\frac{|<x,y>}{||z||}+\frac{|<y,z>|}{||x||}\right)\geq\left|\frac{<x,y><y,z><z,x>}{||x||\,||y||\,||z||}\right| \qquad (b)$$

Proof. (a) Setting into Cesaro's inequality $a=\left|\frac{<x,y>}{||x||\,||y||}\right|$, $b=\left|\frac{<y,z>}{||y||\,||z||}\right|$, and $c=\left|\frac{<z,x>}{||z||\,||x||}\right|$ yields

$$\prod_{cyclic}\left(\left|\frac{<x,y>}{||x||\,||y||}\right|+\left|\frac{<y,z>}{||y||\,||z||}\right|\right)\geq 8\left|\frac{<x,y><y,z><z,x>}{||x||^2\,||y||^2\,||z||^2}\right|$$

from which (a) follows after rearranging terms.

(b) Setting $a=\left|\frac{<x,y>}{||z||}\right|$, $b=\left|\frac{<y,z>}{||x||}\right|$, and $c=\left|\frac{<z,x>}{||y||}\right|$ into Cesaro's inequality again, we have

$$\prod_{cyclic}\left(\left|\frac{<x,y>}{||z||}\right|+\left|\frac{<y,z>}{||x||}\right|\right)\geq 8\left|\frac{<x,y><y,z><z,x>}{||x||\,||y||\,||z||}\right|$$

and (b) follows.

Theorem 8. Let $(X, <\cdot,\cdot>)$ be a real or complex inner product space and let x, y, z be nonzero vectors in X. Then, holds

$$\frac{1}{3}\sum_{cyclic}\left|\frac{<x,y>^2\cos(x,y)}{||z||^4}\right|^{p/3} \geq \prod_{cyclic}|\cos(x,y)|^{p/3}$$

Proof. Setting $a = \left|\frac{<x,y>}{||z||}\right|$, $b = \left|\frac{<y,z>}{||x||}\right|$, and $c = \left|\frac{<z,x>}{||y||}\right|$ into the inequality $a^p + b^p + c^p \geq 3(abc)^{p/3}$, we have

$$\left|\frac{<x,y>}{||z||}\right|^p + \left|\frac{<y,z>}{||x||}\right|^p + \left|\frac{<z,x>}{||y||}\right|^p \geq 3\left|\frac{<x,y><y,z><z,x>}{||x||\,||y||\,||z||}\right|^{p/3}$$

or

$$\frac{1}{3}\sum_{cyclic}\left|\frac{<x,y>}{||z||}\right|^p \geq \left(\prod_{cyclic}|<x,y>|^{p/3}\right)\frac{1}{(||x||\,||y||\,||z||)^{p/3}}$$

Multiplying up both sides of the previous inequality by $(||x||\,||y||\,||z||)^{p/3}$ we get

$$\frac{1}{3}\sum_{cyclic}\left|\frac{||x||\,||y||}{||z||^2}<x,y>^3\right|^{p/3} \geq \prod_{cyclic}|<x,y>|^{p/3}$$

Taking into account that $<x,y> = ||x||\,||y||\cos(x,y)$, we get

$$\frac{1}{3}\sum_{cyclic}\left|\frac{||x||^4\,||y||^4}{||z||^2}\cos^3(x,y)\right|^{p/3} \geq \prod_{cyclic}|||x||\,||y||\cos(x,y)|^{p/3}$$

$$\geq \left(||x||^2\,||y||^2\,||z||^2\right)^{p/3}\prod_{cyclic}|\cos(x,y)|^{p/3}$$

or equivalently,

$$\frac{1}{3}\sum_{cyclic}\left|\frac{||x||^2\,||y||^2}{||z||^4}\cos^3(x,y)\right|^{p/3} = \frac{1}{3}\sum_{cyclic}\left|\frac{<x,y>^2\cos(x,y)}{||z||^4}\right|^{p/3} \geq$$

$$\geq \prod_{cyclic}|\cos(x,y)|^{p/3}$$

This complete the proof.

Theorem 9. Let $(X, < \cdot, \cdot >)$ be a real or complex inner product space and let x, y, z be nonzero vectors in X. Then, if $\alpha, \beta, \gamma > 0$ such that
$\alpha + \beta + \gamma = 1$,holds

$$\sum_{cyclic} \alpha \left| \frac{||x||\,||y||\ < x, y >^{1-\alpha}}{< y, z >^{\beta} < z, x >^{\gamma}} \right| \leq ||x||^{\beta-1} ||y||^{\gamma-1} ||z||^{\alpha-1}$$

Proof. We begin proving the following inequality: if $\alpha, \beta, \gamma > 0$ such that
$\alpha + \beta + \gamma = 1$, the holds

$$a^{\alpha} b^{\beta} c^{\gamma} \leq \alpha\, a + \beta\, b + \gamma\, c$$

Taking into account Jensen's inequality

$$f\left(\sum_{k=1}^{n} q_k x_k\right) \geq \sum_{k=1}^{n} q_k f(x_k)$$

if $f(x) = lnx,\ q_1 = a,\ q_2 = b,\ q_3 = c$ and $x_1 = a,\ x_2 = b,\ x_3 = c$, then

$$ln(\alpha\, a + \beta\, b + \gamma\, c) \geq \alpha\, lna + \beta\, lnb + \gamma\, lnc = ln(a^{\alpha} b^{\beta} c^{\gamma})$$

Setting $a = \left| \dfrac{< x, y >}{||z||} \right|$, $b = \left| \dfrac{< y, z >}{||x||} \right|$, and $c = \left| \dfrac{< z, x >}{||y||} \right|$ into the preceding inequality, we have

$$\left| \frac{< x, y >}{||z||} \right|^{\alpha} \left| \frac{< y, z >}{||x||} \right|^{\beta} \left| \frac{< z, x >}{||y||} \right|^{\gamma} \geq \alpha \left| \frac{< x, y >}{||z||} \right| + \beta \left| \frac{< y, z >}{||x||} \right| + \gamma \left| \frac{< z, x >}{||y||} \right|$$

Multiplying up both sides of the previous inequality by
$\left(< x, y >^{\alpha} < y, z >^{\beta} < z, x >^{\gamma}\right)^{-1}$ we get

$$\alpha \left| \frac{< x, y >^{1-\alpha}\ ||x||\,||y||}{< y, z >^{\beta} < z, x >^{\gamma}} \right| + \beta \left| \frac{< y, z >^{1-\beta}\ ||y||\,||z||}{< x, y >^{\alpha} < z, x >^{\gamma}} \right| + \gamma \left| \frac{< z, x >^{1-\gamma}\ ||x||\,||z||}{< x, y >^{\alpha} < y, z >^{\beta}} \right|$$
$$\leq ||x||^{\beta-1} ||y||^{\gamma-1} ||z||^{\alpha-1}$$

This complete the proof.

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On the Twice Repeated Trapezoid and Hermite's Rules

Vlad Ciobotariu-Boer [4]

ABSTRACT. In this paper, we state some derived quadrature formulae for which the errors of approximation are smaller than in the twice repeated trapezoid rule. We also give examples to show that the bounds obtained within this paper may be tighter than in the one above. In the end, some applications for special means are given.

1. INTRODUCTION

The following inequalities are well known in the literature as the trapezoid inequality (see [1])

$$\left| \int_a^b f(x)\mathrm{d}x - \frac{f(a)+f(b)}{2} \cdot (b-a) \right| \leq \frac{\|f''\|_\infty}{12} \cdot (b-a)^3 \tag{1.1}$$

and the midpoint inequality (see [2], [3])

$$\left| \int_a^b f(x)\mathrm{d}x - f\left(\frac{a+b}{2}\right) \cdot (b-a) \right| \leq \frac{\|f''\|_\infty}{24} \cdot (b-a)^3, \tag{1.2}$$

where the mapping $f : [a,b] \to \mathbb{R}$ $(a, b \in \mathbb{R},\ a < b)$ is supposed to be twice differentiable on the interval (a, b), with the second derivative bounded on (a, b), that is,

$$\|f''\|_\infty := \sup_{x\in(a,b)} |f''(x)| < \infty.$$

In this paper, we point out the twice repeated trapezoid rule, namely

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$$\left|\int_a^b f(x)\mathrm{d}x - \frac{1}{2}\cdot\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right]\cdot(b-a)\right| \leq$$

$$\leq \frac{||f''||_\infty}{48}\cdot(b-a)^3 \tag{1.3}$$

and the twice repeated Hermite's rule on a double point, namely:

$$\left|\int_a^b f(x)\mathrm{d}x - \frac{1}{2}\cdot\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right]\cdot(b-a) + \frac{(b-a)^2}{32}\cdot[f'(b)-f'(a)]\right| \leq$$

$$\leq \frac{||f''||_\infty}{96}\cdot(b-a)^3. \tag{1.4}$$

If we consider that $\Delta : a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$ is a partition of the interval $[a,b]$ and f is as above, then we can approximate the integral $\int_a^b f(x)\mathrm{d}x$ by the quadrature formulae $A_{\mathrm{TR}}(f,\Delta)$ and $A_{\mathrm{HR}}(f,f',\Delta)$, having the remainder terms given by $R_{\mathrm{TR}}(f,\Delta)$ and $R_{\mathrm{HR}}(f,f',\Delta)$, where

$$A_{\mathrm{TR}}(f,\Delta) := \frac{1}{2}\cdot\sum_{i=1}^{n}\left[\frac{f(x_{i-1})+f(x_i)}{2} + f\left(\frac{x_{i-1}+x_i}{2}\right)\right]\cdot h_i,$$

$$A_{\mathrm{HR}}(f,f',\Delta) := \frac{1}{2}\cdot\sum_{i=1}^{n}\left[\frac{f(x_{i-1})+f(x_i)}{2} + f\left(\frac{x_{i-1}+x_i}{2}\right)\right]\cdot h_i -$$

$$-\frac{1}{32}\cdot\sum_{i=1}^{n}[f'(x_i)-f'(x_{i-1})]\cdot h_i^2,$$

and the remainders satisfy the estimations

$$|R_{\mathrm{TR}}(f,\Delta)| \leq \frac{1}{48}\cdot||f''||_\infty\cdot\sum_{i=1}^{n}h_i^3,$$

$$\left|R_{\mathrm{HR}}(f,f',\Delta)\right| \leq \frac{1}{96}\cdot||f''||_\infty\cdot\sum_{i=1}^{n}h_i^3,$$

where $h_i := x_i - x_{i-1}$ for $i = 1,2,\ldots,n$.

In this paper, using some classical result from the Theory of Inequalitites (Hölder's inequality, Grüs inequality, Hermite-Hadamard inequalities), we state some quadrature formulae for which the errors are smaller than the ones given above.

Some applications to special means: arithmetic means, geometric means, identric means, logarithmic means, etc., are also given.

2. SOME INTEGRAL INEQUALITIES

We shall start with the following lemma, which will be useful in the sequel.

Lemma 2.1. Let $f : [a, b] \to \mathbb{R}$ be a twice differentiable mapping on (a, b). Then, we have the identities

$$\frac{1}{2} \cdot \left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a} \cdot \int_a^b f(x)\mathrm{d}x =$$

$$= \frac{1}{16(b-a)} \cdot \int_a^b (x-a)(b-x) \cdot \left[f''\left(\frac{x+a}{2}\right) + f''\left(\frac{x+b}{2}\right)\right] \mathrm{d}x \tag{2.1}$$

and

$$\frac{1}{b-a} \cdot \int_a^b f(x)\mathrm{d}x - \frac{1}{2} \cdot \left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] + \frac{b-a}{32} \cdot [f'(b) - f'(a)] =$$

$$= \frac{1}{16(b-a)} \cdot \int_a^b \left(x - \frac{a+b}{2}\right)^2 \cdot \left[f''\left(\frac{x+a}{2}\right) + f''\left(\frac{x+b}{2}\right)\right] \mathrm{d}x. \tag{2.2}$$

Proof. At first, we show that

$$\int_a^b (x-a)(b-x) \cdot f''\left(\frac{x+a}{2}\right) \mathrm{d}x =$$

$$= 4(b-a) \cdot \left[f\left(\frac{a+b}{2}\right) + f(a)\right] - 16 \cdot \int_a^{\frac{a+b}{2}} f(x)\mathrm{d}x. \tag{2.3}$$

We have succesively

$$\int_a^b (x-a)(b-x)\cdot f''\left(\frac{x+a}{2}\right)\mathrm{d}x =$$

$$= 2(x-a)(b-x)\cdot f'\left(\frac{x+a}{2}\right)\Big|_a^b - 2\cdot\int_a^b (-2x+a+b)\cdot\left[2\cdot f\left(\frac{x+a}{2}\right)\right]'\mathrm{d}x =$$

$$= -4(-2x+a+b)\cdot f\left(\frac{x+a}{2}\right)\Big|_a^b - 8\cdot\int_a^b f\left(\frac{x+a}{2}\right)\mathrm{d}x =$$

$$= 4(b-a)\cdot\left[f\left(\frac{a+b}{2}\right)+f(a)\right] - 16\cdot\int_a^{\frac{a+b}{2}} f(x)\mathrm{d}x.$$

We can also prove that

$$\int_a^b (x-a)(b-x)\cdot f''\left(\frac{x+b}{2}\right)\mathrm{d}x =$$

$$= 4(b-a)\cdot\left[f(b)+f\left(\frac{a+b}{2}\right)\right] - 16\cdot\int_{\frac{a+b}{2}}^b f(x)\mathrm{d}x. \tag{2.4}$$

From (2.3) and (2.4), we obtain(2.1).

In order to prove (2.2), we show that

$$\int_a^b \left(x-\frac{a+b}{2}\right)^2\cdot f''\left(\frac{x+a}{2}\right)\mathrm{d}x = 16\cdot\int_a^{\frac{a+b}{2}} f(x)\mathrm{d}x-$$

$$-4(b-a)\cdot\left[f\left(\frac{a+b}{2}\right)+f(a)\right] + \frac{(b-a)^2}{2}\cdot\left[f'\left(\frac{a+b}{2}\right)-f'(a)\right]. \tag{2.5}$$

By calculation, we have succesively

$$\int_a^b \left(x-\frac{a+b}{2}\right)^2\cdot f''\left(\frac{x+a}{2}\right)\mathrm{d}x = \int_a^b \left(x-\frac{a+b}{2}\right)^2\cdot\left[2f'\left(\frac{x+a}{2}\right)\right]'\mathrm{d}x =$$

$$= \frac{(b-a)^2}{2} \cdot \left[f'\left(\frac{a+b}{2}\right) - f'(a) \right] - 4 \cdot \int\limits_a^b \left(x - \frac{a+b}{2}\right) \cdot \left[2 \cdot f\left(\frac{x+a}{2}\right)\right]' \mathrm{d}x =$$

$$= \frac{(b-a)^2}{2} \cdot \left[f'\left(\frac{a+b}{2}\right) - f'(a) \right] - 4(b-a) \cdot \left[f\left(\frac{a+b}{2}\right) + f(a) \right] +$$

$$+8 \cdot \int\limits_a^b f\left(\frac{x+a}{2}\right) \mathrm{d}x = \frac{(b-a)^2}{2} \cdot \left[f'\left(\frac{a+b}{2}\right) - f'(a) \right] -$$

$$-4(b-a) \cdot \left[f\left(\frac{a+b}{2}\right) + f(a) \right] + 16 \cdot \int_a^{\frac{a+b}{2}} f(x) \mathrm{d}x.$$

We can also prove that

$$\int\limits_a^b \left(x - \frac{a+b}{2}\right)^2 \cdot f''\left(\frac{x+b}{2}\right) \mathrm{d}x = \frac{(b-a)^2}{2} \cdot \left[f'(b) - f'\left(\frac{a+b}{2}\right) \right] -$$

$$-4(b-a) \cdot \left[f\left(\frac{a+b}{2}\right) + f(b) \right] + \int_{\frac{a+b}{2}}^b f(x) \mathrm{d}x. \tag{2.6}$$

From (2.5) and (2.6), we deduce (2.2).

The following lemma also holds.

Lemma 2.2. Let $f : [a, b] \to \mathbb{R}$ be a twice differentiable mapping on (a, b) and suppose that

$$||f''||_\infty := \sup_{x \in (a,b)} |f''(x)| < \infty.$$

Then, we have the estimations:

$$\left| \int\limits_a^b f(x) \mathrm{d}x - \frac{1}{2} \cdot \left[\frac{f(a) + f(b)}{2} + f\left(\frac{a+b}{2}\right) \right] \cdot (b-a) \right| \leq$$

$$\leq \begin{cases} \frac{\|f''\|_\infty}{48} \cdot (b-a)^3 \\ \|f''\|_q \cdot [B(p+1,p+1)]^{\frac{1}{p}} \cdot \left(\frac{b-a}{2}\right)^{2+\frac{1}{p}}, \ \frac{1}{p}+\frac{1}{q}=1, \ p>1 \\ \frac{\|f''\|_1}{32} \cdot (b-a)^2 \end{cases} \tag{2.7}$$

and

$$\left| \int_a^b f(x)\mathrm{d}x - \frac{1}{2} \cdot \left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right) \right] \cdot (b-a) + \frac{(b-a)^2}{32} \cdot [f'(b)-f'(a)] \right| \leq$$

$$\leq \begin{cases} \frac{\|f''\|_\infty}{96} \cdot (b-a)^3 \\ \frac{\|f''\|_q}{4(2p+1)^{\frac{1}{p}}} \cdot \left(\frac{b-a}{2}\right)^{2+\frac{1}{p}}, \ \frac{1}{p}+\frac{1}{q}=1, \ p>1 \\ \frac{\|f''\|_1}{32} \cdot (b-a)^2 \end{cases}, \tag{2.8}$$

where

$$\|f''\|_1 := \int_a^b |f''(t)|\mathrm{d}t, \quad \|f''\|_q := \left(\int_a^b |f''(t)|^q \right)^{\frac{1}{q}}, \ q>1$$

and B is the Beta function of Euler, that is,

$$B(l,s) := \int_0^1 t^{l-1} \cdot (1-t)^{s-1}\mathrm{d}t, \ l,s>0.$$

Proof. From (2.5), we get the equality

$$\frac{1}{2} \cdot \left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right) \right] \cdot (b-a) - \int_a^b f(x)\mathrm{d}x =$$

$$= \frac{1}{16} \cdot \int_a^b (x-a)(b-x) \cdot \left[f''\left(\frac{x+a}{2}\right) + f''\left(\frac{x+b}{2}\right) \right] \mathrm{d}x.$$

Thus

$$\left|\int_a^b f(x)\mathrm{d}x - \frac{1}{2}\cdot\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]\cdot(b-a)\right| \leq$$

$$\leq \frac{1}{16}\cdot\int_a^b (x-a)(b-x)\cdot\left(\left|f''\left(\frac{x+a}{2}\right)\right|+\left|f''\left(\frac{x+b}{2}\right)\right|\right)\mathrm{d}x. \tag{2.9}$$

Note that

$$\int_a^b (x-a)(b-x)\mathrm{d}x = \frac{(b-a)^3}{6}$$

and then

$$\int_a^b (x-a)(b-x)\cdot\left(\left|f''\left(\frac{x+a}{2}\right)\right|+\left|f''\left(\frac{x+b}{2}\right)\right|\right)\mathrm{d}x \leq$$

$$\leq 2\cdot||f''||_\infty\cdot\int_a^b (x-a)(b-x)\mathrm{d}x = \frac{||f''||_\infty}{3}\cdot(b-a)^3.$$

Thus, by (2.9), we get the first inequality in (**??**). Further, by Hölder's and Minkowski's integral inequalitites, we obtain

$$\int_a^b (x-a)(b-x)\left|f''\left(\frac{x+a}{2}\right)+f''\left(\frac{x+b}{2}\right)\right|\mathrm{d}x \leq$$

$$\leq \left(\int_a^b (x-a)^p\cdot(b-x)^p\mathrm{d}x\right)^{\frac{1}{p}}\cdot\left(\int_a^b\left|f''\left(\frac{x+a}{2}\right)+f''\left(\frac{x+b}{2}\right)\right|^q\mathrm{d}x\right)^{\frac{1}{q}} \leq$$

$$\leq \left(\int_a^b (x-a)^p\cdot(b-x)^p\mathrm{d}x\right)^{\frac{1}{p}}\cdot$$

$$\cdot\left[\left(\int_a^b\left|f''\left(\frac{x+a}{2}\right)\right|^q\mathrm{d}x\right)^{\frac{1}{q}}+\left(\int_a^b\left|f''\left(\frac{x+b}{2}\right)\right|^q\mathrm{d}x\right)^{\frac{1}{q}}\right] \leq$$

$$\leq 2^{1+\frac{1}{q}} \cdot ||f''||_q \cdot \int_a^b (x-a)^p \cdot (b-x)^p \mathrm{d}x,$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$ and $||f''||_q$ is as given above.
Now, note that

$$\int_a^b (x-a)^p \cdot (b-x)^p \mathrm{d}x = (b-a)^{2p+1} \cdot \int_0^1 t^p \cdot (1-t)^p \mathrm{d}t = (b-a)^{2p+1} \cdot B(p+1, p+1),$$

where B is the Beta function of Euler, and the second inequality in (2.7) is proved.

Finally, we have

$$\int_a^b (x-a)(b-x) \cdot \left| f''\left(\frac{x+a}{2}\right) + f''\left(\frac{x+b}{2}\right) \right| \mathrm{d}x \leq$$

$$\leq \max_{x \in [a,b]} [(x-a)(b-x)] \cdot \int_a^b \left[\left| f''\left(\frac{x+a}{2}\right) \right| + \left| f''\left(\frac{x+b}{2}\right) \right| \right] \mathrm{d}x =$$

$$= \frac{(b-a)^2}{2} \cdot \int_a^b |f''(t)| \mathrm{d}t = \frac{||f''||_1}{2} \cdot (b-a)^2,$$

and the third inequality in (2.7) is proved.

From (2.2), we deduce the equality

$$\int_a^b f(x)\mathrm{d}x - \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right](b-a) + \frac{(b-a)^2}{32} \cdot [f'(b) - f'(a)] =$$

$$= \frac{1}{16} \cdot \int_a^b \left(x - \frac{a+b}{2}\right)^2 \cdot \left[f''\left(\frac{x+a}{2}\right) + f''\left(\frac{x+b}{2}\right)\right] \mathrm{d}x.$$

Thus

$$\left|\int\limits_a^b f(x)\mathrm{d}x - \frac{1}{2}\cdot\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right]\cdot(b-a) + \frac{(b-a)^2}{32}\cdot[f'(b)-f'(a)]\right| \leq$$

$$\leq \frac{1}{16}\cdot\int\limits_a^b \left(x-\frac{a+b}{2}\right)^2\cdot\left(\left|f''\left(\frac{x+a}{2}\right)\right| + \left|f''\left(\frac{x+b}{2}\right)\right|\right)\mathrm{d}x. \tag{2.10}$$

Noting that

$$\int\limits_a^b \left(x-\frac{a+b}{2}\right)^2\mathrm{d}x = \frac{(b-a)^3}{12},$$

we have

$$\int\limits_a^b \left(x-\frac{a+b}{2}\right)^2\cdot\left(\left|f''\left(\frac{x+a}{2}\right)\right| + \left|f''\left(\frac{x+b}{2}\right)\right|\right)\mathrm{d}x \leq$$

$$\leq 2\cdot||f''||_\infty\cdot\int\limits_a^b \left(x-\frac{a+b}{2}\right)^2\mathrm{d}x = \frac{(b-a)^3}{6}\cdot||f''||_\infty.$$

Thus, by (2.10), we find the first inequality in (2.8). Further, by Hölder's and Minkowski's integral inequalities, we deduce

$$\int\limits_a^b \left(x-\frac{a+b}{2}\right)^2\cdot\left|f''\left(\frac{x+a}{2}\right) + f''\left(\frac{x+b}{2}\right)\right|\mathrm{d}x \leq$$

$$\leq \left(\int\limits_a^b \left(x-\frac{a+b}{2}\right)^{2p}\mathrm{d}x\right)^{\frac{1}{p}}\cdot\left(\int\limits_a^b \left|f''\left(\frac{x+a}{2}\right) + f''\left(\frac{x+b}{2}\right)\right|^q\right)^{\frac{1}{q}} \leq$$

$$\leq \left(\int_a^b \left(x-\frac{a+b}{2}\right)^{2p}\mathrm{d}x\right)^{\frac{1}{p}}\cdot\left[\left(\int\limits_a^b \left|f''\left(\frac{x+a}{2}\right)\right|^q\mathrm{d}x\right)^{\frac{1}{q}} + \left(\int\limits_a^b \left|f''\left(\frac{x+b}{2}\right)\right|^q\mathrm{d}x\right)^{\frac{1}{q}}\right] \leq$$

$$\leq 2^{1+\frac{1}{q}} \cdot ||f''||_q \cdot \left(\int_a^b \left(x - \frac{a+b}{2} \right)^{2p} \mathrm{d}x \right)^{\frac{1}{p}},$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$ and $||f''||_q$ is as given above.
Now, noting that

$$\int_a^b \left(x - \frac{a+b}{2} \right)^{2p} \mathrm{d}x = \frac{(b-a)^{2p+1}}{(2p+1) \cdot 2^{2p}},$$

the second inequality in (2.8) is proved.

In the end, we have

$$\int_a^b \left(x - \frac{a+b}{2} \right)^2 \cdot \left(\left| f'' \left(\frac{x+a}{2} \right) \right| + \left| f'' \left(\frac{x+b}{2} \right) \right| \right) \mathrm{d}x \leq$$

$$\leq \max \left[\left(x - \frac{a+b}{2} \right)^2 \right] \cdot \int_a^b \left(\left| f'' \left(\frac{x+a}{2} \right) \right| + \left| f'' \left(\frac{x+b}{2} \right) \right| \right) \mathrm{d}x \leq$$

$$\leq \frac{(b-a)^2}{2} \cdot \int_a^b |f''(t)| \mathrm{d}t = \frac{||f''||_1}{2} \cdot (b-a)^2,$$

and the third inequality in (2.8) is proved.
Another result of interest is the following lemma:

Lemma 2.3. Let $f : [a,b] \to \mathbb{R}$ be a twice differentiable mapping on (a,b) and assume that

$$m := \inf_{x \in (a,b)} f''(x) > -\infty \quad \text{and} \quad M := \sup_{x \in (a,b)} f''(x) < \infty. \tag{2.11}$$

Then, we have the estimation

$$\left| \int_a^b f(x) \mathrm{d}x - \frac{1}{2} \left[\frac{f(a)+f(b)}{2} + f \left(\frac{a+b}{2} \right) \right] \cdot (b-a) + \frac{(b-a)^2}{48} \cdot [f'(b) - f'(a)] \right| \leq$$

$$\leq \frac{(b-a)^3(M-m)}{128}. \tag{2.12}$$

Proof. We apply the celebrated Grüss' inequality which says that:

$$\left| \frac{1}{b-a} \cdot \int_a^b h(x)g(x)\mathrm{d}x - \frac{1}{(b-a)^2} \cdot \int_a^b h(x)\mathrm{d}x \cdot \int_a^b g(x)\mathrm{d}x \right| \leq$$

$$\leq \frac{(\Phi-\phi)(\Gamma-\gamma)}{4}, \tag{2.13}$$

where g, h are integrable mappings satisfying the conditions $\phi \leq h(x) \leq \Phi$ and $\gamma \leq g(x) \leq \Gamma$ for all $x \in [a,b]$.
Now, if we choose in (2.16)

$$h(x) = \left(x - \frac{a+b}{2}\right)^2, \quad g(x) = f''\left(\frac{x+a}{2}\right) + f''\left(\frac{x+b}{2}\right), \quad x \in [a,b],$$

we get

$$\phi = 0, \quad \Phi = \frac{(b-a)^2}{4}, \quad \gamma = 2m, \quad \text{and} \quad \Gamma = 2M,$$

and we can state that

$$\left| \frac{1}{b-a} \cdot \int_a^b \left(x - \frac{a+b}{2}\right)^2 \cdot \left[f''\left(\frac{x+a}{2}\right) + f''\left(\frac{x+b}{2}\right)\right] \mathrm{d}x - \frac{1}{(b-a)^2} \cdot \right.$$

$$\left. \cdot \int_a^b \left(x - \frac{a+b}{2}\right)^2 \mathrm{d}x \cdot \int_a^b \left[f''\left(\frac{x+a}{2}\right) + f''\left(\frac{x+b}{2}\right)\right] \mathrm{d}x \right| \leq$$

$$\leq \frac{(b-a)^2 \cdot (M-m)}{8}. \tag{2.14}$$

Since

$$\int_a^b \left(x - \frac{a+b}{2}\right)^2 \mathrm{d}x = \frac{(b-a)^3}{12}$$

and

$$\int_a^b \left[f''\left(\frac{x+a}{2}\right)+f''\left(\frac{x+b}{2}\right)\right]\mathrm{d}x = 2\int_a^b f''(x)\mathrm{d}x = 2[f'(b)-f'(a)],$$

then, from (2.14).

$$\left|\int_a^b \left(x-\frac{a+b}{2}\right)^2\cdot\left[f''\left(\frac{x+a}{2}\right)+f''\left(\frac{x+b}{2}\right)\right]\mathrm{d}x-\frac{(b-a)^2}{6}\cdot[f'(b)-f'(a)]\right| \le$$

$$\le \frac{(b-a)^3\cdot(M-m)}{8}.$$

Finally, using the identity (2.2), we have:

$$\left|\int_a^b f(x)\mathrm{d}x-\frac{1}{2}\cdot\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]\cdot(b-a)+\frac{(b-a)^2}{48}\cdot\left[f'(b)-f'(a)\right]\right| \le$$

$$\le \frac{(b-a)^3\cdot(M-m)}{128}.$$

and the lemma is proved.
In the following, using a classical result on convex functions due to Hermite and Hadamard, we state the lemma:

Lemma 2.4. Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable mapping on (a,b) and assume that $-\infty < m \le f''(x) \le M < \infty$ for all $x\in(a,b)$. Then, we have the double inequalities

$$\frac{m}{48}\cdot(b-a)^2 \le \frac{1}{2}\cdot\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\cdot\int_a^b f(x)\mathrm{d}x \le$$

$$\le \frac{M}{48}\cdot(b-a)^2 \tag{2.15}$$

and

$$\frac{m}{96}\cdot(b-a)^2 \le \frac{1}{b-a}\cdot\int\limits_a^b f(x)\mathrm{d}x - \frac{1}{2}\cdot\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]+$$

$$+\frac{b-a}{32}\cdot[f'(b)-f'(a)] \le \frac{M}{96}\cdot(b-a)^2 \tag{2.16}$$

with the estimations

$$\left|\int\limits_a^b f(x)\mathrm{d}x - \frac{1}{2}\cdot\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]\cdot(b-a)+\frac{(m+M)(b-a)^3}{96}\right| \le$$

$$\le \frac{(M-m)(b-a)^3}{96} \tag{2.17}$$

and

$$\left|\int\limits_a^b f(x)\mathrm{d}x - \frac{1}{2}\cdot\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]\cdot(b-a)+\right.$$

$$\left.+\frac{(b-a)^2}{32}\cdot[f'(b)-f'(a)]-\frac{(m+M)(b-a)^3}{192}\right| \le \frac{(M-m)(b-a)^3}{192}. \tag{2.18}$$

Proof. We use the following inequality for convex mappings $g:[a,b]\to\mathbb{R}$ (see [4])

$$\frac{1}{b-a}\cdot\int\limits_a^b g(x)\mathrm{d}x \le \frac{1}{2}\left[\frac{g(a)+g(b)}{2}+g\left(\frac{a+b}{2}\right)\right]. \tag{2.19}$$

Let us choose firstly $g:[a,b]\to\mathbb{R}$, $g(x)=f(x)-\frac{m}{2}\cdot\left(x-\frac{a+b}{2}\right)^2$. Then g is twice differentiable on $[a,b]$ and

$$g'(x)=f'(x)-m\left(x-\frac{a+b}{2}\right),\quad g''(x)=f''(x)-m\ge 0 \text{ on } (a,b),$$

hence, g is convex on $[a,b]$. Thus, we can apply (2.19) for g and we obtain

$$\frac{1}{b-a}\cdot\int\limits_a^b\left[f(x)-\frac{m}{2}\cdot\left(x-\frac{a+b}{2}\right)^2\right]\mathrm{d}x \le$$

$$\leq \frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)-\frac{m(b-a)^2}{8}\right]$$

or

$$\frac{m}{48}\cdot(b-a)^2 \leq \frac{1}{2}\cdot\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\cdot\int\limits_a^b f(x)\mathrm{d}x,$$

which is identical to the first inequality of (2.15.)

The second part in (2.15) follows by (2.19) applied to the convex (and twice differentiable) mapping $g:[a,b]\to\mathbb{R}$, $g(x)=\frac{M}{2}\cdot\left(x-\frac{a+b}{2}\right)^2-f(x)$.

In order to prove (2.16), we can write successively

$$2m \leq f''\left(\frac{x+a}{2}\right)+f''\left(\frac{x+b}{2}\right)\leq 2M,$$

$$2m\cdot\left(x-\frac{a+b}{2}\right)^2 \leq \left(x-\frac{a+b}{2}\right)^2\cdot\left[f''\left(\frac{x+a}{2}\right)+f''\left(\frac{x+b}{2}\right)\right]\leq$$

$$\leq 2M\cdot\left(x-\frac{a+b}{2}\right)^2,$$

$$\frac{m}{8(b-a)}\cdot\int\limits_a^b\left(x-\frac{a+b}{2}\right)^2\mathrm{d}x \leq \frac{1}{16(b-a)}\cdot$$

$$\cdot\int\limits_a^b\left(x-\frac{a+b}{2}\right)^2\cdot\left[f''\left(\frac{x+a}{2}\right)+f''\left(\frac{x+b}{2}\right)\right]\mathrm{d}x \leq \frac{M}{8(b-a)}\cdot\int\limits_a^b\left(x-\frac{a+b}{2}\right)^2\mathrm{d}x$$

or, applying the identity (2.2)

$$\frac{m}{96}\cdot(b-a)^2 \leq \frac{1}{b-a}\cdot\int\limits_a^b f(x)\mathrm{d}x-\frac{1}{2}\cdot\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]+\frac{(b-a)}{32}\cdot$$

$$\cdot[f'(b)-f'(a)]\leq\frac{M}{96}\cdot(b-a)^2.$$

Now, it is straightforward to see that, for $\alpha \leq t \leq \beta$ and thus $\left|t - \frac{\alpha+\beta}{2}\right| \leq \frac{\beta-\alpha}{2}$ on taking $\alpha = \frac{m}{48} \cdot (b-a)^2$ and $\beta = \frac{M}{48} \cdot (b-a)^2$, from (**??**) we get the desired estimation (2.17). If we take $\alpha = \frac{m}{96} \cdot (b-a)^2$ and $\beta = \frac{M}{96} \cdot (b-a)^2$, from (**??**) we find the estimation (**??**)

Remark 2.1. *In the conditions of Lemma 2.1, if the mapping is convex on* $[a,b]$*, then*

$$\frac{1}{2} \cdot \left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] - \frac{(b-a)}{32} \cdot [f'(b) - f'(a)] \leq \frac{1}{b-a} \cdot \int_a^b f(x)\mathrm{d}x \leq$$

$$\leq \frac{1}{2} \cdot \left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right].$$

3. SOME QUADRATURE RULES

In this section, we consider applications of the integral inequalities developed in Section 2.

Theorem 3.1. Let $f : [a,b] \to \mathbb{R}$ be as in Lemma 2.2. If $\Delta : a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$ is a partition of the interval $[a,b]$, then we have

$$\int_a^b f(x)\mathrm{d}x = A_{\mathrm{TR}}(f,\Delta) + R_{\mathrm{TR}}(f,\Delta) \tag{3.1}$$

and

$$\int_a^b f(x)\mathrm{d}x = A_{\mathrm{HR}}(f,f',\Delta) + R_{\mathrm{HR}}(f,f',\Delta), \tag{3.2}$$

where

$$A_{\mathrm{TR}}(f,\Delta) := \frac{1}{2} \cdot \sum_{i=1}^{n} \left[\frac{f(x_{i-1}) + f(x_i)}{2} + f\left(\frac{x_{i-1}+x_i}{2}\right)\right] \cdot h_i,$$

and

$$A_{\mathrm{HR}}(f,f',\Delta) := \frac{1}{2} \cdot \sum_{i=1}^{n} \left[\frac{f(x_{i-1}) + f(x_i)}{2} + f\left(\frac{x_{i-1}+x_i}{2}\right)\right] \cdot h_i -$$

$$-\frac{1}{32}\cdot\sum_{i=1}^{n}[f'(x_i)-f'(x_{i-1})]\cdot h_i^2,$$

are quadrature rules and the remainders $R_{\mathrm{TR}}(f,\Delta)$ and $R_{\mathrm{TR}}(f,f',\Delta)$ satisfy the relations

$$|R_{\mathrm{TR}}(f,\Delta)|\leq$$

$$\leq\begin{cases}\frac{1}{48}\cdot||f''||_\infty\cdot\sum\limits_{i=1}^{n}h_i^3\\ \frac{||f''||_q}{2^{2+\frac{1}{p}}}\cdot[B(p+1,p+1)]^{\frac{1}{p}}\cdot\left(\sum\limits_{i=1}^{n}h_i^{2p+1}\right)^{\frac{1}{p}},\ \frac{1}{p}+\frac{1}{q}=1,\ p>1\\ \frac{1}{32}\cdot||f''||_1\cdot v^2(\Delta)\end{cases}\tag{3.3}$$

and

$$\left|R_{\mathrm{HR}}(f,f',\Delta)\right|\leq\begin{cases}\frac{1}{96}\cdot||f''||_\infty\cdot\sum\limits_{i=1}^{n}h_i^3\\ \frac{||f''||_q}{2^{4+\frac{1}{p}}\cdot(2p+1)^{\frac{1}{p}}}\cdot\left(\sum\limits_{i=1}^{n}h_i^{2p+1}\right)^{\frac{1}{p}},\ \frac{1}{p}+\frac{1}{q}=1,\ p>1\\ \frac{1}{32}\cdot||f''||_1\cdot v^2(\Delta)\end{cases},\tag{3.4}$$

where $h_i:=x_i-x_{i-1}$, $i=1,2,\ldots,n$ and $v(\Delta)=\max\limits_{i=\overline{1,n}}h_i$.

Proof. Applying the first inequality in (2.7), we obtain

$$\left|\int\limits_{x_{i-1}}^{x_i}f(x)\mathrm{d}x-\frac{1}{2}\cdot\left[\frac{f(x_{i-1})+f(x_i)}{2}+f\left(\frac{x_{i-1}+x_i}{2}\right)\right]\cdot h_i\right|\leq\frac{||f''||_\infty}{48}\cdot h_i^3$$

for all $i\in\{1,2,\ldots,n\}$.

Summing over i from 1 to n, we get the first part of (3.3).

The second inequality in (2.7) gives us

$$\left|\int\limits_{x_{i-1}}^{x_i}f(x)\mathrm{d}x-\frac{1}{2}\cdot\left[\frac{f(x_{i-1})+f(x_i)}{2}+f\left(\frac{x_{i-1}+x_i}{2}\right)\right]\cdot h_i\right|\leq$$

$$\leq \left(\frac{h_i}{2}\right)^{2+\frac{1}{p}} \cdot [B(p+1,p+1)]^{\frac{1}{p}} \cdot \left(\int\limits_{x_{i-1}}^{x_i} |f''(t)|^q \mathrm{d}t\right)^{\frac{1}{q}}$$

for all $i \in \{1, 2, \ldots, n\}$.
Summing and using Hölder's discrete inequality, we obtain

$$|R_{\mathrm{TR}}(f,\Delta)| = \left|\int\limits_a^b f(x)\mathrm{d}x - A_{\mathrm{TR}}(f,\Delta)\right| \leq$$

$$\leq [B(p+1,p+1)]^{\frac{1}{p}} \cdot \sum_{i=1}^{n} \left(\frac{h_i}{2}\right)^{\frac{2p+1}{p}} \cdot \left(\int\limits_{x_{i-1}}^{x_i} |f''(t)|^q \mathrm{d}t\right)^{\frac{1}{q}} =$$

$$= [B(p+1,p+1)]^{\frac{1}{p}} \cdot \left(\sum_{i=1}^{n} \left(\frac{h_i}{2}\right)^{2p+1}\right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^{n} \int\limits_{x_{i-1}}^{x_i} |f''(t)|^q \mathrm{d}t\right)^{\frac{1}{q}} =$$

$$= \frac{||f''||_q}{2^{2+\frac{1}{p}}} \cdot [B(p+1,p+1)]^{\frac{1}{p}} \cdot \left(\sum_{i=1}^{n} h_i^{2p+1}\right)^{\frac{1}{p}}.$$

The third inequality in(2.7) gives us

$$\left|\int\limits_{x_{i-1}}^{x_i} f(x)\mathrm{d}x - \frac{1}{2} \cdot \left[\frac{f(x_{i-1}) + f(x_i)}{2} + f\left(\frac{x_{i-1} + x_i}{2}\right)\right] \cdot h_i\right| \leq$$

$$\leq \frac{h_i^2}{32} \cdot \int\limits_{x_{i-1}}^{x_i} |f''(t)|\mathrm{d}t$$

for all $i \in \{1, 2, \ldots, n\}$.
Summing over i from 1 to n, we obtain

$$|R_{\mathrm{TR}}(f,\Delta)| \leq \frac{1}{32} \cdot \sum_{1}^{n} \left(\int\limits_{x_{i-1}}^{x_i} |f''(t)|\mathrm{d}t\right) \cdot h_i^2 \leq$$

$$\leq \frac{1}{32} \cdot \max_{i=\overline{1,n}} h_i^2 \cdot \sum_{i=1}^{n} \left(\int_{x_{i-1}}^{x_i} |f''(t)| \mathrm{d}t \right) = \frac{1}{32} \cdot v^2(\Delta) \cdot ||f''||_1$$

and the relations (3.3) are proved.
Now, applying the first inequality in (2.8), we have

$$\left| \int_{x_{i-1}}^{x_i} f(x)\mathrm{d}x - \frac{1}{2} \cdot \left[\frac{f(x_{i-1}) + f(x_i)}{2} + f\left(\frac{x_{i-1} + x_i}{2} \right) \right] \cdot h_i + \frac{h_i^2}{32} \cdot [f'(x_i) - f'(x_{i-1})] \right| \leq$$

$$\leq \frac{h_i^3}{96} \cdot ||f''||_\infty$$

for all $i \in \{1, 2, \dots, n\}$.
Summing over i from 1 to n, we obtain the first part of (**??**). The second inequality in (2.8) gives us

$$\left| \int_{x_{i-1}}^{x_i} f(x)\mathrm{d}x - \frac{1}{2} \cdot \left[\frac{f(x_{i-1}) + f(x_i)}{2} + f\left(\frac{x_{i-1} + x_i}{2} \right) \right] \cdot h_i + \frac{h_i^2}{32} \cdot [f'(x_i) - f'(x_{i-1})] \right| \leq$$

$$\leq \frac{1}{4(2p+1)^{\frac{1}{p}}} \cdot \left(\frac{h_i}{2} \right)^{2+\frac{1}{p}} \cdot \left(\int_{x_{i-1}}^{x_i} |f''(t)|^q \mathrm{d}t \right)^{\frac{1}{q}}$$

for all $i \in \{1, 2, \dots, n\}$.
Summing and using Hölder's discrete inequality, we find

$$|R_{\mathrm{HR}}(f, f', \Delta)| = \left| \int_a^b f(x)\mathrm{d}x - A_{\mathrm{HR}}(f, f', \Delta) \right| \leq$$

$$\leq \frac{1}{2^{4+\frac{1}{p}} \cdot (2p+1)^{\frac{1}{p}}} \cdot \sum_{i=1}^{n} h_i^{2+\frac{1}{p}} \cdot \left(\int_{x_{i-1}}^{x_i} |f''(t)|^q \mathrm{d}t \right)^{\frac{1}{q}} \leq$$

$$\leq \frac{1}{2^{4+\frac{1}{p}} \cdot (2p+1)^{\frac{1}{p}}} \cdot \left(\sum_{i=1}^{n} h_i^{2p+1} \right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |f''(t)|^q \mathrm{d}t \right)^{\frac{1}{q}} =$$

$$= \frac{||f''||_q}{2^{4+\frac{1}{p}} \cdot (2p+1)^{\frac{1}{p}}} \cdot \left(\sum_{i=1}^{n} h_i^{2p+1} \right)^{\frac{1}{p}}.$$

The third inequality in (2.8) gives us

$$\left| \int_{x_{i-1}}^{x_i} f(x)\mathrm{d}x - \frac{1}{2} \cdot \left[\frac{f(x_{i-1}) + f(x_i)}{2} + f\left(\frac{x_{i-1}+x_i}{2} \right) \right] \cdot h_i + \frac{h_i^2}{32} \cdot [f'(x_i) - f'(x_{i-1})] \right| \leq$$

$$\leq \frac{h_i^2}{32} \cdot \int_{x_{i-1}}^{x_i} |f''(t)|\mathrm{d}t$$

for all $i \in \{1, 2, \ldots, n\}$.
Summing over i from 1 to n, we obtain

$$\left| R_{\mathrm{HR}}(f, f', \Delta) \right| \leq \frac{1}{32} \cdot \sum_{i=1}^{n} h_i^2 \cdot \left(\int_{x_{i-1}}^{x_i} |f''(t)|\mathrm{d}t \right) \leq$$

$$\leq \frac{1}{32} \cdot v^2(\Delta) \cdot ||f''||_1.$$

and the theorem is completely proved.
We shall now investigate the case when we have an equidistant partitioning of the interval $[a, b]$, given by

$$\Delta : x_i = a + \frac{b-a}{n} \cdot i, \quad i = 0, 1, \ldots, n.$$

The following result is a consequence of Theorem 3.1.

Corollary 3.1. Let $f : [a, b] \to \mathbb{R}$ be a twice differentiable mapping and $||f''||_\infty < \infty$. Then, we have

$$\int_a^b f(x)\mathrm{d}x = A_{\mathrm{TR},n}(f) + R_{\mathrm{TR},n}(f)$$

and

$$\int_a^b f(x)\mathrm{d}x = A_{\mathrm{HR},n}(f,f') + R_{\mathrm{HR},n}(f,f')$$

where

$$A_{\mathrm{TR},n}(f) = \frac{b-a}{2n}\cdot$$

$$\cdot\sum_{i=1}^{n}\left[\frac{f\left(a+\frac{b-a}{n}\cdot(i-1)\right)+f\left(a+\frac{b-a}{n}\cdot i\right)}{2}+f\left(a+\frac{b-a}{2n}\cdot(2i-1)\right)\right]$$

and

$$A_{\mathrm{HR},n}(f,f') = \frac{b-a}{2n}\cdot$$

$$\cdot\sum_{i=1}^{n}\left[\frac{f\left(a+\frac{b-a}{n}\cdot(i-1)\right)+f\left(a+\frac{b-a}{n}\cdot i\right)}{2}+f\left(a+\frac{b-a}{2n}\cdot(2i-1)\right)\right]-$$

$$-\frac{(b-a)^2}{32n^2}\cdot[f'(b)-f'(a)]$$

and the remainders $R_{\mathrm{TR},n}(f)$ and $R_{\mathrm{HR},n}(f,f')$ satisfy the estimations

$$|R_{\mathrm{TR},n}(f)| \le \begin{cases} \frac{(b-a)^3\cdot\|f''\|_\infty}{48n^2} \\ \frac{\|f''\|_q}{n^2}\cdot\left(\frac{b-a}{2}\right)^{2+\frac{1}{p}}\cdot[B(p+1,p+1)]^{\frac{1}{p}}, \quad \frac{1}{p}+\frac{1}{q}=1,\ p>1 \\ \frac{(b-a)^2\cdot\|f''\|_1}{32n^2} \end{cases}$$

and

$$|R_{\mathrm{HR},n}(f,f')| \le \begin{cases} \frac{(b-a)^3\cdot\|f''\|_\infty}{96n^2} \\ \frac{\|f''\|_q}{4(2p+1)^{\frac{1}{p}}}\cdot\left(\frac{b-a}{2}\right)^{2+\frac{1}{p}}, \quad \frac{1}{p}+\frac{1}{q}=1,\ p>1 \\ \frac{(b-a)^2\cdot\|f''\|_1}{32n^2} \end{cases}$$

for all $n \geq 1$.

Theorem 3.2. Let $f : [a, b] \to \mathbb{R}$ be as in Lemma 2.3 and let Δ be an arbitrary partition of the interval $[a, b]$. Then, we have

$$\int_a^b f(x)\mathrm{d}x = \overline{A}_{\mathrm{HR}}(f, f', \Delta) + \overline{R}_{\mathrm{HR}}(f, f', \Delta) \tag{3.5}$$

where

$$\overline{A}_{\mathrm{HR}}(f, f', \Delta) := \frac{1}{2} \cdot \sum_{i=1}^{n} \left[\frac{f(x_{i-1}) + f(x_i)}{2} + f\left(\frac{x_{i-1} + x_i}{2}\right) \right] \cdot h_i - \\ - \frac{1}{48} \cdot \sum_{i=1}^{n} [f'(x_i) - f'(x_{i-1})] \cdot h_i^2$$

is a derived rule and the remainder term $\overline{R}_{\mathrm{HR}}(f, f', \Delta)$ satisfies the estimation

$$|\overline{R}_{\mathrm{HR}}(f, f', \Delta)| \leq \frac{M - m}{128} \cdot \sum_{i=1}^{n} h_i^3, \tag{3.6}$$

where the h_i are as above.

Proof. Writing the inequality (2.12) on the intervals $[x_{i-1}, x_i]$ $(i = 1, 2, \ldots, n)$, we obtain

$$\left| \int_{x_{i-1}}^{x_i} f(x)\mathrm{d}x - \frac{1}{2} \cdot \left[\frac{f(x_{i-1}) + f(x_i)}{2} + f\left(\frac{x_{i-1} + x_i}{2}\right) \right] \cdot h_i + \frac{1}{48} \cdot [f'(x_i) - f'(x_{i-1})] \cdot h_i^2 \right| \leq \\ \leq \frac{M - m}{128} \cdot h_i^3$$

for all $i \in \{1, 2, \ldots, n\}$.
Summing over i from 1 to n, we deduce easily the desired estimation (3.6).

Remark 3.1. *As*

$$0 \leq M - m \leq 2 \cdot ||f''||_\infty,$$

then

$$\frac{M-m}{128} \leq \frac{||f''||_\infty}{64} < \frac{||f''||_\infty}{48},$$

and so, the approximation of the integral $\int\limits_a^b f(x)dx$ *by the use of* $\overline{A}_{HR}(f, f', \Delta)$ *is better then that provided by the formulae* $A_{TR}(f, \Delta)$ *for every partition* Δ *of the interval* $[a, b]$*.*
The following corollary of Theorem 3.2 holds:

Corollary 3.2. Let $f : [a, b] \to \mathbb{R}$ be as in Lemma 2.3. Thus we have

$$\int\limits_a^b f(x)\mathrm{d}x = \overline{A}_{\mathrm{HR},n}(f, f') + \overline{R}_{\mathrm{HR},n}(f, f'),$$

where

$$\overline{A}_{\mathrm{HR},n}(f, f') = \frac{b-a}{2n} \cdot \sum_{i=1}^{n} \left[\frac{f\left(a + \frac{b-a}{n} \cdot (i-1)\right) + f\left(a + \frac{b-a}{n} \cdot i\right)}{2} + \right.$$

$$\left. + f\left(a + \frac{b-a}{2n} \cdot (2i-1)\right)\right] - \frac{(b-a)^2}{48n^2} \cdot [f'(b) - f'(a)]$$

and the remainder $\overline{R}_{\mathrm{HR},n}(f, f')$ satisfies the estimation

$$\left|\overline{R}_{\mathrm{HR},n}(f, f')\right| \leq \frac{(M-m)(b-a)^3}{128n^2}$$

for all $n \geq 1$.
Now, applying Lemma 2.4, we state the following quadrature formulae:

Theorem 3.3. Let f be as in Lemma 2.4. If Δ is a partition of the interval $[a, b]$, then we have

$$\int\limits_a^b f(x)\mathrm{d}x = A_{\mathrm{TR},m,M}(f, \Delta) + R_{\mathrm{TR},m,M}(f, \Delta) \tag{3.7}$$

and

$$\int\limits_a^b f(x)\mathrm{d}x = A_{\mathrm{HR},m,M}(f, f', \Delta) + R_{\mathrm{HR},m,M}(f, f', \Delta) \tag{3.8}$$

where

$$A_{\mathrm{TR},m,M}(f,\Delta)=\frac{1}{2}\cdot\sum_{i=1}^{n}\left[\frac{f(x_{i-1})+f(x_i)}{2}+f\left(\frac{x_{i-1}+x_i}{2}\right)\right]\cdot h_i-\frac{m+M}{96}\cdot\sum_{i=1}^{n}h_i^3$$

and

$$A_{\mathrm{HR},m,M}(f,f',\Delta)=\frac{1}{2}\cdot\sum_{i=1}^{n}\left[\frac{f(x_{i-1})+f(x_i)}{2}+f\left(\frac{x_{i-1}+x_i}{2}\right)\right]\cdot h_i-$$

$$-\frac{1}{32}\cdot\sum_{i=1}^{n}h_i^2\cdot[f'(x_i)-f'(x_{i-1})]+\frac{m+M}{192}\cdot\sum_{i=1}^{n}h_i^3$$

are two quadrature formulae and the remainder terms $R_{\mathrm{TR},m,M}(f,\Delta)$ and $R_{\mathrm{HR},m,M}(f,f',\Delta)$ satisfy the estimations

$$|R_{\mathrm{TR},m,M}(f,\Delta)|\le\frac{M-m}{96}\cdot\sum_{i=1}^{n}h_i^3 \tag{3.9}$$

and

$$|R_{\mathrm{HR},m,M}(f,f',\Delta)|\le\frac{M-m}{192}\cdot\sum_{i=1}^{n}h_i^3. \tag{3.10}$$

Proof. Applying the inequalitites (2.17) and (**??**) on $[x_{i-1},x_i]$, we get

$$\left|\int\limits_{x_{i-1}}^{x_i}f(x)\mathrm{d}x-\frac{1}{2}\left[\frac{f(x_{i-1})+f(x_i)}{2}+f\left(\frac{x_{i-1}+x_i}{2}\right)\right]\cdot h_i+\frac{m+M}{96}\cdot h_i^3\right|\le$$

$$\le\frac{M-m}{96}\cdot h_i^3$$

and

$$\left|\int\limits_{x_{i-1}}^{x_i}f(x)\mathrm{d}x-\frac{1}{2}\left[\frac{f(x_{i-1})+f(x_i)}{2}+f\left(\frac{x_{i-1}+x_i}{2}\right)\right]\cdot h_i+\right.$$

$$+\frac{h_i^2}{32}\cdot[f'(x_i)-f'(x_{i-1})]-\frac{m+M}{192}\cdot h_i^3\Bigg|\le\frac{M-m}{192}\cdot h_i^3$$

for all $i\in\{1,2,\ldots,n\}$.
Summing over i from 1 to n, we deduce easily the desired estimations (3.9) and (3.10).

Corollary 3.3. Let f be as above. Then, we have:

$$\int_a^b f(x)\mathrm{d}x=A_{\mathrm{TR},m,M,n}(f)+R_{\mathrm{TR},m,M,n}(f)$$

and

$$\int_a^b f(x)\mathrm{d}x=A_{\mathrm{HR},m,M,n}(f,f')+R_{\mathrm{HR},m,M,n}(f,f')$$

where

$$A_{\mathrm{TR},m,M,n}(f)=\frac{1}{2}\cdot\sum_{i=1}^{n}\left[\frac{f\left(a+\frac{b-a}{n}\cdot(i-1)\right)++f\left(a+\frac{b-a}{n}\cdot i\right)}{2}+\right.$$

$$\left.+f\left(a+\frac{b-a}{2n}\cdot(2i-1)\right)\right]-\frac{M+m}{96n^2}\cdot(b-a)^3$$

and

$$A_{\mathrm{HR},m,M,n}(f,f')=\frac{1}{2}\cdot\sum_{i=1}^{n}\left[\frac{f\left(a+\frac{b-a}{n}\cdot(i-1)\right)+f\left(a+\frac{b-a}{n}\cdot i\right)}{2}+\right.$$

$$\left.+f\left(a+\frac{b-a}{2n}\cdot(2i-1)\right)\right]-\frac{(b-a)^2}{32}\cdot[f'(b)-f'(a)]+\frac{M+m}{192n^2}\cdot(b-a)^3$$

with the remainder terms $R_{\mathrm{TR},m,M,n}(f)$ and $R_{\mathrm{HR},m,M,n}(f,f')$ satisfying the estimations

$$|R_{\mathrm{TR},m,M,n}(f)|\le\frac{M-m}{96n^2}\cdot(b-a)^3$$

and

$$|R_{\mathrm{HR},m,M,n}(f,f')| \leq \frac{M-m}{192n^2} \cdot (b-a)^3.$$

Remark 3.2. *As* $0 \leq M-m \leq 2 \cdot ||f''||_\infty$*, the approximation given by* $A_{HR,m,M,n}(f,f')$ *to the integral* $\int\limits_a^b f(x)dx$ *is better than the first given by* $A_{TR,n}(f)$.

4. APPLICATIONS FOR SOME SPECIAL MEANS

Let us recall the following means:

a) The arithmetic mean: $A = A(a,b) := \frac{a+b}{2},\ a,b \geq 0$;

b) The geometric mean: $G = G(a,b) := \sqrt{ab},\ a,b \geq 0$;

c) The harmonic mean: $H = H(a,b) := \frac{2}{\frac{1}{a}+\frac{1}{b}},\ a,b > 0$;

d) The logarithmic mean: $L = L(a,b) := \begin{cases} a, \text{if } a = b \\ \frac{b-a}{\ln(b)-\ln(a)}, \text{if } a \neq b \end{cases},\ a,b > 0$;

e) The identric mean: $I = I(a,b) := \begin{cases} a, \text{if } a = b \\ \frac{1}{\mathrm{e}} \cdot \left(\frac{a^a}{b}\right)^{\frac{1}{b-a}}, \text{if } a \neq b \end{cases}$;

f) The p-logarithmic mean:
$L = L_p(a,b) := \begin{cases} a, \text{if } a = b \\ \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}, \text{if } a \neq b \end{cases},\ p \in \mathbb{R} - \{-1, 0\}$.
The following inequalities are known in the literature

$$H \leq G \leq L \leq I \leq A.$$

It is also well known that L_p is monotonically increasing for $p \in \mathbb{R}$ (assuming that $L_0 := I$ and $L_{-1} := L$).

4.1. Results for the rules (2.7) The inequalities (2.7) are equivalent to

$$\left|\frac{1}{b-a} \cdot \int\limits_a^b f(x)\mathrm{d}x - \frac{1}{2} \cdot \left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right]\right| \leq$$

$$\leq \begin{cases} \frac{\|f''\|_\infty}{48} \cdot (b-a)^2 \\ \frac{\|f''\|_q}{2} \cdot [B(p+1,p+1)]^{\frac{1}{p}} \cdot \left(\frac{b-a}{2}\right)^{1+\frac{1}{p}}, \ \frac{1}{p}+\frac{1}{q}=1, \ p>1. \\ \frac{\|f''\|_1}{32} \cdot (b-a) \end{cases} \tag{4.1}$$

We can now apply (2.7) to deduce some inequalities for the special means given above using some particular mappings as follows.

i) Consider the mapping $f:(0,+\infty)\to\mathbb{R}$, $f(x)=x^r$, where $r\in\mathbb{R}-\{0,1\}$. If $a,b\in(0,+\infty)$ with $a<b$, we have

$$\frac{1}{b-a}\cdot\int_a^b f(x)\mathrm{d}x = L_r^r(a,b),$$

$$\frac{f(a)+f(b)}{2} = A(a^r,b^r),$$

$$f\left(\frac{a+b}{2}\right) = A^r(a,b),$$

$$\|f''\|_\infty = |r(r-1)|\cdot\begin{cases} b^{r-2}, \text{if } r\in[2,\infty) \\ a^{r-2}, \text{if } r\in(-\infty,2)-\{-1,0\} \end{cases},$$

$$\|f''\|_q = |r(r-1)|\cdot(b-a)^{\frac{1}{q}}\cdot L_{q(r-1)}^{r-1}(a,b),$$

$$\|f''\|_1 = |r(r-1)|\cdot(b-a)\cdot L_{r-1}^{r-1}(a,b).$$

Thus, the inequalities (4.1) give us

$$|L_r^r(a,b)-\frac{1}{2}\cdot[A(a^r,b^r)+A^r(a,b)]|\leq$$

$$\leq \begin{cases} \frac{|r(r-1)|\cdot\delta_r(a,b)}{48}\cdot(b-a)^2 \\ \frac{1}{2^{2+\frac{1}{p}}}\cdot|r(r-1)|\cdot(b-a)^2\cdot L_{q(r-1)}^{r-1}(a,b)\cdot[B(p+1,p+1)]^{\frac{1}{p}}, \quad \frac{1}{p}+\frac{1}{q}=1, p>1, \\ \frac{|r(r-1)|\cdot L_{r-1}^{r-1}(a,b)\cdot(b-a)^2}{32} \end{cases}$$

where

$$\delta_r(a,b) := \begin{cases} b^{r-2}, \text{if } r \in [2,\infty) \\ a^{r-2}, \text{if } r \in (-\infty,2)-\{-1,0\}. \end{cases}$$

ii) Consider now the mapping $f : (0,+\infty) \to \mathbb{R}$, $f(x) = \frac{1}{x}$, and $a, b \in (0,+\infty)$, $a < b$. Then, we have

$$\frac{1}{b-a} \cdot \int_a^b f(x)\mathrm{d}x = L_{-1}^{-1}(a,b),$$

$$\frac{f(a)+f(b)}{2} = \frac{A(a,b)}{G^2(a,b)},$$

$$f\left(\frac{a+b}{2}\right) = \frac{1}{A(a,b)},$$

$$||f''||_\infty = \frac{2}{a^3},$$

$$||f''||_q = 2(b-a)^{\frac{1}{q}} \cdot L_{-3q}^{-1}(a,b),$$

$$||f''||_1 = 2(b-a) \cdot L_{-3}^{-3}(a,b).$$

Then, the inequalities (4) give us

$$\left|\frac{1}{L} - \frac{1}{2}\left[\frac{A}{G^2} + \frac{1}{A}\right]\right| = \left|\frac{1}{L} - \frac{1}{2}\left(\frac{1}{H} + \frac{1}{A}\right)\right| = \left|\frac{1}{L} - \frac{1}{H(H,A)}\right| \leq$$

$$\leq \begin{cases} \frac{(b-a)^2}{24a^2} \\ \frac{(b-a)^2}{2^{1+\frac{1}{p}}} \cdot L_{-3q}^{-1}(a,b) \cdot [B(p+1,p+1)]^{\frac{1}{p}}, \quad \frac{1}{p}+\frac{1}{q}=1, p>1. \\ \frac{(b-a)^2}{16} \cdot L_{-3}^{-3}(a,b) \end{cases}$$

iii) Let us consider the mapping $f : (0,+\infty) \to \mathbb{R}$, $f(x) = \ln x$, and $a, b \in (0,+\infty)$, $a < b$. Thus, we have

$$\frac{1}{b-a} \cdot \int_a^b f(x)\mathrm{d}x = \ln I(a,b),$$

$$\frac{f(a)+f(b)}{2} = \ln G(a,b),$$

$$f\left(\frac{a+b}{2}\right) = \ln A(a,b),$$

$$||f''||_\infty = \frac{1}{a^2},$$

$$||f''||_q = (b-a)^{\frac{1}{q}} \cdot L_{-2q}^{-2}(a,b),$$

$$||f''||_1 = (b-a) \cdot L_{-2}^{-2}(a,b).$$

Then, the inequalities (4.1) give us

$$\left|\ln \frac{I}{\sqrt{AG}}\right| \leq \begin{cases} \frac{(b-a)^2}{48a^2} \\ \frac{(b-a)^2}{2^{1+\frac{1}{p}}} \cdot L_{-2q}^{-2}(a,b) \cdot [B(p+1,p+1)]^{\frac{1}{p}}, & \frac{1}{p}+\frac{1}{q}=1, p>1. \\ \frac{(b-a)^2}{32} \cdot L_{-2}^{-2}(a,b) \end{cases}$$

Results for the rule (2.8). The inequalities (2.8) are equivalent to

$$\left|\frac{1}{b-a} \cdot \int_a^b f(x)\mathrm{d}x - \frac{1}{2} \cdot \left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] + \frac{b-a}{32} \cdot [f'(b)-f'(a)]\right| \leq$$

$$\leq \begin{cases} \frac{||f''||_\infty}{96} \cdot (b-a)^2 \\ \frac{||f''||_q}{8(2p+1)^{\frac{1}{p}}} \cdot \left(\frac{b-a}{2}\right)^{1+\frac{1}{p}}, & \frac{1}{p}+\frac{1}{q}=1,\ p>1. \\ \frac{||f''||_1}{32} \cdot (b-a) \end{cases} \tag{4.2}$$

Applying (4.2) to the mappings chosen for (4.1), we obtain successively:

i) For the mapping $f(x) = x^r$, $r \in \mathbb{R} - \{0,1\}$ on the interval $[a,b] \subset (0,+\infty)$. Noting that

$$\frac{f'(b)-f'(a)}{b-a} = r(r-1) \cdot L_{r-2}^{r-2},$$

the inequalitites (4.2) become

$$\left|L_r^r(a,b)-\frac{1}{2}\cdot\left[A(a^r,b^r)+A^r(a,b)\right]+r(r-1)(b-a)^2\cdot L_{r-2}^{r-2}(a,b)\right|\leq$$

$$\leq\begin{cases}\frac{|r(r-1)|\cdot\delta_r(a,b)}{96}\cdot(b-a)^2\\ \frac{|r(r-1)|\cdot L_{q(r-1)}^{r-1}(a,b)}{2^{4+\frac{1}{p}}\cdot(2p+1)^{\frac{1}{p}}}\cdot(b-a)^2, \quad \frac{1}{p}+\frac{1}{q}=1, p>1.\\ \frac{|r(r-1)|\cdot L_{r-1}^{r-1}(a,b)}{32}\cdot(b-a)^2\end{cases}$$

ii) For the mapping $f(x)=\frac{1}{x}$ on the interval $[a,b]\subset(0,+\infty)$.
Noting that

$$\frac{f'(b)-f'(a)}{b-a}=\frac{2A(a,b)}{G^4(a,b)},$$

the inequalities (4.2) become

$$\left|\frac{1}{L}-\frac{1}{2}\cdot\left(\frac{A}{G^2}+\frac{1}{A}\right)+\frac{(b-a)^2}{16}\cdot\frac{A}{G^4}\right|=\left|\frac{1}{L}-\frac{1}{H(H,A)}+\frac{(b-a)^2}{16HG^2}\right|\leq$$

$$\leq\begin{cases}\frac{(b-a)^2}{48a^2}\\ \frac{(b-a)^2\cdot L_{-3q}^{-1}(a,b)}{2^{3+\frac{1}{p}}\cdot(2p+1)^{\frac{1}{p}}}, \quad \frac{1}{p}+\frac{1}{q}=1, p>1.\\ \frac{(b-a)^2\cdot L_{-3}^{-3}(a,b)}{16}\end{cases}$$

iii) For the mapping $f(x)=\ln x$ on the interval $[a,b]\subset(0,+\infty)$.
Noting that

$$\frac{f'(b)-f'(a)}{b-a}=-\frac{1}{G^2},$$

the inequalities (4.2) become

$$\left|\ln\frac{I}{\sqrt{AG}}-\frac{(b-a)^2}{32G^2}\right|\leq\begin{cases}\frac{(b-a)^2}{96a^2}\\ \frac{(b-a)^2\cdot L_{-2q}^{-2}(a,b)}{2^{4+\frac{1}{p}}\cdot(2p+1)^{\frac{1}{p}}}, \quad \frac{1}{p}+\frac{1}{q}=1, p>1.\\ \frac{(b-a)^2\cdot L_{-2}^{-2}(a,b)}{32}\end{cases}$$

Remark 4.1. *If we use the inequalities (2.12), (2.17) and (2.18), we can deduce similar results. We shall omit the details.*

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On an Extension of the Hilbert Integral Inequality

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ABSTRACT. In this paper it is shown that an extension of the Hilbert integral inequality can be built by introducing a parameter λ $(\lambda > -1)$. And the constant factor expressed by Γ-function is proved to be the best possible. As applications, some equivalent forms are given.

1. INTRODUCTION

Let $f(x),\ g(x) \in L^2(0, +\infty)$. Then

$$\int_0^\infty \int_0^\infty \frac{f(x)\, g(y)}{x+y} dx dy \le \pi \left\{ \int_0^\infty f^2(x)\, dx \right\}^{\frac{1}{2}} \left\{ \int_0^\infty g^2(x)\, dx \right\}^{\frac{1}{2}} \tag{1.1}$$

where the constant factor π is the best possible. And the equality contained in (1.1) holds if and only if $f(x) = 0$, or $g(x) = 0$.

This is the famous Hilbert integral inequality, see [1],[2]. Owing to the importance of the Hilbert inequality and the Hilbert type inequality in analysis and applications, some mathematicians have been studying them. Recently, various improvements and extensions of (1.1) appear in a great deal of papers (see [3] etc.).

Specially, Gao and Hsu enumerated the research articles more than 40 in the paper [4].

For convenience, we define $\left(\ln \frac{x}{y}\right)^0 = 1$, when $x = y$. The purpose of the present paper is to establish the Hilbert-type integral inequality of the form

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$$\int_0^\infty\int_0^\infty \frac{\left|\ln\frac{x}{y}\right|^\lambda f(x)\,g(y)}{x+y}dxdy \leq C\left\{\int_0^\infty f^2(x)\,dx\right\}^{\frac{1}{2}}\left\{\int_0^\infty g^2(x)\,dx\right\}^{\frac{1}{2}} \tag{1.2}$$

where $\lambda > -1$. We will give the constant factor C, and will prove the constant factor C in (1.2) to be the best possible, and then give some important and especial results, and study some equivalent forms of them. Evidently, the inequality (1.2) is an extension of (1.1). The new inequality established is significant in theory and applications.

2. LEMMAS

In order to prove our main results, we need the following lemmas.

Lemma 2.1. Let a be a positive number and $\lambda > -1$. Then

$$\int_0^\infty x^\lambda e^{-ax}dx = \frac{\Gamma(\lambda+1)}{a^{\lambda+1}}, \tag{2.1}$$

where $\Gamma(z)$ is Γ-function.
Proof. According to the definition of Γ-function, we obtain immediately (2.1). This result can be also found in the paper [12] (pp. 226, formula 1053).

Lemma 2.2. Let a be a positive number. Then

$$\int_0^\infty \frac{x}{\cosh ax}dx = \frac{2G}{a^2} \tag{2.2}$$

where G is Catalan constant, i.e. $G = 0.915965594\cdots$.
Proof. Let $\lambda > -1$. Expanding the hyperbolic secant function $\frac{1}{\cosh ax}$, and then using Lemma 2.1 we have

$$\int_0^\infty \frac{x^\lambda}{\cosh ax}dx = 2\int_0^\infty \frac{x^\lambda e^{-ax}}{1+e^{-2ax}}dx = 2\int_0^\infty x^\lambda e^{-ax}\sum_{k=1}^\infty (-1)^{k+1}e^{-2(k-1)ax}dx =$$

$$= 2\sum_{k=1}^\infty (-1)^{k+1}\int_0^\infty x^\lambda e^{-(2k-1)ax}dx = \frac{2\Gamma(\lambda+1)}{a^{\lambda+1}}\sum_{k=1}^\infty \frac{(-1)^{k+1}}{(2k-1)^{\lambda+1}} =$$

$$= \frac{2\Gamma(\lambda+1)}{a^{\lambda+1}} G(\lambda) \tag{2.3}$$

where the function $G(\lambda)$ is defined by

$$G(\lambda) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)^{\lambda+1}} \tag{2.4}$$

Let $\lambda = 1$.Then $\Gamma(\lambda+1) = 1$. In accordance with the definition of the Catalan constant (see [10], pp.503.), i.e.

$$G\,(1) = G = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)^2} = 0.915965594 \cdots .$$

We obtain from (2.3) the equality (2.2) at once.

Lemma 2.3. Let $\lambda > -1$. Then

$$\int_0^1 t^{-\frac{1}{2}} \left(\ln \frac{1}{t}\right)^{\lambda} \frac{1}{1+t} dt = 2^{\lambda+1}\Gamma(\lambda+1)G(\lambda). \tag{2.5}$$

where the function $G(\lambda)$ is defined by (2.4).

Proof. Substitution $x = \ln \frac{1}{t}$, it is easy to deduce that

$$\int_0^1 t^{-\frac{1}{2}} \left(\ln \frac{1}{t}\right)^{\lambda} \frac{1}{1+t} dt = \int_0^{\infty} \frac{x^{\lambda} e^{-\frac{1}{2}x}}{1+e^{-x}} dx = \int_0^{\infty} \frac{x^{\lambda}}{e^{\frac{1}{2}x} + e^{-\frac{1}{2}x}} dx = \frac{1}{2} \int_0^{\infty} \frac{x^{\lambda}}{\cosh \frac{1}{2}x} dx.$$

By using (2.3), the equality (2.5) follows.

Lemma 2.4. With the assumptions as Lemma 2.3, then

$$\int_0^{\infty} t^{-\frac{1}{2}} \left|\ln \tfrac{1}{t}\right|^{\lambda} \tfrac{1}{1+t} dt = 2^{\lambda+2}\Gamma(\lambda+1)G(\lambda) \tag{2.6}$$

where $G(\lambda)$ is defined by (2.4).

Proof. It is easy to deduce that

$$\int_0^{\infty} t^{-\frac{1}{2}} \left|\ln \frac{1}{t}\right|^{\lambda} \frac{1}{1+t} dt = \int_0^{1} t^{-\frac{1}{2}} \left|\ln \frac{1}{t}\right|^{\lambda} \frac{1}{1+t} dt + \int_1^{\infty} t^{-\frac{1}{2}} \left|\ln \frac{1}{t}\right|^{\lambda} \frac{1}{1+t} dt =$$

$$= \int_0^1 t^{-\frac{1}{2}} \left|\ln \frac{1}{t}\right|^{\lambda} \frac{1}{1+t} dt + \int_0^1 v^{-\frac{1}{2}} \left|\ln v\right|^{\lambda} \frac{1}{1+v} dv =$$

$$= \int_0^1 t^{-\frac{1}{2}} \left(\ln \frac{1}{t}\right)^{\lambda} \frac{1}{1+t} dt + \int_0^1 v^{-\frac{1}{2}} \left(\ln \frac{1}{v}\right)^{\lambda} \frac{1}{1+v} dv = 2\int_0^1 t^{-\frac{1}{2}} \left(\ln \frac{1}{t}\right)^{\lambda} \frac{1}{1+t} dt.$$

By Lemma 2.3, the equality (2.6) follows at once.

3. THEOREMS AND THEIR PROOFS

In this section, we will prove our assertions by using the above Lemmas.

Theorem 3.1. Let $f$and g be two real functions, and$\lambda > -1$. If $0 \le \int_0^\infty f^2(x)dx < +\infty$ and $0 \le \int_0^\infty g^2(x)dx < +\infty$, then

$$\int_0^\infty \int_0^\infty \frac{\left|\ln \frac{x}{y}\right|^{\lambda} f(x)\, g(y)}{x+\ y} dxdy \ \le\ C \left\{ \int_0^\infty f^2(x)\, dx \right\}^{\frac{1}{2}} \left\{ \int_0^\infty g^2(x)\, dx \right\}^{\frac{1}{2}} \tag{3.1}$$

where the constant factor C is defined by

$$C = 2^{\lambda+2}\Gamma(\lambda+1)G(\lambda) \tag{3.2}$$

and the function $G(\lambda)$ is defined by (2.4) and $\Gamma(z)$is Γ-function. And the constant factor C in (3.1) is the best possible. And the equality in (3.1) holds if and only if $f(x) = 0$, or $g(x) = 0$.

Proof. We can apply the Cauchy inequality to estimate the left-hand side of (3.1) as follows.

$$\int_0^\infty \int_0^\infty \frac{\left|\ln \frac{x}{y}\right|^{\lambda} f(x)g(y)}{x+\ y} dxdy \le$$

$$\le \left(\int_0^\infty \omega(x) f^2(x)dx \right)^{\frac{1}{2}} \left(\int_0^\infty \omega(x) g^2(x)dx \right)^{\frac{1}{2}}, \tag{3.3}$$

where $\omega(x) = \int\limits_0^\infty \frac{\left|\ln\frac{x}{y}\right|^\lambda}{x+y}\left(\frac{x}{y}\right)^{\frac{1}{2}} dy$.
By proper substitution of variable, and then by Lemma 2.4, it is easy to deduce that

$$\omega(x) = \int\limits_0^\infty \frac{\left|\ln\frac{x}{y}\right|^\lambda}{x\left(1+\left(\frac{y}{x}\right)\right)}\left(\frac{x}{y}\right)^{\frac{1}{2}} dy = \int\limits_0^\infty t^{-\frac{1}{2}}\left|\ln t\right|^\lambda \frac{1}{1+t}dt = C \qquad (3.4)$$

where the constant factor C is defined by (2.6).
It follows from (3.3) and (3.4) that

$$\int\limits_0^\infty\int\limits_0^\infty \frac{\left|\ln\frac{x}{y}\right|^\lambda f(x)\,g(y)}{x+y}dxdy \le$$

$$\le C\left\{\int\limits_0^\infty f^2(x)\,dx\right\}^{\frac{1}{2}}\left\{\int\limits_0^\infty g^2(x)\,dx\right\}^{\frac{1}{2}}, \qquad (3.5)$$

If (3.5) takes the form of the equality, then there exist a pair of non-zero constants c_1 and c_2 such that

$$c_1\frac{\left|\ln\frac{x}{y}\right|^\lambda}{x+y}f^2(x)\left(\frac{x}{y}\right)^{\frac{1}{2}} = c_2\frac{\left|\ln\frac{x}{y}\right|^\lambda}{x+y}g^2(y)\left(\frac{y}{x}\right)^{\frac{1}{2}}, \quad \text{a.e. on } (0,+\infty)\times(0,+\infty)$$

Then we have

$$c_1x\, f^2(x) = c_2y\, g^2(y) = C_0 \quad (\text{constant}) \;\; \text{a.e. on } (0,+\infty)\times(0,+\infty)$$

Without losing the generality, we suppose that $c_1 \neq 0$, then

$$\int_0^\infty f^2(x)\,dx = \frac{C_0}{c_1}\int_0^\infty x^{-1}dx.$$

This contradicts that$0 < \int_0^\infty f^2(x)\,dx < +\infty$. It is obvious that the equality in (3.5) holds if and only if $f(x) = 0$, or $g(x) = 0$. It follows that the inequality (3.1) is valid.
It remains to need only to show that C in (3.1) is the best possible.
$\forall \quad 0 < \varepsilon < 1$.

Define two functions by

$$\tilde{f}(x) = \begin{cases} 0 & x \in (0,\ 1) \\ x^{-\frac{1+\varepsilon}{2}} & x \in [1,\ \infty) \end{cases} \quad \text{and } \tilde{g}(y) = \begin{cases} 0 & y \in (0,\ 1) \\ y^{-\frac{1+\varepsilon}{2}} & y \in [1,\ \infty) \end{cases}$$

It is easy to deduce that

$$\int_0^{+\infty} \tilde{f}^2(x)\,dx = \int_0^{+\infty} \tilde{g}^2(y)\,dy = \frac{1}{\varepsilon}\ .$$

If C in (3.1) is not the best possible, then there exists $C^* > 0$, such that $C^* < C$ and

$$H(\lambda) = \int_0^\infty \int_0^\infty \frac{\left|\ln \frac{x}{y}\right|^\lambda \tilde{f}(x)\,\tilde{g}(y)}{x\ +\ y} dxdy \le\ C^* \left(\int_0^\infty \tilde{f}^2(x)\,dx\right)^{\frac{1}{2}} \left(\int_0^\infty \tilde{g}^2(y)\,dy\right)^{\frac{1}{2}}.$$

$$=\ C^* \left(\int_1^\infty \tilde{f}^2(x)\,dx\right)^{\frac{1}{2}} \left(\int_1^\infty \tilde{g}^2(y)\,dy\right)^{\frac{1}{2}} = \frac{C^*}{\varepsilon}. \qquad (3.6)$$

On the other hand, we have

$$H(\lambda) = \int_0^\infty \int_0^\infty \frac{\left|\ln \frac{x}{y}\right|^\lambda \tilde{f}(x)\,\tilde{g}(y)}{x\ +\ y} dxdy =$$

$$= \int_1^\infty \int_1^\infty \frac{\left\{x^{-\frac{1+\varepsilon}{2}}\right\}\left\{\left|\ln \frac{x}{y}\right|^\lambda y^{-\frac{1+\varepsilon}{2}}\right\}}{x\ \ +\ y} dxdy =$$

$$= \int_1^\infty \left\{\int_1^\infty \frac{\left|\ln \frac{x}{y}\right|^\lambda y^{-\frac{1+\varepsilon}{2}}}{x\ \left(1\ +\ \frac{y}{x}\right)} dy\right\} \left\{x^{-\frac{1+\varepsilon}{2}}\right\} dx =$$

$$= \int_1^\infty \left\{\int_{1/x}^\infty \frac{\left|\ln \frac{1}{t}\right|^\lambda\ t^{-\frac{1+\varepsilon}{2}}}{1\ +\ t} dt\right\} \left\{x^{-1-\varepsilon}\right\} dx =$$

$$= \int_{1}^{\infty}\left\{\int_{1/x}^{1}\frac{|\ln t|^{\lambda}\ \ t^{-\frac{1+\varepsilon}{2}}}{1\ +\ t}dt\right\}\left\{x^{-1-\varepsilon}\right\}dx+\int_{1}^{\infty}\left\{\int_{1}^{\infty}\frac{|\ln t|^{\lambda}\ \ t^{-\frac{1+\varepsilon}{2}}}{1\ +\ t}dt\right\}\left\{x^{-1-\varepsilon}\right\}dx$$

$$= \int_{0}^{1}\left\{\int_{1/t}^{\infty}x^{-1-\varepsilon}dx\right\}\frac{|\ln t|^{\lambda}\ \ t^{-\frac{1+\varepsilon}{2}}}{1\ +\ t}dt+\int_{1}^{\infty}\left\{\int_{1}^{\infty}\frac{|\ln t|^{\lambda}\ \ t^{-\frac{1+\varepsilon}{2}}}{1\ +\ t}dt\right\}\left\{x^{-1-\varepsilon}\right\}dx$$

$$= \frac{1}{\varepsilon}\int_{0}^{1}\frac{|\ln t|^{\lambda}\ \ t^{-\frac{1-\varepsilon}{2}}}{1\ +\ t}dt+\frac{1}{\varepsilon}\int_{1}^{\infty}\frac{|\ln t|^{\lambda}\ \ t^{-\frac{1+\varepsilon}{2}}}{1\ +\ t}dt. \qquad (3.7)$$

When ε is sufficiently small, we obtain from (3.7) that

$$H(\lambda)=\frac{1}{\varepsilon}\left(\int_{0}^{1}\frac{|\ln t|^{\lambda}\ \ t^{-\frac{1}{2}}}{1\ +\ t}dt+\circ_{1}(1)\right)+\frac{1}{\varepsilon}\left(\int_{1}^{\infty}\frac{|\ln t|^{\lambda}\ \ t^{-\frac{1}{2}}}{1\ +\ t}dt+\circ_{2}(1)\right)=$$

$$=\frac{1}{\varepsilon}\left(\int_{0}^{\infty}\frac{|\ln t|^{\lambda}\ \ t^{-\frac{1}{2}}}{1\ +\ t}dt+\circ(1)\right)\qquad(\varepsilon\to 0)$$

By £¨3.4), we have

$$H(\lambda)=\frac{1}{\varepsilon}\left(C+\circ(1)\right).\qquad(\varepsilon\to 0)\qquad(3.8)$$

Evidently, the inequality (3.8) is in contradiction with (3.6). Therefore, the constant factor C in (3.1) is the best possible**.** Thus the proof of Theorem is completed.

Based on Theorem 3.1, we have the following important results.

Theorem 3.2. If $0\le\int_{0}^{\infty}f^{2}(x)dx<+\infty$ and $0\le\int_{0}^{\infty}g^{2}(x)dx<+\infty$, then

$$\int_{0}^{\infty}\int_{0}^{\infty}\frac{\left|\ln\frac{x}{y}\right|f\left(x\right)g\left(y\right)}{x\ +\ y}dxdy\ \ \le\ \ 8G\left\{\int_{0}^{\infty}f^{2}\left(x\right)dx\right\}^{\frac{1}{2}}\left\{\int_{0}^{\infty}g^{2}\left(x\right)dx\right\}^{\frac{1}{2}}.\qquad(3.9)$$

where G is the Catalan constant. And the constant factor $8G$ in (3.9) is the best possible. And the equality in (3.9) holds if and only if $f(x) = 0$, or $g(x) = 0$.

Theorem 3.3. Let $\lambda = 2n \quad (n \in N_0)$ If $0 \le \int_0^\infty f^2(x)dx < +\infty$ and $0 \le \int_0^\infty g^2(x)dx < +\infty$, then

$$\int_0^\infty \int_0^\infty \frac{\left(\ln \frac{x}{y}\right)^{2n} f(x)\, g(y)}{x+\ y} dxdy \ \le$$

$$\le \ \left(\pi^{2n+1} E_n\right) \left\{ \int_0^\infty f^2(x)\, dx \right\}^{\frac{1}{2}} \left\{ \int_0^\infty g^2(x)\, dx \right\}^{\frac{1}{2}}, \qquad (3.10)$$

where $E_0 = 1$ and the $E_{n's}$ are the Euler numbers viz.
$E_1 = 1,\ \ E_2 = 5,\ \ E_3 = 61,\ \ E_4 = 1385,$ *etc.*. And the constant factor $\pi^{2n+1}E_n$ in (3.10) is the best possible. And the equality in (3.10) holds if and only if $f(x) = 0$, or $g(x) = 0$.

Proof. We need only to show that the constant factor in (3.10) is true. When $\lambda = 2n$, it is known from (3.10) that

$$C = 2^{\lambda+2}\Gamma(\lambda+1)G(\lambda) =^{2n+2} \Gamma(2n+1)G(2n) = 2^{2n+2}\,(2n)! \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)^{2n+1}}.$$

According to the paper [11] (pp. 231.), we have

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)^{2n+1}} = \frac{\pi^{2n+1}}{2^{2n+2}(2n)!} E_n.$$

where the $E_{n's}$ are the Euler numbers viz.
$E_1 = 1,\ \ E_2 = 5,\ \ E_3 = 61,\ \ E_4 = 1385,$ *etc.* Notice that $\sum\limits_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)} = \frac{\pi}{4}$,hence we can define $E_0 = 1$. Whence we obtain $C = \pi^{2n+1}E_n$.

4. SOME APPLICATIONS

As applications, we will build some new inequalities.

Theorem 4.1. Let f be a real function, and$\lambda > -1$. If $0 \le \int_0^\infty f^2(x)dx < +\infty$, then

$$\int\limits_0^\infty \left\{ \int\limits_0^\infty \frac{\left|\ln \frac{x}{y}\right|^\lambda}{x\ +\ y} f\left(x\right)dx \right\}^2 dy \leq\ C^2 \int\limits_0^\infty f^2\left(x\right)dx, \tag{3.11}$$

where C is defined by (3.2) and the constant factor C^2 in (4.1) is the best possible. And the inequality (4.1) is equivalent to (3.1). And the equality in (4.1) holds if and only if $f(x) = 0$.

Proof. First, we assume that the inequality (3.1) is valid. Setting a real function $g(y)$as

$$g(y) = \int\limits_0^\infty \frac{\left|\ln \frac{x}{y}\right|^\lambda}{x\ +\ y} f\left(x\right)dx, \quad y \in (0,\ +\infty)$$

By using (3.1), we have

$$\int\limits_0^\infty \left\{ \int\limits_0^\infty \frac{\left|\ln \frac{x}{y}\right|^\lambda}{x\ +\ y} f\left(x\right)dx \right\}^2 dy = \int\limits_0^\infty \int\limits_0^\infty \frac{\left|\ln \frac{x}{y}\right|^\lambda}{x\ +\ y} f\left(x\right)g(y)dxdy \leq$$

$$\leq C \left\{ \int\limits_0^\infty f^2(x)dx \right\}^{\frac{1}{2}} \left\{ \int\limits_0^\infty g^2(y)dy \right\}^{\frac{1}{2}}$$

$$= C \left\{ \int\limits_0^\infty f^2(x)dx \right\}^{\frac{1}{2}} \left\{ \int\limits_0^\infty \left(\int_0^\infty \frac{\left|\ln \frac{x}{y}\right|^\lambda}{x\ +\ y} f(x)dx \right)^2 dy \right\}^{\frac{1}{2}} \tag{4.2}$$

It follows from (4.2) that the inequality (4.1) is valid after some simplifications.

On the other hand, assume that the inequality (4.1) keeps valid, by applying in turn the Cauchy inequality and (4.1), we have

$$\int\limits_0^\infty \int\limits_0^\infty \frac{\left|\ln \frac{x}{y}\right|^\lambda}{x\ +\ y} f\left(x\right)g\left(y\right)dxdy = \int\limits_0^\infty \left\{ \int\limits_0^\infty \frac{\left|\ln \frac{x}{y}\right|^\lambda}{x\ +\ y} f\left(x\right)dx \right\} g\left(y\right)dy \leq$$

$$\leq \left\{ \int_0^\infty \left(\int_0^\infty \frac{\left|\ln \frac{x}{y}\right|^\lambda}{x\ +\ y} f\left(x\right) dx \right)^2 dy \right\}^{\frac{1}{2}} \left\{ \int_0^\infty g^2\left(y\right) dy \right\}^{\frac{1}{2}} \leq$$

$$\leq \left\{ C^2 \int_0^\infty f^2\left(x\right) dx \right\}^{\frac{1}{2}} \left\{ \int_0^\infty g^2\left(y\right) dy \right\}^{\frac{1}{2}}$$

$$= C \left\{ \int_0^\infty f^2\left(x\right) dx \right\}^{\frac{1}{2}} \left\{ \int_0^\infty g^2\left(y\right) dy \right\}^{\frac{1}{2}}. \tag{4.3}$$

Therefore the inequality (4.1) is equivalent to (3.1).
If the constant factor C^2 in (4.1) is not the best possible, then it is known from (4.3) that the constant factor C in (3.1) is also not the best possible. This is a contradiction. It is obvious that the equality in (4.1) holds if and only if $f(x) = 0$. Theorem is proved.

Theorem 4.2. Let f be a real function. If $0 \leq \int_0^\infty f^2(x)dx < +\infty$, then

$$\int_0^\infty \left\{ \int_0^\infty \frac{\left|\ln \frac{x}{y}\right|}{x\ +\ y} f\left(x\right) dx \right\}^2 dy \leq\ 64G^2 \int_0^\infty f^2\left(x\right) dx \tag{4.4}$$

where G is the Catalan constant and the constant factor $64G^2$ in (4.4) is the best possible. And the inequality (4.4) is equivalent to (3.9). And the equality in (3.1) holds if and only if $f(x) = 0$.

Theorem 4.3. Let f be a real function, and $n \in N_0$. If $0 < \int_0^\infty f^2(x)dx < +\infty$, then

$$\int_0^\infty \left\{ \int_0^\infty \frac{\left|\ln \frac{x}{y}\right|^{2n}}{x\ +\ y} f\left(x\right) dx \right\}^2 dy \leq\ \left(\pi^{2n+1} E_n\right)^2 \int_0^\infty f^2\left(x\right) dx,$$

where $E_0 = 1$ and the $E_{n's}$ are the Euler numbers viz.
$E_1 = 1,\ \ E_2 = 5,\ \ E_3 = 61,\ \ E_4 = 1385,$ *etc.*. And the constant factor

$\left(\pi^{2n+1}E_n\right)^2$ in (4.5) is the best possible. And the inequality (4.5) is equivalent to (3.10) . And the equality in (4.5) holds if and only if $f(x)=0$.

The proofs of Theorems 4.2 and 4.3 are similar to one of Theorem 4.1. Hence they are omitted.

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Some inequalities and Khintichine theorem for intuitionistic fuzzy random variables

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ABSTRACT. In this paper the theoretical framework of intuitionistic fuzzy random variable is introduced. Standard stochastic inequalities such as Markovs inequality, Chebyshevs inequality are established for intuitionistic fuzzy random variables. We have also established Chebyshevs theorem and Khintehines theorem for intuitionistic fuzzy random variables.

1. INTRODUCTION

Atanassov [1,2] introduced the concept of intuitionistic fuzzy sets, which is a generalization of fuzzy sets. In the recent years intuitionistic fuzzy sets have been realized as the most fitting theoretical framework to deal vagueness. Initially fuzzy sets have been advocated as a way of grasping imprecision in probabilities. The imprecision inherent in probability models make the distinction of probability models with standard models.

Impression in probabilities is needed to reflect the amount of information on which they are based. Imprecise probability models are easier to determine and extract than precise ones. Impression can stem from incomplete information as well as from incomplete assessment. It may be due to the conflict between several types of information. In this context intuitionistic fuzzy set is the unique set that manifests a coherent association between the universes related to available information and non-available information.

When fuzzy sets are employed to determine uncertainity one must ensure appropriateness of the membership functions. In case of random phenomena, where dimness of perception is prevalent random excitation and fuzziness

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have to be simultaneously dealtwith. To accomplish this task full fledged fuzzy parameters are necessary. As an added advantage intuitionistic fuzzy sets naturally accommodate a hesitation part in its conceptualization. The hesitation part is the balancing idea of membership functions and non-membership functions that evades several conflicts that arise in the formulation of fuzzy sets. The compartmentalizations such as membership function, non-membership function and hesitation part of intuitionistic fuzzy sets are more feasible than any other theory to deal with uncertainity where randomness and fuzziness are in co-existence. Along this line Earnest Lazarus Piriyakumar et al., [15] have introduced the concept of intuitionistic fuzzy random variables based on the formulation of Kwakernaaks [6] fuzzy random variables and studied random walks for intuitionistic fuzzy random variables.

Fuzzy random variable is viewed as the imprecise observation of the outcomes in a random experiment. The co-existence of randomness and fuzziness in the same framework compelled the researchers to employ fuzzy random variables to handle imprecise probability theory. The concept of fuzzy random variable is introduced at the end of the 1970s to handle situations where the outcomes of a random experiment are modelled by fuzzy sets. A fuzzy random variable is a mapping that associates a fuzzy set to each element of the universe. In this paper, the intuitionistic fuzzy random variable is considered as a mapping that associates an intuitionistic fuzzy set to each element of the universe.

In this paper expectation is considered as an intuitionistic fuzzy subset of the final space and it plays the role of the average value of the intuitionistic fuzzy random variable. As a natural corollary to the notion of expectation we can define the variance of the intuitionistic fuzzy random variable as the mean of the squares of the distances from the images of the intuitionistic fuzzy random variable to the (intuitionistic fuzzy) expectation. To put it precisely variance of an intuitionistic fuzzy random variable is fixed as an intuitionistic fuzzy number that quantifies the degree of dispersion of the images of the intuitionistic fuzzy random variable.

Different approaches to the conceptualization of fuzzy random variables have been developed in the literature, most widely considered being that introduced by Kwakernaak [6] and Kruse and Meyer [7] and the one by Puri and Ralescu [11]. According to Kwakernaak [6] fuzzy random variables are a particular kind of fuzzy sets whose end points of ?-cuts are amenable to satisfy measurability conditions. H.C. Wu [7-11] has finetuned the work of Kwakernaak [6] in a series of papers, by invoking strong measurability conditions. H.C. Wu [7-11] has introduced various rudiments of fuzzy random variables, such as fuzzy probability distribution function, fuzzy probability density function, fuzzy expectation and fuzzy variance, in a well built theoretical setting. In this paper we adopt the theoretical notions of H.C. Wu [7-11]. Following the work of Kwakernaak [6] and H.C. Wu [7-11], an intuitionistic fuzzy random variable is viewed as a particular kind of intuitionistic fuzzy set whose end points of ?-cuts are subjected to satisfy measurability conditions.

This paper is organized as follows. The preliminaries and related definitions are briefly introduced in section 2. The theoretical framework of intuitionistic fuzzy random variables is introduced in section 3. Stochastic inequalities such as Markovs inequality, Chebyshevs inequality for intuitionistic fuzzy random variables, Chebyshevs theorem and Khintchines theorem for intuitionistic fuzzy random variables are introduced in section 4.

2. PRELIMINARIES

Let $f(x)$ be a real valued function on a topological space. If the set $\{x; f\left(x\right) \geq \alpha\}$ is closed for every α, $f(x)$ is said to be upper semi continuous. $f(x)$ is said to be lower semi continuous of $-f(x)$ is upper semi continuous.

Theorem 2.1. [5] Let S be a compact set in R^n. If f(x) is upper semi continuous on S than $f(x)$ assumes maximum over S and if $f(x)$ is lower semi continuous on S then $f(x)$ assumes minimum over S.
Let X be an ordinary finite non-empty set. An intuitionistic fuzzy set in X is an expression given by

$$A = \{< x, \mu_A\left(x\right), V_A\left(x\right) >; xX\}$$

where $\mu_A : X \rightarrow [0,1]$, $V_A : X \rightarrow [0,1]$ with condition $0 \leq \mu_A\left(x\right) + V_A\left(x\right) \leq 1$ for all x in X. The numbers $\mu_A\left(x\right)$ and $V_A\left(x\right)$ denote respectively the membership degree and the nonmembership degree of the element x in A. We call $\pi_A\left(x\right) = 1 - \mu_A\left(x\right) - V_A\left(x\right)$ the hesitation part of the element x in A. $\pi_A\left(x\right)$ denotes the measure of non-determinacy. The operations of intuitionistic fuzzy sets are defined as follows. For each intuitionistic fuzzy set A, B, we say $A \leq B$ if and only if $\mu_A\left(x\right) \leq \mu_B\left(x\right)$ and $V_A\left(x\right) \geq V_B\left(x\right)$ for all $x \in X$. $A = B$ if and only if $A \leq B$ and $B \leq A$.

$A \cap B = \{< x, \min(\mu_A(x), \mu_B(x)), \max(V_A(x), V_B(x)) >; x \in X\}$
$A \cup B = \{< x, \max(\mu_A(x), \mu_B(x)), \min(V_A(x), V_B(x)) >; x \in X\}$
The complementary of an intuitionistic fuzzy set A is

$$A^c = \{< x, V_A(x), \mu A(x) >; x \in X\}.$$

Definition 2.1. [15] An intuitionistic fuzzy number is an intuitionistic fuzzy set with the real line R as its basic set. It is specified as the mapping $(\mu, v) : R \to [0, 1]$ satisfying the following conditions.
1. μ is upper semi continuous and V is lower semi continuous.
2. There exists $x \in R$, such that $\mu(x) = 1$ and $V(x) = 0$.
3. For $x_1, x_2 \in R$, $\lambda \in [0, 1]$

$$\mu\left(\lambda x_1 + (1 - \lambda) x_2\right) \leq v\left(x_1\right) \vee v\left(x_2\right)$$

and

$$V\left(\lambda x_1 + (1 - \lambda) x_2\right) \leq v\left(x_1\right) \vee v\left(x_2\right)$$

In the case of fuzzy sets if we want to exhibit an element $x \in X$, that typically belongs to a fuzzy set A, it is required that its membership value to be greater than some threshold $\alpha \in (0, 1]$. This has generated a nested sequence of α-level sets. A fuzzy set is tractable from these α-level sets whose membership function can be expressed in terms of the characteristic functions of its α-cuts since an intuitionistic fuzzy set is endowed with membership and non-membership functions, the formulation of level sets of an intuitionistic fuzzy set, need to be two folded. The (α, β) level set of an intuitionistic fuzzy set where α represents the membership function and β represents the non-membership function is defined as follows.

Definition 2.2. The set of elements that belong to the intuitionistic fuzzy set A with membership at least to the degree $\alpha \in (0, 1]$ and non-membership at least to the degree $\beta \in [0, 1)$ is known as the (α, β) level set of an intuitionistic fuzzy set A. It is defined as

$$\beta$$
$$\alpha A = \{x; \mu_A\left(x\right) \geq \alpha \wedge V_A\left(x\right) \geq \beta\}$$

The (α, β) level set of the intuitionistic fuzzy set A is the common region of the two level sets. If $\alpha > \beta$ then the (α, β) level set of the intuitionistic fuzzy set A is

$$^{\beta}A=\{x\in X;V_{A}(x)\geq\beta\}$$

If $\beta>\alpha$ then the (α,β) level set of the intuitionistic fuzzy set A is

$$_{\alpha}A=\{x\in X;\mu_{A}(x)\geq\alpha\}\,.$$

Without loss of generality, throughout this paper we assume that $\alpha>\beta$. i.e. The membership value of belongingness is greater than that of non-belongingness.

Definition 2.3. If $\widetilde{a}$ is a closed intuitionistic fuzzy number then the (α,β) level set of $\widetilde{a}$ with $\alpha>\beta$ is a closed interval denoted as ${}_{\alpha}\overset{\beta}{\widetilde{a}=\left[{}^{\beta}\widetilde{a}^{L},{}^{\beta}\widetilde{a}^{U}\right]}$ we can see that if $\widetilde{a}$ is a bounded intuitionistic fuzzy number then $\{x;\mu_{\widetilde{a}}(x)\geq 0\}$ is a compact interval.

Definition 2.4. $\widetilde{a}$ is called a canonical intuitionistic fuzzy number if it is a closed and bounded intuitionistic fuzzy number and its membership function is strictly increasing on the interval $\left[{}_{0}\widetilde{a}^{L},{}_{1}\widetilde{a}^{L:}\right]$ and strictly decreasing on the interval $\left[{}_{1}\widetilde{a}^{U},{}_{0}\widetilde{a}^{U:}\right]$.

Theorem 2.2. Suppose that $\widetilde{a}$ is a canonical intuitionistic fuzzy number. For an (α,β) level set of the intuitionistic fuzzy set $\widetilde{a}$, with $\alpha>\beta$. Let $g(\alpha)={}^{\beta}\widetilde{a}^{L}$ and $h(\alpha)={}^{\beta}\widetilde{a}^{U}$. Then $g(\alpha)$ and $h(\alpha)$ are continuous for each $\alpha,\beta\in[0,1]$, $\alpha+\beta\leq 1$ and $\alpha>\beta$. Following the principle of resolution identity proposed by Zadeh [14] and Negoita and Ralescu [17], and as a natural corollary to the principle of resolution of identity the following result is stated. For an intuitionistic fuzzy set A, $_{\alpha}A$ and $^{\beta}A$ are defined as follows.

$$_{\alpha}A=\{x\in X;\mu_{A}(x)\geq\alpha\}\tag{2.1}$$

$$^{\beta}A=\{x\in X;V_{A}(x)\geq\beta\}$$

Theorem 2.3. Let A be an intuitionistic fuzzy set with membership function μ_{A} and non-membership function V_{A}. Then for the (α,β) level set of A,

$$\mu_{A}(x)=\sup_{\alpha\in[0,1]}\alpha 1_{\alpha A}(x)\ \text{ and }\ V_{A}(x)=\sup_{\beta\in[0,1]}\beta 1_{\beta A}(x)\tag{2.2}$$

Where $1_{(*)}$ denote the indicator function of $(\bullet)$.

Definition 2.5. Let $\widetilde{A}$ and $\widetilde{B}$ be two intuitionistic fuzzy numbers in R. The operations on intuitionistic fuzzy numbers are defined as follows.
Let $\left[{}^{\beta}\widetilde{A}^{\,L},{}^{\beta}\widetilde{A}^{\,U}\right]$ and $\left[{}^{\beta}\widetilde{B}^{\,L},{}^{\beta}\widetilde{B}^{\,U}\right]$ be the (α,β) level set of $\widetilde{A}$ and $\widetilde{B}$. Then

$$\widetilde{A}+\widetilde{B}=\left[{}^{\beta}\widetilde{A}^{\,L}+{}^{\beta}\widetilde{B}^{\,L},{}^{\beta}\widetilde{A}^{\,U}+{}^{\beta}\widetilde{B}^{\,U}\right]$$

$$A-B=\left[{}^{\beta}\widetilde{A}^{\,L}-{}^{\beta}\widetilde{B}^{\,U},{}^{\beta}\widetilde{A}^{\,U}-{}^{\beta}\widetilde{B}^{\,L}\right]$$

$$A\bullet B=\left[{}^{\beta}\widetilde{A}^{\,L}\,{}^{\beta}\widetilde{B}^{\,L},{}^{\beta}\widetilde{A}^{\,U}\,{}^{\beta}\widetilde{B}^{\,U}\right]$$

$$A\div B=\left[{}^{\beta}\widetilde{A}^{\,L}\div{}^{\beta}\widetilde{B}^{\,U},{}^{\beta}\widetilde{A}^{\,U}\div{}^{\beta}\widetilde{B}^{\,L}\right]$$

3. INTUITIONISTIC FUZZY RANDOM VARIABLES

In this section based on the theoretical frame work of fuzzy random variables by H.C. Wu [7-11] we provide the theoretical details of intuitionistic fuzzy random variables.

Let x be a real number. This real number x can induce an intuitionistic fuzzy number $\widetilde{x}$ with the membership function $\mu_{\widetilde{x}}(r)$ and non-membership function $V_{\widetilde{x}}(r)$ such that $\mu_{\widetilde{x}}(x)=1$, $V_{\widetilde{x}}(x)=0$, $\mu_{\widetilde{x}}(r)<1$ and $V_{\widetilde{x}}<1$ for $r\neq x$, subject to the condition that $\mu_{\widetilde{x}}(x)+V_{\widetilde{x}}(x)\leq 1$ we denote the intuitionistic fuzzy real number induced by the real number x by $\widetilde{x}$.

Let F_R be a set of intuitionistic fuzzy real numbers induced by the real number system R. Let $\tilde{}$ be a relation defined on F_R. We say $\widetilde{x}^1\sim\widetilde{x}^2$ if and only if $\widetilde{x}^1$ and $\widetilde{x}^2$ are induced by the same real number x. Clearly $\tilde{}$ is an equivalence relation. This equivalence relation induces equivalence classes $[\widetilde{x}]=\{\widetilde{a};\widetilde{a}\sim\widetilde{x}\}$. The quotient set $F_{R/\tilde{}}$ is the set of all equivalence classes. Then the Quotient set and the real number system R are one and the same. This claim is justified by the mapping $R\rightarrow F_{R/\tilde{}}$ with the specification $x\rightarrow[\widetilde{x}]$.

We call $F_{R/\tilde{}}$ as the intuitionistic fuzzy real number system. For practical reasons we consider only one element from each equivalence class $[\widetilde{x}]$ to form the intuitionistic fuzzy real number system. We define

$(F_{R/\tilde{}})_R = \{\widetilde{a}|\widetilde{a} \in [\widetilde{x}]\,,\ \widetilde{a}$ is the only element from $[\widetilde{x}]\}$.

Definition 3.1. Let (X, M) be a measurable space and (R, B) be a Borel measurable space.

(i) [4] Let $f : X \to P(R)$ (Power set of R) be a set valued function.

Then f is called measurable if and only if $\{(x, y); y \in f(x)\}$ is a measurable subset of MXB.

(ii) $\widetilde{f}(x)$ is called an intuitionistic fuzzy valued function if $\widetilde{f}X \to F$ (the set of all fuzzy numbers). If $\widetilde{f}$ is an intuitionistic fuzzy valued function then ${}^{\beta}_{\alpha\widetilde{f}}$ is a set valued function for all $\alpha, \beta \in [0, 1]$, where ${}^{\beta}_{\alpha\widetilde{f}(x)}$ is the (α, β) level set of the intuitionistic fuzzy number $\widetilde{f}(x)$ for any $x \in X$. $\widetilde{f}$ is called measurable if ${}^{\beta}_{\alpha\widetilde{f}}$ (set valued) is measurable for each $\alpha, \beta \in [0, 1]$, and $\alpha + \beta \leq 1$.

Let F_{c1} denote the set of all closed intuitionistic fuzzy numbers. $\widetilde{f}$ is called a closed intuitionistic fuzzy valued function if

$$\widetilde{f} : X \to F_{c1}$$

Then ${}^{\beta}_{\alpha\widetilde{f}(x)=[{}^{\beta}\widetilde{f}^{\,L}(x);{}^{\beta}\widetilde{f}^{\,U}(x)]}$ where $x \in X$, $\alpha, \beta \in [0, 1]$ $\alpha + \beta \leq 1$ and $\alpha > \beta$.

The following theorem is an extension of the proposition 3.3 by H.C.Wu [8].

Theorem 3.1. Let $\widetilde{f}(x)$ be a closed intuitionistic fuzzy valued function defined on X. Then the following two statements are equivalent.

(i) For the (α, β) level set of $\widetilde{f}(x)$ and for $\alpha > \beta$, ${}^{\beta}\widetilde{f}^{\,L}(x)$ and ${}^{\beta}\widetilde{f}^{\,U}(x)$ are (real valued) measurable for all $\alpha, \beta \in [0, 1]$.

(ii) $\widetilde{f}(x)$ is (intuitionistic fuzzy valued) measurable and one of ${}^{\beta}\widetilde{f}^{\,L}(x)$ and ${}^{\beta}\widetilde{f}^{\,U}(x)$ is (real valued) measurable for all α, β with $\alpha + \beta \leq 1$.

Definition 3.2. Let $\widetilde{f}(x)$ be a closed intuitionistic fuzzy valued function. $\widetilde{f}(x)$ is called strongly measurable if one of the following conditions is satisfied.

(i) For the (α, β) level set ${}^{\beta}\widetilde{f}^{\,L}(x)$ and ${}^{\beta}\widetilde{f}^{\,U}(x)$; $\alpha > \beta$ are (real valued) measurable.

(ii) $\widetilde{f}(x)$ is (intuitionistic fuzzy valued) measurable and one of ${}^{\beta}\widetilde{f}^{\,L}(x)$ and ${}^{\beta}\widetilde{f}^{\,U}(x)$ (real valued) is measurable for all $\alpha, \beta \in [0, 1]$, $\alpha + \beta \leq 1$ and $\alpha > \beta$.

Definition 3.3. Let $(F_{R/\tilde{}})_R$ be a canonical intuitionistic fuzzy real number system.

Let $X : \Omega \rightarrow (F_{R/\sim})_R$ be a closed intuitionistic fuzzy valued function. X is called an intuitionistic fuzzy random variable if the mapping X is strongly measurable.
The object of the definition is to make the end points of each (α, β) level set $(\alpha > \beta)$, ${}^{\beta}X^{\ L}$ and ${}^{\beta}X^{\ U}$ as random variables for all $\alpha, \beta \in [0, 1]$ and $\alpha + \beta \leq 1$.

Theorem 3.2. Let $(F_{R/\sim})_R$ be a canonical intuitionistic fuzzy real number system, and $X : \Omega \rightarrow (F_{R/\sim})_R$ be a closed intuitionistic fuzzy valued function. X is an intuitionistic fuzzy random variable if and only if ${}^{\beta}X^{\ L}$ and ${}^{\beta}X^{\ U}$ are random variables for all $\alpha, \beta \in [0, 1]$ with $\alpha > \beta$ and $\alpha + \beta \leq 1$.
Let the end points of the (α, β) level set of the intuitionistic fuzzy random variable with $\alpha > \beta$, viz., ${}^{\beta}X^{\ L}$ and ${}^{\beta}X^{\ U}$ have the same continuous probability density function $f(x)$. Let $\widetilde{x}$ be any intuitionistic fuzzy observation of the intuitionistic fuzzy random variable $X(X(\omega) = \widetilde{x})$.
Then the (α, β) level set with $\alpha \geq \beta$ and $\alpha + \beta \leq 1$ of $\widetilde{x}$ is given by

$$\begin{gathered}\beta\\ \alpha\widetilde{x} = [{}^{\beta}X^{L},\ {}^{\beta}X^{U}]\end{gathered}$$

where ${}^{\beta}\widetilde{x}^{\ L}$ and ${}^{\beta}\widetilde{x}^{\ U}$ are the observations of ${}^{\beta}X^{\ L}$ and ${}^{\beta}X^{\ U}$ respectively.
From theorem 2.2 ${}^{\beta}X^{\ L}(\omega) ={}^{\beta}\widetilde{x}^{\ L}$ and ${}^{\beta}X^{\ U}(\omega) ={}^{\beta}\widetilde{x}^{\ U}$ are continuous with respect to β for fixed ω. For any real number $x \in \left[{}^{\beta}\widetilde{x}^{L}, {}^{\beta}\widetilde{x}^{U}\right]$, $x ={}^{\gamma}\widetilde{x}^{L}$ or $x ={}^{\gamma}\widetilde{x}^{U}$ for some $r \geq \beta$. Thus for any $x \in \left[{}^{\beta}\widetilde{x}^{L}, {}^{\beta}\widetilde{x}^{U}\right]$ we can associate a probability density $f(x)$ with x. If we construct an interval

$$\begin{aligned}\beta&\\ \alpha D = \Bigg[&\min\left\{\min_{\beta\leq\alpha\leq\gamma} f\left({}^{\gamma}\widetilde{x}^{L}\right), \min_{\beta\leq\alpha\leq\gamma} f\left({}^{\gamma}\widetilde{x}^{U}\right)\right\},\\ &\max\left\{\max_{\beta\leq\alpha\leq\gamma} f\left({}^{\gamma}\widetilde{x}^{L}\right), \max_{\beta\leq\alpha\leq\gamma} f\left({}^{\gamma}\widetilde{x}^{U}\right)\right\}\Bigg] \qquad (3.1)\end{aligned}$$

then this interval will contain all of the probability density associated with each $x \in \left[{}^{\beta}\widetilde{x}^{L}, {}^{\beta}\widetilde{x}^{U}\right]$. The membership function of the intuitionistic fuzzy probability density of $\widetilde{x}$ denoted as $\widetilde{f}\left(\widetilde{x}\right)$ is defined as

$$\mu_{\widetilde{f(\widetilde{x})}}(\gamma) = \sup_{\beta\leq\alpha\leq\gamma} \alpha \circ 1_{\alpha\ \mathrm{D}}\ (\gamma)$$

The non membership function of the intuitionistic fuzzy probability density of $\widetilde{x}$ denoted as $\widetilde{f}\left(\widetilde{x}\right)$ is defined as

$$V_{\widetilde{f}(\widetilde{x})}(\gamma) = \inf_{\beta\leq\alpha\leq\gamma} \beta \circ 1^{\beta}_{\ \mathrm{D}}\ (\gamma) \tag{3.2}$$

Let X be an intuitionistic fuzzy random variable. Then by the theorem ${}^{\beta}X$ ^{L} and ${}^{\beta}X\ {}^{U}$ which are the end points of the (α, β) level set with $\alpha \geq \beta$ are random variables. Since ${}^{\gamma}X^{L}$ and ${}^{\gamma}X^{U} \in \left[{}^{\beta}X^{L}, {}^{\beta}X^{U}\right]$ for $\gamma \geq \beta$. The intuitionistic fuzzy expectation of the intuitionistic fuzzy random variable is defined as follows.
Let

$$\beta$$

$$\alpha E = \left[\min\left\{\inf_{\beta\leq\alpha\leq\gamma} E\left({}^{\gamma}X^{L}\right), \inf_{\beta\leq\alpha\leq\gamma} E\left({}^{\gamma}X^{U}\right)\right\},\right.$$

$$\left.\max\left\{\sup_{\beta\leq\alpha\leq\gamma} f\left({}^{\gamma}X^{L}\right), \sup_{\beta\leq\alpha\leq\gamma} f\left({}^{\gamma}X^{U}\right)\right\}\right] \tag{3.3}$$

The membership function and the non-membership function of the intuitionistic fuzzy expectation of the intuitionistic fuzzy random variable X is defined as

$$\mu_{E(x)}^{(\gamma)} = \sup_{\beta\leq\alpha\leq\gamma} \alpha 1_{\alpha\ \mathrm{E}}(\gamma), \qquad V_{E(x)}^{(\gamma)} = \inf_{\beta\leq\alpha\leq\gamma} \beta 1^{\beta}_{\ \mathrm{E}}(\gamma) \tag{3.4}$$

The intuitionistic fuzzy variance of the intuitionistic fuzzy random variables X is formulated as follows.
Let

$$\beta$$

$$\alpha V = \left[\min\left\{\inf_{\beta\leq\alpha\leq\gamma} V\left({}^{\gamma}X^{L}\right), \inf_{\beta\leq\alpha\leq\gamma} V\left({}^{\gamma}X^{U}\right)\right\},\right.$$

$$\left.\max\left\{\sup_{\beta\leq\alpha\leq\gamma} V\left({}^{\gamma}X^{L}\right), \sup_{\beta\leq\alpha\leq\gamma} V\left({}^{\gamma}X^{U}\right)\right\}\right] \tag{3.5}$$

The membership and non-membership function of the intuitionsitic fuzzy variance is specified as

$$\mu_{V(x)}^{(\gamma)} = \sup_{\beta\leq\alpha\leq\gamma} \alpha 1_{\alpha\ \mathrm{V}}(\gamma), \qquad V_{V(x)}^{(\gamma)} = \inf_{\beta\leq\alpha\leq\gamma} \beta 1^{\beta}_{\ \mathrm{V}}(\gamma) \tag{3.6}$$

4. MAIN RESULTS

In this section, with the support of the theoretical settings developed in earlier sections, standard stochastic inequalities such as Markov‘s inequality and Chebyshev‘s inequality for intuitionstic fuzzy random variables are established. Using these inequalities Chebyshev‘s theorem and Khintchine‘s theorem for intuitionistic fuzzy random variables are proved.

Theorem. 4.1. (Markov‘s inequality) Let X be a non-negative intuitionistic fuzzy random variable and ${}^{\beta}_{\alpha X}$ be the (α, β) level set of X with $\alpha \geq \beta$. Then for any canonical intuitionistic non-negative fuzzy number $\widetilde{a}$

$$P\left\{X \geq \widetilde{a}\right\} \subseteq \frac{E\left(X\right)}{\widetilde{a}}$$

Proof. By stipulation X is a non-negative intuitionistic fuzzy random variable. $\therefore$ The end points of the (α, β) level set viz., ${}^{\beta}X^{L} \geq 0$ and ${}^{\beta}X^{U} \geq 0$ for each $\alpha, \beta \in$ [0,1], $\alpha + \beta \leq 1$ and $\alpha \geq \beta$. Then

$$\left({}^{\beta}\widetilde{a}^{L} 1^{\beta}_{\left[{}^{\beta}x^{L} \geq a\right]} \quad V^{\beta}\widetilde{a}^{U} 1_{\left[{}^{\beta}x^{U} \geq a\right]}\right) \leq^{\beta} X^{L} \vee^{\beta} X^{U}$$

Taking expectations both sides

$$E\left({}^{\beta}\widetilde{a}^{L} 1^{\beta}_{\left[{}^{\beta}x^{L} \geq a\right]} \quad V^{\beta}\widetilde{a}^{U} 1_{\left[{}^{\beta}x^{U} \geq a\right]}\right) \leq^{\beta} X^{L} \vee^{\beta} X^{U}$$

$${}^{\beta}\widetilde{a}^{L} P\left\{\left|{}^{\beta}X^{L} - a\right| \geq 0\right\} \vee^{\beta} \widetilde{a}^{U} P\left\{\left|{}^{\beta}X^{U} - a\right| \geq 0\right\} \leq E\left({}^{\beta}X^{L}\right) \vee E\left({}^{\beta}X^{U}\right)$$

$$P\left\{\left|{}^{\beta}X^{L} - a\right| \geq 0\right\} \vee P\left\{\left|{}^{\beta}X^{U} - a\right| \geq 0\right\} \leq \frac{E\left({}^{\beta}X^{L}\right)}{{}^{\beta}\widetilde{a}^{L}} \vee \frac{E\left({}^{\beta}X^{U}\right)}{{}^{\beta}\widetilde{a}^{U}}$$

This is true for each (α, β) level set of the intuitionistic fuzzy random variable X.

$$\therefore P\left\{X \geq \widetilde{a}\right\} \leq \frac{E\left(X\right)}{\widetilde{a}}.$$

Theorem. 4.2. (Chebyshev‘s inequality) If X is an intuitionistic fuzzy random variable with finite intuitionistic fuzzy expectation $E(X)$ and intuitionistic fuzzy variance $V(X)$ then for any non-negative $k \in R$.

$$\left\{P\left(X - E\left(X\right)\right) \geq k\right\} \leq \frac{V\left(X\right)}{k^{2}}$$

Proof. $\left({}^{\beta}X^{L}-E\left({}^{\beta}X^{L}\right)\right)^{2}$ and $\left({}^{\beta}X^{U}-E\left({}^{\beta}X^{U}\right)\right)^{2}$ are non-negative random variables. Then by Markovs inequality

$$P\left\{\left({}^{\beta}X^{L}-E\left({}^{\beta}X^{L}\right)\right)^{2}\vee\left({}^{\beta}X^{U}-E\left({}^{\beta}X^{U}\right)\right)^{2}\geq k^{2}\right\}\leq$$

$$\leq\frac{E\left({}^{\beta}X^{L}-E\left({}^{\beta}X^{L}\right)\right)^{2}}{k^{2}}\vee\frac{E\left({}^{\beta}X^{U}-E\left({}^{\beta}X^{U}\right)\right)^{2}}{k^{2}}\tag{4.1}$$

$\left({}^{\beta}X^{L}-E\left({}^{\beta}X^{L}\right)\right)^{2}\geq k^{2}$ if and only if $\left|{}^{\beta}X^{L}-E\left({}^{\beta}X^{L}\right)\right|\geq k$ and
$\left({}^{\beta}X^{U}-E\left({}^{\beta}X^{U}\right)\right)^{2}\geq k^{2}$ if and only if $\left|{}^{\beta}X^{U}-E\left({}^{\beta}X^{U}\right)\right|\geq k$.
$\therefore$ (4.1) implies

$$P\left\{\left|{}^{\beta}X^{L}-E\left({}^{\beta}X^{L}\right)\right|\vee\left|{}^{\beta}X^{U}-E\left({}^{\beta}X^{U}\right)\right|\geq k\right\}\leq$$

$$\leq\frac{E\left({}^{\beta}X^{L}-E\left({}^{\beta}X^{L}\right)\right)^{2}}{k^{2}}\vee\frac{E\left({}^{\beta}X^{U}-E\left({}^{\beta}X^{U}\right)\right)^{2}}{k^{2}}=\frac{V\left({}^{\beta}X^{L}\right)}{k^{2}}\vee\frac{V\left({}^{\beta}X^{U}\right)}{k^{2}}$$

This is true for each (α,β) level set of the intuitionistic fuzzy random variable X.

$$\{P\left(X-E\left(X\right)\right)\geq k\}\leq\frac{V\left(X\right)}{k^{2}}$$

Theorem 4.3. (Chebyshev‘s Theorem) If $X_1,X_2,........X_n,.....$ is a sequence of pairwise independent intuitionistic fuzzy random variables with finite intuitionistic fuzzy variances, each of its end points of the (α,β) level sets are bounded by the same positive constant c and $\alpha+\beta\leq 1$ then for any $\in>0$.

$$\lim_{n\to\infty}P\left\{\left|Y_n-\widetilde{p}\right|<\in\right\}=1$$

When $Y_n=X_1+X_2+.........+X_n$ and

$$\widetilde{p}=\frac{1}{n}\sum_{k=1}^{n}E\left(X_k\right)$$

Proof. Since $X_1,X_2,........X_n$ are pairwise independent intuitionistic fuzzy random variables.

$$V\left({}^{\beta}{Y_n}^{L}\right)\vee V\left({}^{\beta}{Y_n}^{U}\right)=\frac{1}{n^{2}}\sum_{k=1}^{n}V\left({}^{\beta}{X_K}^{L}\right)\vee\frac{1}{n^{2}}\sum_{k=1}^{n}V\left({}^{\beta}{X_K}^{U}\right)\leq\frac{c}{n}$$

By Chebyshev‘s inequality

$$P\left\{\left(\left|\frac{1}{n}\sum_{k=1}^{n} {}^{\beta}X_K^{L} - \frac{1}{n}\sum_{k=1}^{n} E\left({}^{\beta}X_K^{L}\right)\right| \vee \left|\frac{1}{n}\sum_{k=1}^{n} {}^{\beta}X_K^{U} - \frac{1}{n}\sum_{k=1}^{n} E^{\beta}X_K^{U}\right|\right) <\in\right\} \geq$$

$$\geq \left(1-\frac{var\left({}^{\beta}Y_n{}^{L}\right)}{\in^2}\right) \vee \left(1-\frac{var\left({}^{\beta}Y_n{}^{U}\right)}{\in^2}\right) \geq 1-\frac{c}{n\in^2}$$

Letting $n \to \infty$

$$\lim_{n\to\infty} P\left\{\left(\left|{}^{\beta}Y_n{}^{L} -{}^{\beta}\widetilde{p}^{L}\right| \vee \left|{}^{\beta}Y_n{}^{U} -{}^{\beta}\widetilde{p}^{U}\right|\right) <\in\right\} = 1$$

This is true for each (α, β) level set of Y_n and $\widetilde{p}$
$\therefore \lim_{n\to\infty} P\left\{\left|Y_n-\widetilde{p}\right| <\in\right\} = 1.$

Theorem 4.4. (Khintchine‘s theorem) Let $\{X_k\}$ be pairwise independent and identically distributed intuitionistic fuzzy random variables and let $E(X_k) = \widetilde{M}$ where $\widetilde{M}$ is a finite intuitionistic fuzzy number. Then for any $\in > 0$.

$$\lim_{n\to\infty} P\left\{\left|Y_n-\widetilde{M}\right| <\in\right\} = 0, \text{ where } Y_n = \frac{1}{n}\sum_{k=1}^{n} X_k.$$

Proof. We define the intuitionistic fuzzy number $\widetilde{O}$ as follows.

$$\mu_{\widetilde{O}}(x) = \begin{cases} 1 & ; \quad x = 0 \\ 0 & ; \quad x \neq 0 \end{cases}, \quad V_{\widetilde{O}}(x) = \begin{cases} 0 & ; \quad x = 0 \\ 1 & ; \quad x \neq 0 \end{cases}$$

Without loss of generality we assume $\widetilde{M} = \widetilde{O}$. We define a new intuitionistic fuzzy random variable as follows.

$$\begin{aligned} x_K^* &= X_k \quad \text{for } \left|{}^{\beta}X_K^{L}\right| \leq k \quad \text{and } \left|{}^{\beta}X_K^{U}\right| \leq k \\ &= 0 \quad \text{for } \left|{}^{\beta}X_K^{L}\right| > k \quad \text{and } \left|{}^{\beta}X_K^{U}\right| > k \end{aligned}$$

If $\widetilde{f}(x)$ is the intuitionistic fuzzy distribution function of the intuitionistic fuzzy random variable X_k then

$$E\left({}^{\beta}Y_n{}^{L}\right) = \frac{1}{n}\sum_{k=1}^{n} E\left({}^{\beta}X_k{}^{*L}\right) = \frac{1}{n}\sum_{k=1}^{n} \int_{-k}^{k} {}^{\beta}\widetilde{x}^{L}\, {}^{\beta}\widetilde{f}^{L}\left({}^{\beta}\widetilde{x}^{L}\right) d\, {}^{\beta}\widetilde{x}^{L}$$

$$E\left({}^{\beta}Y_n{}^{U}\right)=\frac{1}{n}\sum_{k=1}^{n}E\left({}^{\beta}X_k{}^{*U}\right)=\frac{1}{n}\sum_{k=1}^{n}\int\limits_{-k}^{k}{}^{\beta}\widetilde{x}^{U}\,{}^{\beta}\widetilde{f}^{U}\left({}^{\beta}\widetilde{x}^{U}\right)d\,{}^{\beta}\widetilde{x}^{U}$$

Letting $k \to \infty$ we note

$$\int\limits_{-\infty}^{\infty}{}^{\beta}\widetilde{x}^{L}\,{}^{\beta}\widetilde{f}^{L}\left({}^{\beta}\widetilde{x}^{L}\right)d\,{}^{\beta}\widetilde{x}^{L}={}^{\beta}\,\widetilde{M}^{L}={}^{\beta}\,\widetilde{O}^{L}$$

and

$$\int\limits_{-\infty}^{\infty}{}^{\beta}\widetilde{x}^{U}\,{}^{\beta}\widetilde{f}^{U}\left({}^{\beta}\widetilde{x}^{U}\right)d\,{}^{\beta}\widetilde{x}^{U}={}^{\beta}\,\widetilde{M}^{U}={}^{\beta}\,\widetilde{O}^{U}$$

This shows that $\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{k=1}^{n}E\left({}^{\beta}X_k{}^{*L}\right)=0$, $\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{k=1}^{n}E\left({}^{\beta}X_k{}^{*U}\right)=0.$
For the intuitionistic fuzzy variance $V\left(X_k^{*}\right).$

$$V\left({}^{\beta}X_k{}^{*L}\right)\leq E\left({}^{\beta}X_k{}^{*L}\right)^{2}=\int\limits_{-k}^{k}\left({}^{\beta}\widetilde{x}^{L}\right)^{2}\,{}^{\beta}\widetilde{f}^{L}\left({}^{\beta}\widetilde{x}^{L}\right)d\,{}^{\beta}\widetilde{x}^{L}$$

$$V\left({}^{\beta}X_k{}^{*U}\right)\leq E\left({}^{\beta}X_k{}^{*U}\right)^{2}=\int\limits_{-k}^{k}\left({}^{\beta}\widetilde{x}^{U}\right)^{2}\,{}^{\beta}\widetilde{f}^{U}\left({}^{\beta}\widetilde{x}^{U}\right)d\,{}^{\beta}\widetilde{x}^{U}$$

$\therefore \frac{1}{n^2}\sum\limits_{k=1}^{n}E\left({}^{\beta}X_k{}^{*L}\right)\leq\frac{1}{n}\int\limits_{-n}^{n}\left({}^{\beta}\widetilde{x}^{L}\right)^{2}\,{}^{\beta}\widetilde{f}^{L}\left({}^{\beta}\widetilde{x}^{L}\right)d\,{}^{\beta}\widetilde{x}^{L}\leq$
$\frac{1}{\sqrt{n}}\int\limits_{-\sqrt{n}}^{\sqrt{n}}\left|{}^{\beta}\widetilde{x}^{L}\right|\,{}^{\beta}\widetilde{f}^{L}\left({}^{\beta}\widetilde{x}^{L}\right)d\,{}^{\beta}\widetilde{x}^{L}+\int\limits_{\left|{}^{\beta}\widetilde{x}^{L}\right|>\sqrt{n}}\left|{}^{\beta}\widetilde{x}^{L}\right|\,{}^{\beta}\widetilde{f}^{L}\left({}^{\beta}\widetilde{x}^{L}\right)d\,{}^{\beta}\widetilde{x}^{L}$
and $\frac{1}{n^2}\sum\limits_{k=1}^{n}V\left({}^{\beta}X_k{}^{*|U}\right)\leq$
$\leq\frac{1}{\sqrt{n}}\int\limits_{-\sqrt{n}}^{\sqrt{n}}\left|{}^{\beta}\widetilde{x}^{U}\right|\,{}^{\beta}\widetilde{f}^{U}\left({}^{\beta}\widetilde{x}^{U}\right)d\,{}^{\beta}\widetilde{x}^{U}+\int\limits_{\left|{}^{\beta}\widetilde{x}^{U}\right|>\sqrt{n}}\left|{}^{\beta}\widetilde{x}^{U}\right|\,{}^{\beta}\widetilde{f}^{U}\left({}^{\beta}\widetilde{x}^{U}\right)d\,{}^{\beta}\widetilde{x}^{U}$
Consequently we obtain

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{k=1}^{n}var\left({}^{\beta}X_k{}^{*L}\right)=0$$

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{k=1}^{n} var\left({}^{\beta}X_k{}^{*U}\right)=0.$$

We introduce another intuitionistic fuzzy random variable $Y^*_{n,r}$ such that

$${}^{\beta}Y_{n,r}^{*L}=\frac{1}{n}\left(\sum_{k=1}^{r}{}^{\beta}X_k{}^{L}+\sum_{k=r+1}^{n}{}^{\beta}X_k{}^{*L}\right)$$

and

$${}^{\beta}Y_{n,r}^{*U}=\frac{1}{n}\left(\sum_{k=1}^{r}{}^{\beta}X_k{}^{U}+\sum_{k=r+1}^{n}{}^{\beta}X_k{}^{*U}\right)$$

we note that the sequence of intuitionistic fuzzy random variables
$X_1, X_2, ...X_n$ is such that as $n\to\infty$.
$\frac{1}{n^2}Var\left(\sum\limits_{i=1}^{n}{}^{\beta}X_i{}^{L}\right)$tends to zero then for any positive constant $\in$

$$\lim_{n\to\infty}P\left\{\left|\frac{1}{n}\sum_{i=1}^{n}{}^{\beta}X_i{}^{L}-\frac{1}{n}\sum_{i=1}^{n}E\left({}^{\beta}X_i{}^{L}\right)\right|<\in\right\}=1.$$

similarly we obtain

$$\lim_{n\to\infty}P\left\{\left|\frac{1}{n}\sum_{i=1}^{n}{}^{\beta}X_i{}^{U}-\frac{1}{n}\sum_{i=1}^{n}E\left({}^{\beta}X_i{}^{U}\right)\right|<\in\right\}=1.$$

Letting $n\to\infty$ we note that

$$\lim_{n\to\infty}P\left\{\left|{}^{\beta}Y_{n,r}^{*L}\right|>\in\right\}=0 \text{ and } \lim_{n\to\infty}P\left\{\left|{}^{\beta}Y_{n,r}^{*U}\right|>\in\right\}=0$$

Now

$$P\left({}^{\beta}Y_{n,r}^{*L}\neq^{\beta}Y_n{}^{L}\right)\leq\sum_{k=r+1}^{n}p\left({}^{\beta}Y_{n,r}^{*L}\neq^{\beta}X_k^{*L}\right)=$$

$$=\sum_{k=r+1}^{n}\int\limits_{\left|{}^{\beta}\widetilde{x}^{L}\right|>k}{}^{\beta}\widetilde{f}^{L}\left({}^{\beta}\widetilde{x}^{L}\right)d^{\beta}\widetilde{x}^{L}\leq\int\limits_{\left|{}^{\beta}\widetilde{x}^{L}\right|>r}\left|{}^{\beta}\widetilde{x}^{L}\right|^{\beta}\widetilde{f}^{L}\left({}^{\beta}\widetilde{x}^{L}\right)d^{\beta}\widetilde{x}^{L}$$

In a similar fashion we can derive

$$P\left({}^{\beta}Y_{n,r}^{*U} \neq^{\beta} Y_{n}^{U}\right) \leq \int\limits_{\left|{}^{\beta}\widetilde{x}^{L}\right|>r} \left|{}^{\beta}\widetilde{x}^{U}\right|^{\beta} \widetilde{f}^{UL}\left({}^{\beta}\widetilde{x}^{U}\right) d^{\beta}\widetilde{x}^{U}$$

If r is sufficiently large, then for any $\delta > 0$

$$P\left(\left({}^{\beta}Y_{n,r}^{*L} \neq^{\beta} Y_{n}^{L}\right) \vee \left({}^{\beta}Y_{n,r}^{*U} \neq^{\beta} Y_{n}^{U}\right)\right) < \delta$$

Hence for any $\in > 0$

$$P\left(\left(\left|{}^{\beta}Y_{n}^{L}\right| \vee \left|{}^{\beta}Y_{n}^{U}\right|\right) > \in\right) =$$

$$= P\left\{\left(\left|{}^{\beta}Y_{n}^{L}\right| \vee \left|{}^{\beta}Y_{n}^{U}\right|\right) > \in, {}^{\beta}Y_{n,r}^{*L} \vee^{\beta} Y_{n,r}^{*U} \neq^{\beta} Y_{n,r}^{L} \vee^{\beta} Y_{n,r}^{U}\right\} +$$

$$+P\left\{\left(\left|{}^{\beta}Y_{n}^{L}\right| \vee \left|{}^{\beta}Y_{n}^{U}\right|\right) > \in, {}^{\beta}Y_{n,r}^{*L} \vee^{\beta} Y_{n,r}^{*U} =^{\beta} Y_{n,r}^{L} \vee^{\beta} Y_{n,r}^{U}\right\}$$

and so $\lim\limits_{n\to\infty} \sup P\left\{\left|{}^{\beta}Y_{n}^{L}\right| > \in\right\} < \delta$.
Since $\delta > 0$ is arbitrary, we obtain

$$\lim_{n\to\infty} P\left\{\left|{}^{\beta}Y_{n}^{L}\right| > \in\right\} = 0 \tag{4.2}$$

Similarly

$$\lim_{n\to\infty} P\left\{\left|{}^{\beta}Y_{n}^{U}\right| > \in\right\} = 0 \tag{4.3}$$

(4.2) and (4.3) are true for each (α, β) level set of the intuitionistic fuzzy random variable which completes the proof.

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About H. Guggenheimer‘s inequality

Mihály Bencze[7]

ABSTRACT. In this paper we will present a refinement of H. Guggenheimer‘s inequality, along with some relations connected with this inequality.

INTRODUCTION

H. Guggenheier in American Mathematical Monthly 73 (1966), 668 presented the following inequality

$$\sum\left(\frac{r_a}{w_a}\right)^t \geq 3$$

valid in any triangle for all $t > 0$.

MAIN RESULTS

Theorem 1. In all triangle ABC holds

$$\sum\left(\frac{r_a}{w_a}\right)^{2t} \geq 3\left(1+\frac{(4R+r)\left((4R+r)^2-3s^2\right)}{3s^2r}\right)^t$$

for all $t \geq 1$.

Proof. Starting from $r_a = \sqrt{\frac{s(s-b)(s-c)}{s-a}}$, $w_a = \frac{2\sqrt{bc}}{b+c}\sqrt{s\left(s-a\right)}$ we obtain $\frac{r_a}{w_a} = \frac{\sqrt{(s-b)(s-c)}}{s-a}\cdot\frac{b+c}{2\sqrt{bc}} \geq \frac{\sqrt{(s-b)(s-c)}}{s-a} = \frac{r\sqrt{s}}{(s-a)\sqrt{s-a}}$. From Jensen‘s inequality we get

$$\sum\left(\frac{r_a}{w_a}\right)^{2t} \geq \sum\left(\frac{r\sqrt{s}}{\left(s-a\right)\sqrt{s-a}}\right)^{2t} = \sum\left(\frac{r^2s}{\left(s-a\right)^3}\right)^t \geq$$

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$$\geq 3\left(\frac{r^2 s}{3}\sum\frac{1}{(s-a)^3}\right)^t = 3\left(1+\frac{(4R+r)\left((4R+r)^2-3s^2\right)}{3s^2r}\right)^t,$$

because

$$\sum\frac{1}{s-a}=\frac{4R+r}{sr},\sum\frac{1}{(s-a)(s-b)}=\frac{1}{r^2},\sum\frac{1}{(s-a)^2}=$$
$$=\frac{(4R+r)^2-2s^2}{s^2r},$$

$$\sum\frac{1}{(s-a)^3}=\frac{3}{\prod(s-a)}+\left(\sum\frac{1}{s-a}\right)\left(\sum\frac{1}{(s-a)^2}-\sum\frac{1}{(s-a)(s-b)}\right)=$$
$$=\frac{3}{sr^2}+\frac{(4R+r)\left((4R+r)^2-3s^2\right)}{s^3r^3}.$$

This is a refinement of H. Guggenheier‘s inequality for $t \geq 1$.
How can we give a refinement for $t \in [0,1)$?
If $t \geq 0$ then we have:

$$\sum\left(\frac{r_a}{w_a}\right)^t \geq \sum\left(\frac{r\sqrt{s}}{(s-a)\sqrt{s-a}}\right)^t.$$

In [1] is proved that

$$x+y+z \geq 3\sqrt[3]{xyz}+\frac{1}{3}\left((x-y)^2+(y-z)^2+(z-x)^2\right), \text{ therefore}$$

$$\sum\left(\frac{r_a}{w_a}\right)^t \geq 3+\frac{\left(r^2s\right)^t}{3}\sum\left(\frac{1}{(s-a)^{\frac{3t}{2}}}-\frac{1}{(s-b)^{\frac{3t}{2}}}\right)^2$$

And we have prove the following:

Theorem 2. In all triangle ABC holds

$$\sum\left(\frac{r_a}{w_a}\right)^t \geq 3+\frac{\left(r^2s\right)^t}{3}\sum\left(\frac{1}{(s-a)^{\frac{3t}{2}}}-\frac{1}{(s-b)^{\frac{3t}{2}}}\right)^2$$

for all $t \geq 0$.

This is an another refinement of H. Guggenheimer's inequality.

Theorem 3. In all triangle ABC holds

1). $\sum \frac{\sqrt{s-a}r_a}{w_a} \geq \frac{4R+r}{\sqrt{s}}$ 2). $\sum \frac{(s-a)^2 r_a^2}{w_a^2} \geq r(4R+r)$

3). $\sum \frac{(s-a)r_a^2}{w_a^2} \geq \frac{(4R+r)^2-2s^2}{s}$ 4). $\sum \frac{r_a r_b}{\sqrt{s-c}w_a w_b} \geq \frac{\sqrt{s}}{r}$

Proof. We have:

1). $\sum \frac{\sqrt{s-a}r_a}{w_a} \geq r\sqrt{s}\sum \frac{1}{s-a} = \frac{4R+r}{\sqrt{s}}$

2). $\sum \frac{(s-a)^2 r_a^2}{w_a^2} \geq r^2 s \sum \frac{1}{s-a} = r(4R+r)$

3). $\sum \frac{(s-a)r_a^2}{w_a^2} \geq r^2 s \sum \frac{1}{(s-a)^2} = \frac{(4R+r)^2-2s^2}{s}$

4). We have $\frac{\sqrt{(s-a)(s-b)r_a r_b}}{w_a w_b} \geq \frac{r^2 s}{(s-a)(s-b)}$ or $\frac{r_a r_b}{\sqrt{s-c}w_a w_b} \geq \frac{r\sqrt{s}}{(s-a)(s-b)}$, therefore $\sum \frac{r_a r_b}{\sqrt{s-c}w_a w_b} \geq r\sqrt{s}\sum \frac{1}{(s-a)(s-b)} = \frac{\sqrt{s}}{r}$

Theorem 4. In all triangle ABC holds

1). $\sum \frac{1}{w_a^2} \geq \frac{1}{2Rr}$ 2). $\sum \frac{1}{w_a^4} \geq \frac{s^2-r^2-4Rr}{8s^2R^2r^2}$

3). $\sum \frac{1}{w_a^2 w_b^2} \geq \frac{s^2+r^2+4Rr}{16s^2R^2r^2}$ 4). $\sum \frac{1}{w_a^4 w_b^4} \geq \frac{\left(s^2+r^2+4Rr\right)^2-16s^2Rr}{256s^4R^4r^4}$

Proof. We have:

$$w_a = \frac{2\sqrt{bc}}{b+c}\sqrt{s(s-a)} \leq \frac{2\sqrt{bc}(s+s-a)}{2(b+c)} = \sqrt{bc}, \text{ therefore}$$

1). $\sum \frac{1}{w_a^2} \geq \sum \frac{1}{bc} = \frac{1}{2Rr}$

2). $\sum \frac{1}{w_a^4} \geq \sum \frac{1}{b^2c^2} = \frac{s^2-r^2-4Rr}{8s^2R^2r^2}$

3). $\sum \frac{1}{w_a^2 w_b^2} \geq \frac{1}{abc}\sum \frac{1}{a} = \frac{s^2+r^2+4Rr}{16s^2R^2r^2}$

4). $\frac{1}{w_a^4 w_b^4} \geq \frac{1}{a^2b^2c^2}\sum \frac{1}{a^2} = \frac{\left(s^2+r^2+4Rr\right)^2-16s^2Rr}{256s^4R^4r^4}$

Theorem 5. In all triangle ABC holds

$$\max\left\{\sum \frac{w_a^2}{b}; \sum \frac{w_a^2}{c}; \sum \frac{w_a w_b}{\sqrt{ab}}\right\} \leq 2s$$

Proof. We have $\sum \frac{w_a^2}{b} \leq \sum c = 2s$, $\sum \frac{w_a^2}{c} \leq \sum b = 2s$, $\sum \frac{w_a w_b}{\sqrt{ab}} \leq \sum c = 2s$

Theorem 6. In all triangle ABC holds

1). $\sum \frac{a}{\sqrt{s-a}} \geq \frac{2(4R+r)}{\sqrt{s}}$

2). $\sum \frac{a^2}{s-a} \geq \frac{4\left((4R+r)^2-2s^2\right)}{s}$

3). $\sum \frac{a^3}{(s-a)\sqrt{s-a}} \geq \frac{8\left((4R+r)^3-12s^2R\right)}{s\sqrt{s}}$

Proof. We have $r_a = \sqrt{\frac{s(s-b)(s-c)}{s-a}} \leq \sqrt{\frac{s}{s-a}} \cdot \frac{s-b+s-c}{2} = \frac{a}{2}\sqrt{\frac{s}{s-a}}$ or $\frac{a}{\sqrt{s-a}} \geq \frac{2r_a}{\sqrt{s}}$, therefore

1). $\sum \frac{a}{\sqrt{s-a}} \geq \frac{2}{\sqrt{s}} \sum r_a = \frac{2(4R+r)}{\sqrt{s}}$

2). $\sum \frac{a^2}{s-a} \geq \frac{4}{s} \sum r_a^2 = \frac{4\left((R+r)^2-2s^2\right)}{s}$

3). $\sum \frac{a^3}{(s-a)\sqrt{s-a}} \geq \frac{8}{s\sqrt{s}} \sum r_a^3 = \frac{8\left((4R+r)^3-12s^2R\right)}{s\sqrt{s}}$

Theorem 7. In all triangle ABC holds

$$\frac{64s^2Rr^2}{27R^2+8Rr+4r^2} \leq w_a w_b w_c \leq \frac{8s^2Rr}{R+14r}$$

Proof. We have $w_a = \frac{2\sqrt{bc}}{b+c}\sqrt{s\,(s-a)}$, therefore

$$\prod w_a = \prod \frac{2\sqrt{bc}}{b+c}\sqrt{s\,(s-a)} = \frac{16s^2Rr^2}{s^2+r^2+2Rr} \text{ but}$$

$$3\sqrt{3}r \leq s \leq \frac{3\sqrt{3}R}{2}, \text{ so}$$

$$\prod w_a \leq \frac{16s^2Rr^2}{27r^2+r^2+2Rr} = \frac{8s^2Rr}{R+14r} \text{ and}$$

$$\prod w_a \geq \frac{16s^2Rr^2}{\frac{27R^2}{4}+r^2+2Rr} = \frac{64s^2Rr^2}{27R^2+8Rr+4r^2}$$

Theorem 8. In all triangle ABC holds

1). $\sum \frac{r_a w_a}{\sqrt{bc}} \leq \frac{2s\left(s^2-r^2-Rr\right)}{s^2+r^2+2Rr}$

2). $\sum \frac{r_a^2 w_a^2}{bc} \leq \frac{s^2\left(s^2-3r^2-4Rr\right)+r^2\left(6R^2+2Rr+r^2\right)}{(s^2+r^2+2Rr)^2}$

3). $\sum \frac{r_a^3 w_a^3}{(bc)^{\frac{3}{2}}} \leq s^3\left(\frac{6Rr}{s^2+r^2+2Rr} + \frac{8\left(s^2-r^2-Rr\right)^3}{(s^2+r^2+2Rr)^3} - \frac{6\left(s^2-r^2-Rr\right)\left(s^2+r^2-2Rr\right)}{(s^2+r^2+2Rr)^2}\right)$

Proof. We have

$$r_a w_a = \frac{2\sqrt{bc}}{b+c} s\sqrt{(s-b)\,(s-c)} \leq \frac{a\sqrt{bc}\cdot s}{b+c} \text{ or}$$

$\frac{r_a w_a}{\sqrt{bc}} \leq \frac{as}{b+c}$ and $\sum \frac{a}{b+c} = \frac{2\left(s^2-r^2-Rr\right)}{s^2+r^2+2Rr}$, $\sum \frac{ab}{(b+c)(c+a)} = \frac{s^2+r^2-2Rr}{s^2+r^2+2Rr}$,
$\sum \left(\frac{a}{b+c}\right)^2 = \left(\sum \frac{a}{b+c}\right)^2 - 2\sum \frac{ab}{(b+c)(c+a)} = \frac{s^2\left(s^2-3r^2-4Rr\right)+r^2\left(6R^2+2Rr+r^2\right)}{(s^2+r^2+2Rr)^2}$;

$\sum\left(\frac{a}{b+c}\right)^3 = 3\prod\frac{a}{b+c} + \left(\sum\frac{a}{b+c}\right)^3 - 3\left(\sum\frac{a}{b+c}\right)\sum\frac{ab}{(b+c)(c+a)} =$
$\frac{6Rr}{s^2+r^2+2Rr} + \frac{8\left(s^2-r^2-Rr\right)^3}{(s^2+r^2+2Rr)^3} - \frac{6\left(s^2-r^2-Rr\right)\left(s^2+r^2-2Rr\right)}{(s^2+r^2+2Rr)^2}$
1). $\sum\frac{r_a w_a}{\sqrt{bc}} \le s\sum\frac{a}{b+c} = \frac{2s\left(s^2-r^2-Rr\right)}{s^2+r^2+2Rr}$
2). $\sum\frac{r_a^2 w_a^2}{bc} \le s^2\sum\left(\frac{a}{b+c}\right)^2 = \frac{s^2\left(s^2-3r^2-4Rr\right)+r^2\left(4R^2+2Rr+r^2\right)}{(s^2+r^2+2Rr)^2}$
3). $\sum\frac{r_a^3 w_a^3}{(bc)^{\frac{3}{2}}} \le s^3\sum\left(\frac{a}{b+c}\right)^3 =$
$s^3\left(\frac{6Rr}{s^2+r^2+2Rr} + \frac{8\left(s^2-r^2-Rr\right)^3}{(s^2+r^2+2Rr)^3} - \frac{6\left(s^2-r^2-Rr\right)\left(s^2+r^2-2Rr\right)}{(s^2+r^2+2Rr)^2}\right)$

Theorem 9. In all triangle ABC holds
1). $\sum r_a w_a \le s^2$
2). $\sum r_a^2 w_a^2 \le \frac{1}{2}s^2\left(s^2-r^2-4Rr\right)$
3). $\sum r_a^3 w_a^3 \le \frac{1}{4}s^4\left(s^2-3r^2-6Rr\right)$
Proof. We have

$$r_a w_a = \frac{2\sqrt{bc}}{b+c}\sqrt{(s-b)(s-c)} \le \frac{a\sqrt{bc}s}{b+c} \le \frac{as}{2}, \text{ therefore}$$

1). $\sum r_a w_a \le \frac{s}{2}\sum a = s^2$
2). $\sum r_a^2 w_a^2 \le \frac{s^2}{4}\sum a^2 = \frac{1}{2}s^2\left(s^2-r^2-4Rr\right)$
3). $\sum r_a^3 w_a^3 \le \frac{s^3}{8}\sum a^3 = \frac{1}{4}s^4\left(s^2-3r^2-6Rr\right)$

Theorem 10. In all triangle ABC holds
1). $7s^2Rr - \frac{1}{2}\sum ab^3 - 4sRr\left(s^2-r^2-4Rr\right) + \frac{1}{4}\left(s^2+r^2+4Rr\right)^2 \le$
$\sum w_a^2 w_b^2 \le s^2r(4R+r)$
2). $\sum w_a^4 w_b^4 \le s^4r^2\left((4R+r)^2-2s^2\right)$
3). $\sum w_a^6 w_b^6 \le s^7\left(\left(s^2-2r^2-8Rr\right)^2 - r(8R-r)\right)$
Proof. We have $w_a^2 = bc - \frac{a^2bc}{(b+c)^2} \le s(s-a)$ and $w_a^2 \ge bc - \frac{1}{4}a^2$, therefore
1). $7s^2Rr - \frac{1}{2}\sum ab^3 - 4sRr\left(s^2-r^2-4Rr\right) + \frac{1}{4}\left(s^2+r^2+4Rr\right)^2 =$
$= \sum\left(bc-\frac{1}{4}a^2\right)\left(ca-\frac{1}{4}b^2\right) \le \sum w_a^2 w_b^2 \le s^2\sum(s-a)(s-b) = s^2r(4R+r)$
2). $\sum w_a^4 w_b^4 \le s^4\sum(s-a)^2(s-b)^2 = s^4r^2\left((4R+r)^2-2s^2\right)$
3). $\sum w_a^6 w_b^6 \le s^6\sum(s-a)^3(s-b)^3 =$
$s^6\left(3\prod(s-a)+\sum(s-a)\right)\left(\sum(s-a)^2 - 2\sum(s-a)(s-b)\right) =$
$s^7\left(\left(s^2-2r^2-8Rr\right)^2 - r(8R-r)\right)$

Theorem 11. In all triangle ABC holds
1). $\frac{1}{2}\left(s^2+3r^2\right)+6Rr \le \sum w_a^2 \le s^2$
2). $\frac{7}{8}\left(s^2+r^2+4Rr\right)^2+\frac{1}{4}\left(s^2-r^2-4Rr\right)^2-19s^2Rr \le \sum w_a^4 \le s^2\left(s^2-2r^2-8Rr\right)$
3). $\left(s^2+r^2+4Rr\right)^3-\frac{s^2Rr}{4}\left(96s^2+108r^2+349Rr\right)-\frac{1}{8}\left(s^2-r^2-4Rr\right)^3+\frac{3}{32}\left(s^2-r^2-4Rr\right)\left(s^2+r^2+4Rr\right)^2 \le \sum w_a^6 \le s^4\left(s^2-12Rr\right)$
Proof. We starting from

$$bc-\frac{1}{4}a^2 \le w_a^2 \le s\left(s-a\right),\ \text{therefore}$$

1). $\frac{1}{2}\left(s^2+3r^2\right)+6Rr=\sum\left(bc-\frac{1}{4}a^2\right)\le\sum w_a^2\le\sum s\left(s-a\right)=s^2$
2). $\frac{7}{8}\left(s^2+r^2+4Rr\right)^2+\frac{1}{4}\left(s^2-r^2-4Rr\right)^2-19s^2Rr=\sum\left(bc-\frac{1}{4}a^2\right)^2\le \sum w_a^4\le\sum s^2\left(s-a\right)^2=s^2\left(s^2-2r^2-8Rr\right)$
3). $\sum a^3b^3=\left(\sum ab\right)^3-3abc\prod\left(a+b\right)= \left(s^2+r^2+4Rr\right)^3-24s^2Rr\left(s^2+r^2+2Rr\right),$
$\sum a^6=3a^2b^2c^2+\left(\sum a^2\right)\left(\left(\sum a^2\right)^2-3\sum a^2b^2\right)=48s^2R^2r^2+ 8\left(s^2-r^2-4Rr\right)^3-6\left(s^2-r^2-4Rr\right)\left(\left(s^2+r^2+4Rr\right)^2-16s^2Rr\right),$
therefore $\left(s^2+r^2+4Rr\right)^3-\frac{s^2Rr}{4}\left(96s^2+108r^2+349Rr\right)- \frac{1}{8}\left(s^2-r^2-4Rr\right)^3+\frac{3}{32}\left(s^2-r^2-4Rr\right)\left(s^2+r^2+4Rr\right)^2=\sum\left(bc-\frac{1}{4}a^2\right)^3\le \sum w_a^6\le\sum s^3\left(s-a\right)^3=s^4\left(s^2-12Rr\right)$

Theorem 12. In all triangle ABC holds

$$\sum\frac{1}{\sqrt{h_a}}\ge\frac{1}{2s\sqrt{2Rr}}\sum\left(\sqrt{r_a}+\sqrt{r_b}\right)\left(\sqrt{r_b}+\sqrt{r_c}\right)$$

Proof. We have $\frac{r_b}{h_a}=\frac{1}{2}\left(1+\frac{r_b}{r_c}\right)$, therefore
$\frac{r_br_c}{h_ah_b}=\frac{1}{4}\left(1+\frac{r_b}{r_c}\right)\left(1+\frac{r_c}{r_a}\right)\ge\frac{1}{4}\left(1+\sqrt{\frac{r_br_c}{r_cr_a}}\right)^2=\frac{1}{4}\left(1+\sqrt{\frac{r_b}{r_a}}\right)^2$ and
$\sqrt{\frac{r_br_c}{h_ah_b}}\sqrt{\frac{r_cr_a}{h_bh_c}}\ge\frac{1}{4}\left(1+\sqrt{\frac{r_b}{r_a}}\right)\left(1+\sqrt{\frac{r_c}{r_b}}\right)$ or
$\frac{1}{\sqrt{h_b}}\ge\frac{1}{2s\sqrt{2Rr}}\left(\sqrt{r_a}+\sqrt{r_b}\right)\left(\sqrt{r_b}+\sqrt{r_c}\right)$, therefore
$\sum\frac{1}{\sqrt{h_a}}\ge\frac{1}{2s\sqrt{2Rr}}\sum\left(\sqrt{r_a}+\sqrt{r_b}\right)\left(\sqrt{r_b}+\sqrt{r_c}\right).$

Theorem 13. In all triangle ABC holds
1). $\sum\sqrt{r_a}\ge\frac{2s\sqrt{r}}{R}$ 2). $\sum\sqrt{r_ar_b}\ge\frac{s^2+r^2+4Rr}{2R}$
3). $\prod\left(\sqrt{r_a}+\sqrt{r_b}\right)\le\frac{4sR}{\sqrt{r}}$

4). $\prod\left(\sqrt[m]{r_a}+\sqrt[m]{r_b}\right)\le\sqrt[n]{2^{3n-1}s^2R}$ for all $n\ge 2$.
Proof. We have $\frac{r_a}{h_a}=\frac{1}{2}\left(\frac{r_a}{r_b}+\frac{r_a}{r_c}\right)\ge\frac{r_a}{\sqrt{r_br_c}}$ or $\sqrt{r_br_c}\ge h_a$, therefore
1). $\sqrt{r_br_c}\sqrt{r_cr_a}\ge h_ah_b$ or $\sqrt{r_c}\ge\frac{1}{s\sqrt{r}}h_ah_b$, therefore
$\sum\sqrt{r_c}\ge\frac{1}{s\sqrt{r}}\sum h_ah_b=\frac{2s\sqrt{r}}{R}$
2). $\sum\sqrt{r_br_c}\ge\sum h_a=\frac{s^2+r^2+4Rr}{2R}$, therefore $\sum\sqrt{r_c}\ge\frac{1}{s\sqrt{r}}\sum h_ah_b=\frac{2s\sqrt{r}}{R}$
3). $\sqrt{\frac{r_br_c}{h_ah_b}}\ge\frac{1}{2}\left(1+\sqrt{\frac{r_b}{r_c}}\right)$, therefore $\prod\left(1+\sqrt{\frac{r_b}{r_a}}\right)\le\prod\sqrt{\frac{r_br_c}{h_ah_b}}=8\prod\frac{r_a}{h_a}$ or
$\prod\left(\sqrt{r_a}+\sqrt{r_b}\right)\le 8\prod\frac{r_a\sqrt{r_a}}{h_a}=\frac{4sR}{\sqrt{r}}$
4). $\prod\left(\sqrt[n]{r_a}+\sqrt[n]{r_b}\right)\le 8\prod\sqrt[n]{\frac{r_a+r_b}{2}}=\sqrt[n]{2^{3n-1}s^2R}$

Theorem 14. In all triangle *ABC* holds

$$\sum\frac{1}{h_a-\lambda r}\ge\frac{3}{(3-\lambda)\,r}$$

for all $\lambda\in[0,2]$.
Proof. We have $\frac{1}{h_a}=\frac{1}{r}$ so $\sum\frac{h_a-\lambda r}{h_a}=3-\lambda$ and from
$\left(\sum\frac{h_a-\lambda r}{h_a}\right)\left(\sum\frac{h_a}{h_a-\lambda r}\right)\ge 9$ we get:

$$\sum\frac{h_a}{h_a-\lambda r}\ge\frac{9}{3-\lambda}\text{ and}$$

$$\sum\frac{\lambda r}{h_a-\lambda r}=\sum\frac{h_a-(h_a-\lambda r)}{h_a-\lambda r}=\sum\frac{h_a}{h_a-\lambda r}-3=\frac{3\lambda}{3-\lambda}$$

so

$$\sum\frac{1}{h_a-\lambda_r}\ge\frac{3}{(3-\lambda)\,r}$$

If $\lambda=2$ the we obtain the inequality of E.A. Bokov (Matematika v. scole, 1966, Nr. 4, 77).

Theorem 15. In all triangle *ABC* holds
1). $\frac{1}{2}\left(s^2+3r^2+12Rr\right)\le\sum m_a^2\le\frac{1}{2}\left(3s^2+r^2+4Rr\right)$
2). $\frac{7}{8}\left(s^2+r^2+4Rr\right)^2+\frac{1}{4}\left(s^2-r^2-4Rr\right)^2-18s^2Rr\le\sum m_a^4\le$
$\frac{7}{8}\left(s^2+r^2+4Rr\right)^2+\frac{1}{4}\left(s^2-r^2-4Rr\right)^2-10s^2Rr$
3). $\left(s^2+r^2+4Rr\right)^3-\frac{1}{8}\left(s^2-r^2-4Rr\right)^3+$
$+\frac{3}{32}\left(s^2-r^2-4Rr\right)\left(s^2+r^2+4Rr\right)^2-\frac{1}{4}s^2Rr\left(128s^2+140r^2+479Rr\right)\le$
$\sum m_a^6\le\left(s^2+r^2+4Rr\right)^3+$

$+\frac{1}{8}\left(s^2-r^2-4Rr\right)^3-\frac{3}{32}\left(s^2-r^2-4Rr\right)\left(s^2+r^2+4Rr\right)^2-$
$\frac{1}{4}s^2Rr\left(116s^2+152r^2+263Rr\right)$

Proof. Because $bc-\frac{a^2}{4}\leq\frac{2\left(b^2+c^2\right)-a^2}{4}=m_a^2\leq bc+\frac{a^2}{4}$ and
$\sum a^3b^3=\left(s^2+r^2+4Rr\right)^3-32s^2Rr\left(s^2+r^2\right)-80s^2R^2r^2$ and
$\sum a^6=48s^2R^2r^2+8\left(s^2-r^2-4Rr\right)^3-$
$6\left(s^2-r^2-4Rr\right)\left(\left(s^2+r^2+4Rr\right)^2-16s^2Rr\right)$ we get

1). $\frac{1}{2}\left(s^2+3r^2+12Rr\right)=\sum\left(bc-\frac{a^2}{4}\right)\leq\sum m_a^2\leq\sum\left(bc+\frac{a^2}{4}\right)=$
$\frac{1}{2}\left(3s^2+r^2+4Rr\right)$

2). $\frac{7}{8}\left(s^2+r^2+4Rr\right)^2+\frac{1}{4}\left(s^2-r^2-4Rr\right)^2-18s^2Rr=\sum\left(bc-\frac{a^2}{4}\right)^2\leq$
$\sum m_a^4\leq\sum\left(bc+\frac{a^2}{4}\right)^2=\frac{7}{8}\left(s^2+r^2+4Rr\right)^2+\frac{1}{4}\left(s^2-r^2-4Rr\right)^2-10s^2R$

3). $\left(s^2+r^2+4Rr\right)^3-\frac{1}{8}\left(s^2-r^2-4Rr\right)^3+$
$\frac{3}{32}\left(s^2-r^2-4Rr\right)\left(s^2+r^2+4Rr\right)^2-\frac{1}{4}s^2Rr\left(128s^2+140r^2+479sRr\right)=$
$\sum\left(bc-\frac{a^2}{4}\right)^3\leq\sum m_a^6\leq\sum\left(bc+\frac{a^2}{4}\right)^3=\left(s^2+r^2+4Rr\right)^3+$
$+\frac{1}{8}\left(s^2-r^2-4Rr\right)^3-\frac{3}{32}\left(s^2-r^2-4Rr\right)\left(s^2+r^2+4Rr\right)^2-$
$\frac{1}{4}s^2Rr\left(116s^2+152r^2+263Rr\right)$

Theorem 16. In all triangle ABC holds

1). $\frac{7}{8}\left(s^2+r^2+4Rr\right)^2+\frac{1}{4}\left(s^2-r^2-4Rr\right)^2-18s^2Rr\leq\sum m_a^2w_a^2\leq$
$\left(s^2+r^2+4Rr\right)^2-20s^2Rr$

2). $\left(s^2+r^2+4Rr\right)^3-\frac{1}{8}\left(s^2-r^2-4Rr\right)^3+$
$\frac{3}{32}\left(s^2-r^2-4Rr\right)\left(s^2+r^2+4Rr\right)^2-\frac{1}{4}s^2Rr\left(128s^2+140r^2+479Rr\right)\leq$
$\leq\sum m_a^2w_a^4\leq\left(s^2+r^2+4Rr\right)^3-32s^2Rr\left(s^2+r^2\right)-68s^2R^2r^2$

3). $\left(s^2+r^2+4Rr\right)^3-\frac{1}{8}\left(s^2-r^2-4Rr\right)^3+$
$\frac{3}{32}\left(s^2-r^2-4Rr\right)\left(s^2+r^2+4Rr\right)^2-\frac{1}{4}s^2Rr\left(128s^2+140r^2+479Rr\right)\leq$
$\sum m_a^4w_a^2\leq\left(s^2+r^2+4Rr\right)^3-\frac{1}{2}s^2Rr\left(63s^2+67r^2+118Rr\right)$

Proof. We have $bc-\frac{a^2}{4}\leq w_a^2\leq bc$, therefore

1). $\frac{7}{8}\left(s^2+r^2+4Rr\right)^2+\frac{1}{4}\left(s^2-r^2-4Rr\right)^2-18s^2Rr=\sum\left(bc-\frac{a^2}{4}\right)^2\leq$
$\sum m_a^2w_a^2\leq\sum\left(bc+\frac{a^2}{4}\right)bc=\left(s^2+r^2+4Rr\right)^2-20s^2Rr$

2). $\left(s^2+r^2+4Rr\right)^3-\frac{1}{8}\left(s^2-r^2-4Rr\right)^3+$
$\frac{3}{32}\left(s^2-r^2-4Rr\right)\left(s^2+r^2+4Rr\right)^2-\frac{1}{2}s^2Rr\left(128s^2+140r^2+479Rr\right)=$
$=\sum\left(bc-\frac{a^2}{4}\right)^3\leq\sum m_a^2w_a^2\leq\sum\left(bc+\frac{a^2}{4}\right)b^2c^2=$

$\left(s^2+r^2+4Rr\right)^3-32Rr\left(s^2+r^2\right)-68s^2R^2r^2$
3). $\left(s^2+r^2+4Rr\right)^3-\frac{1}{8}\left(s^2-r^2-4Rr\right)^3+$
$\frac{3}{32}\left(s^2-r^2-4Rr\right)\left(s^2+r^2+4Rr\right)^2-\frac{1}{4}s^2Rr\left(128s^2+140r^2+479Rr\right)=$
$=\sum\left(bc-\frac{a^2}{4}\right)^3\leq\sum m_a^4w_a^2\leq\sum\left(bc+\frac{a^2}{4}\right)^2bc=$
$\left(s^2+r^2+4Rr\right)^3-\frac{1}{2}s^2Rr\left(63s^2+67r^2+118Rr\right)$

Theorem 17. In all triangle ABC holds

$$\sum m_a^{2t}\geq 3\left(\frac{1}{2}\left(s^2-r^2-4Rr\right)\right)^t$$

for all $t\in(-\infty,0]\cup[1,+\infty)$ and for $t\in(0,1)$ holds the reverse inequality.
Proof. The function $f(x)=x^t$ is convex for $t\in(-\infty,0]\cup[1,+\infty)$ and concave for $t\in(0,1)$. Using the Jensen‘s inequality, we get:

$$\sum m_a^{2t}\geq 3\left(\frac{1}{3}\sum m_a^2\right)^t=3\left(\frac{1}{2}\left(s^2-r^2-4Rr\right)\right)^t$$

for $t\in(-\infty,0]\cup[1,+\infty)$ and for $t\in(0,1)$ holds the reverse inequality.

Theorem 18. In all triangle ABC holds

$$\sum\left(r_aw_a\right)^{2t}\leq 3\left(\frac{s^2r\left(4R+r\right)}{3}\right)^t$$

for all $t\in[0,1]$.
Proof. The function $f(x)=x^t$ is concave for $t\in[0,1]$ and from Jensen‘s inequality, we get:

$$\sum\left(r_aw_a\right)^{2t}\leq 3\left(\frac{1}{3}\sum r_a^2w_a^2\right)^t\leq 3\left(\frac{s^2}{3}\sum(s-a)(s-b)\right)^t=$$

$$=3\left(\frac{s^2r\left(4R+r\right)}{3}\right)^t$$

For $t=1$ we obtain the L. Carlitz‘s inequality (E. 1628, American Mathematical Monthly, 70, 1963 and 71, 1964).

Theorem 19. In all triangle ABC holds

$$\sum w_a^{2t}\geq 3\left(\frac{s^2+3r^2+12Rr}{6}\right)^t$$

for all $t \in [1, +\infty)$ and

$$\sum w_a^{2t} \le 3\left(\frac{s^2 + r^2 + 4Rr}{3}\right)^t$$

for all $t \in [0, 1]$.
Proof. We have

$$\sum w_a^{2t} \ge 3\left(\frac{1}{3}\sum w_a^2\right)^t \ge 3\left(\frac{1}{3}\sum\left(bc - \frac{a^2}{4}\right)\right)^t = 3\left(\frac{s^2 + 3r^2 + 12Rr}{6}\right)^t$$

for $t \ge 1$ and

$$\sum w_a^{2t} \le 3\left(\frac{1}{3}\sum w_a^2\right)^t \le 3\left(\frac{1}{3}\sum bc\right)^t = 3\left(\frac{s^2 + r^2 + 4Rr}{3}\right)^t$$

for all $t \in [0, 1]$.

Theorem 20. In all triangle ABC holds

$$\sum w_a^{2t} \le 3\left(\frac{s^2}{3}\right)^t$$

for all $t \in [0, 1]$.
Proof. We have

$$w_a \le \sqrt{s(s-a)} \text{ etc, therefore}$$

$$\sum w_a^{2t} \le 3\left(\frac{1}{3}w_a^2\right)^t \le 3\left(\frac{1}{3}\sum s(s-a)\right)^t = 3\left(\frac{s^2}{3}\right)^t$$

Theorem 21. In all triangle ABC holds
1). $\sum a w_a^2 \ge 2s\left(-s^2 + 3r^2 + 12Rr\right)$
2). $2s\left(-s^2 + 3r^2 + 12Rr\right) \le \sum a m_a^2 \le 2s\left(s^2 - 3r^2\right)$
3).
$\sum w_a^2 w_b^2 \ge \left(s^2 - r^2 - 4Rr\right)^2 - \frac{7}{16}\left(s^2 + r^2 + 4Rr\right)^2 + s^2\left(-s^2 + 3r^2 + 21Rr\right)$
4). $w_a^2 w_b^2 w_c^2 \ge$
$\frac{1}{8}\left(s^2 - r^2 - 4Rr\right)^3 - \frac{3}{8}\left(s^2 - r^2 - 4Rr\right)\left(s^2 + r^2 + 4Rr\right)^2 + s^2 Rr\left(4s^2 + \frac{27}{4}Rr\right)$
Proof. We have
1). $\sum a w_a^2 \ge \sum a\left(bc - \frac{a^2}{4}\right) = 2s\left(-s^2 + 3r^2 + 12Rr\right)$

2). $2s\left(-s^2+3r^2+12Rr\right)=\sum a\left(bc-\frac{a^2}{4}\right)\leq\sum am_a^2\leq\sum a\left(bc+\frac{a^2}{4}\right)=$
$2s\left(s^2-3r^2\right)$
3). $\sum w_a^2w_b^2\geq\sum\left(bc-\frac{a^2}{4}\right)\left(ca-\frac{b^2}{4}\right)=$
$\left(s^2-r^2-4Rr\right)^2-\frac{7}{16}\left(s^2+r^2+4Rr\right)^2+s^2\left(-s^2+3r^2+21Rr\right)$
4). $w_a^2w_b^2w_c^2\geq\prod\left(bc-\frac{a^2}{4}\right)=$
$\frac{1}{8}\left(s^2-r^2-4Rr\right)^3-\frac{3}{8}\left(s^2-r^2-4Rr\right)\left(s^2+r^2+4Rr\right)^2+s^2Rr\left(4s^2+\frac{27}{4}Rr\right)$

Theorem 22. In all triangle ABC holds
1). $\max\left\{\sum\frac{w_a}{(s-b)\sqrt{s-a}};\sum\frac{w_a}{(s-c)\sqrt{s-a}}\right\}\leq\frac{4R+r}{r\sqrt{s}}$
2). $\min\left\{\sum\frac{1}{(s-b)w_a^2};\sum\frac{1}{(s-c)w_a^2}\right\}\geq\frac{1}{sr^2}$
3). $\min\left\{\sum\frac{ab}{(s-b)w_a^2};\sum\frac{ac}{(s-c)w_a^2}\right\}\geq\frac{s^2+r^2-8Rr}{sr^2}$
4). $\sum\frac{a^2}{w_a^2}\geq\frac{4(R-r)}{r}$
5). $\min\left\{\sum\frac{(s-b)^2}{(s-c)w_a^2};\sum\frac{(s-b)^2}{(s-b)w_a^2}\right\}\geq\frac{s^2-12Rr}{sr^2}$
6). $\min\left\{\sum\frac{c}{(s-b)w_a^2};\sum\frac{b}{(s-c)w_a^2}\right\}\geq\frac{2(4R+r)}{s^2r}$
7). $\sum\frac{a}{w_a^2}\geq\frac{2(2R-r)}{sr}$
Proof. We have $w_a\leq\sqrt{s(s-a)}$, therefore
1). $\sum\frac{w_a}{(s-b)\sqrt{s-a}}\leq\sqrt{s}\sum\frac{1}{s-b}=\frac{4R+r}{r\sqrt{s}}$, $\sum\frac{w_a}{(s-c)\sqrt{s-a}}\leq\sqrt{s}\sum\frac{1}{s-c}=\frac{4R+r}{r\sqrt{s}}$
2). $\sum\frac{1}{(s-b)w_a^2}\geq\frac{1}{s}\sum\frac{1}{(s-a)(s-b)}=\frac{1}{sr^2}$, $\sum\frac{1}{(s-c)w_a^2}\geq\frac{1}{s}\sum\frac{1}{(s-a)(s-b)}=\frac{1}{sr^2}$
3). $\sum\frac{ab}{(s-b)w_a^2}\geq\frac{1}{s}\sum\frac{ab}{(s-a)(s-b)}=\frac{s^2+r^2-8Rr}{sr^2}$,
$\sum\frac{ac}{(s-c)w_a^2}\geq\frac{1}{s}\sum\frac{ac}{(s-a)(s-c)}=\frac{s^2+r^2-8Rr}{sr^2}$
4). $\sum\frac{a^2}{w_a^2}\geq\frac{1}{s}\sum\frac{a^2}{s-a}=\frac{4(R-r)}{r}$
5). $\sum\frac{(s-b)^2}{(s-c)w_a^2}\geq\frac{1}{s}\sum\frac{(s-b)^2}{(s-a)(s-c)}=\frac{s^2-12Rr}{sr^2}$,
$\sum\frac{(s-c)^2}{(s-b)w_a^2}\geq\frac{1}{2}\sum\frac{(s-c)^2}{(s-a)(s-b)}=\frac{s^2-12Rr}{sr^2}$
6). $\sum\frac{b}{(s-c)w_a^2}\geq\frac{1}{s}\sum\frac{b}{(s-a)(s-c)}=\frac{2(4R+r)}{s^2r}$,
$\sum\frac{c}{(s-b)w_a^2}\geq\frac{1}{s}\sum\frac{c}{(s-a)(s-b)}=\frac{2(4R+r)}{s^2r}$
7). $\sum\frac{a}{w_a^2}\geq\frac{1}{s}\sum\frac{a}{s-a}=\frac{2(2R-r)}{sr}$

Theorem 23. In all triangle ABC holds
1). $\sum\frac{w_a^2}{a}\leq\frac{s\left(s^2+r^2-8Rr\right)}{4Rr}$
2). $\sum\frac{w_a^2w_b^2}{ab}\leq\frac{s^2(2R-r)}{2R}$

3). $\sum \frac{w_a^2 w_b^2}{c} \le \frac{sr\left(s^2+(4R+r)^2\right)}{4R}$

4). $\sum \frac{w_a^2 w_b^2}{s-c} \le s\left((4R+r)^2-2s^2\right)$

5). $\sum \frac{a w_a^2}{(s-b)(s-c)} \le \frac{2s(2R-r)}{r}$

6). $\sum \frac{w_a^4}{(s-b)(s-c)} \le \frac{s^2\left(s^2-12Rr\right)}{r^2}$

Proof. We have

1). $\sum \frac{w_a^2}{a} \le s\sum \frac{s-a}{a} = \frac{s\left(s^2+r^2-8Rr\right)}{4Rr}$

2). $\sum \frac{w_a^2 w_b^2}{ab} \le s^2\sum \frac{(s-a)(s-b)}{ab} = \frac{s^2(2R-r)}{2R}$

3). $\sum \frac{w_a^2 w_b^2}{c} \le s^2\sum \frac{(s-a)(s-b)}{c} = \frac{sr\left(s^2+(4R+r)^2\right)}{4R}$

4). $\sum \frac{w_a^2 w_b^2}{s-c} \le s^2\sum \frac{(s-a)(s-b)}{s-c} = s\left((4R+r)^2-2s^2\right)$

5). $\sum \frac{a w_a^2}{(s-b)(s-c)} \le s\sum \frac{a(s-a)}{(s-b)(s-c)} = \frac{2s(2R-r)}{r}$

6). $\sum \frac{w_a^4}{(s-b)(s-c)} \le s^2\sum \frac{(s-a)^2}{(s-b)(s-c)} = \frac{s^2\left(s^2-12Rr\right)}{r^2}$

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New identities and inequalities for the identric mean

Mihály Bencze[8]

ABSTRACT. In this paper we present some new inequalities for the identric mean $I\left(a,b\right)=\frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}$ if $0<a<b$ and $I\left(a,b\right)=a$ if $a=b$.

MAIN RESULTS

Theorem 1. If $1\leq a<b$, then

$$e^3\left(\frac{a^2+4a+1}{b^2+4b+1}\right)^{\frac{6}{b-a}}\left(\frac{\left(b+2-\sqrt{3}\right)\left(a+2+\sqrt{3}\right)}{\left(b+2+\sqrt{3}\right)\left(a+2-\sqrt{3}\right)}\right)^{\frac{3\sqrt{3}}{b-a}}\leq I\left(a,b\right)\leq$$

$$\leq\left(\frac{a}{b}\right)^{\frac{1}{3(b-a)}}\exp\left(\frac{b+a+2}{6}-\frac{2arctg\frac{b-a}{1+ab}}{3\left(b-a\right)}\right)$$

Proof. Using the Theorem 1 from [5] we have:

$$\frac{3\left(x^2-1\right)}{x^2+4x+1}\leq\ln x\leq\frac{\left(x^3-1\right)\left(x+1\right)}{3x\left(x^2+1\right)}$$

therefore

$$\int\limits_a^b\frac{3\left(x^2-1\right)}{x^2+4x+1}dx\leq\int\limits_a^b\ln xdx\leq\int\limits_a^b\frac{\left(x^3-1\right)\left(x+1\right)}{3x\left(x^2+1\right)}dx \text{ or}$$

$$\ln\left(e^{3(b-a)}\left(\frac{a^2+4a+1}{b^2+4b+1}\right)^6\left(\frac{\left(b+2-\sqrt{3}\right)\left(a+2+\sqrt{3}\right)}{\left(b+2-\sqrt{3}\right)\left(a+2-\sqrt{3}\right)}\right)^{3\sqrt{3}}\right)\leq$$

$$\leq\ln\left(I\left(a,b\right)\right)^{b-a}\leq\ln\left(e^{\frac{b^2-a^2}{6}}\cdot e^{\frac{b-a}{3}}\sqrt[3]{\frac{a}{b}}e^{-\frac{3}{2}arctg\frac{b-a}{1+ab}}\right)$$

which prove the desired result.

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Theorem 2. If $1 \leq a < b$, then

$$I\left(a,b\right) \leq \frac{1}{b-a}\exp\left(\int_a^b \sqrt[3]{\frac{2\left(x-1\right)^3}{x\left(x+1\right)}}dx\right)$$

Proof. Using the Theorem 2 from [1] we get:

$$\ln x \leq \sqrt[3]{\frac{2\left(x-1\right)^3}{x\left(x+1\right)}}$$

therefore

$$\int_a^b \ln x dx \leq \int_a^b \sqrt[3]{\frac{2\left(x-1\right)^3}{x\left(x+1\right)}}dx \text{ or}$$

$$I\left(a,b\right) \leq \frac{1}{b-a}\exp\left(\int_a^b \sqrt[3]{\frac{2\left(x-1\right)^3}{x\left(x+1\right)}}dx\right)$$

Theorem 3. If $1 \leq a < b$, then

$$I\left(a,b\right) \leq \exp\left(\frac{1}{2}-\frac{1}{b-a}\ln\frac{b+1}{a+1}+\frac{1}{2\left(b-a\right)}\int_a^b \frac{x-1}{x+1}\sqrt{\frac{2x^2+5x+2}{x}}dx\right)$$

and if $0 \leq a < b \leq 1$, then holds the reverse inequality.
Proof. Using the Theorem 3 from [1] we get:

$$\ln x \leq \frac{x-1}{2\left(x+1\right)}\left(1+\sqrt{\frac{2x^2+5x+2}{x}}\right)$$

for $x \geq 1$ and the reverse for $0 < x \leq 1$, therefore

$$\int_a^b \ln x dx \leq \int_a^b \frac{x-1}{2\left(x+1\right)}dx + \int_a^b \frac{x-1}{2\left(x+1\right)}\sqrt{\frac{2x^2+5x+2}{x}}dx$$

and after then we obtain the desired result.

Theorem 4. If $1 \leq a < b$, then

$$\exp\left(\frac{1}{b-a}\int\limits_a^b F_1(x)\,dx\right) \leq I(a,b) \leq \exp\left(\frac{1}{b-a}\int\limits_a^b F_2(x)\,dx\right)$$

when

$$F_1(x) = \frac{2(x-1)}{x+1}\left(1 + \frac{x-1}{12} - \frac{x-1}{12x} - \frac{(x-1)^3}{360} + \frac{(x-1)^3}{360x^2}\right)$$

and

$$F_2(x) = \frac{2(x-1)}{x+1}\left(1 + \frac{x-1}{12} - \frac{x-1}{12x} - \frac{(x-1)^3}{360+\alpha(x)} + \frac{(x-1)^3}{(360+\alpha(x))\,x^3}\right)$$

where $\alpha(x) = \frac{\left(x^2+5x+1\right)(x-1)^2}{x^2}$.

Proof. Using the Theorem 4 from [1] we get:

$$F_1(x) \leq \ln x \leq F_2(x)\,, \text{ therefore}$$

$$\int\limits_a^b F_1(x)\,dx \leq \int\limits_a^b \ln x dx \leq \int\limits_a^b F_2(x)\,dx \text{ etc.}$$

Theorem 5. If $1 \leq a < b$, then

$$\exp\left(2 - \frac{4}{b-a}\ln\frac{b+1}{a+1}\right) \leq I(a,b) \leq \exp\left(\frac{2\left(\sqrt[3]{b}+\sqrt[3]{a}\right)}{3\left(\sqrt[3]{b^2}+\sqrt[3]{ba}+\sqrt[3]{a^2}\right)} - \frac{2}{\sqrt{b}+\sqrt{a}}\right)$$

Proof. Starting from the inequalities

$$\frac{2(x-1)}{x+1} \leq \ln x \leq \sqrt{x} - \frac{1}{\sqrt{x}}$$

for $x \geq 1$ we obtain:

$$\int\limits_a^b \frac{2(x-1)}{x+1}dx \leq \int\limits_a^b \ln x dx \leq \int\limits_a^b \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right) dx \text{ etc.}$$

Theorem 6. If $1 \le a < b$, then

$$\exp\left(\sum_{k=0}^{n-1}\left(\frac{1}{k+1}-\frac{n}{(k+1)^2(b-a)}\ln\frac{(k+1)(b-1)+n}{(k+1)(a-1)+n}\right)\right) \le I(a,b) \le$$

$$\le \exp\left(\frac{b+a-2}{2n}-\frac{n}{b-a}\sum_{k=1}^{n-1}\frac{1}{k^2}\ln\frac{k(b-1)+n}{k(a-1)+n}\right)$$

Proof. We start from the Theorem 5 from [1] which say:

$$(x-1)\sum_{k=0}^{n-1}\frac{1}{n+(k+1)(x-1)} \le \ln x \le (x-1)\sum_{k=0}^{n-1}\frac{1}{n+k(k-1)}$$

Theorem 7. If $1 \le a < b$, then

$$I(a,b) \ge \exp\left(\frac{1}{b-a}\int_a^b \frac{\left(x^2-1\right)dx}{\sqrt{2\left(x^4+1\right)-(x-1)^2}}\right)$$

Proof. We start with inequality

$$\ln x \ge \frac{x^2-1}{\sqrt{2\left(x^4+1\right)-(x-1)^2}}$$

where $x \ge 1$ (see [2]).

Theorem 8. If $0 < a < b$, then

$$I(a,b) \le \exp\left(\frac{3\left(\sqrt[3]{b}+\sqrt[3]{a}\right)\left(\sqrt[3]{b^2}+\sqrt[3]{a^2}\right)}{4\left(\sqrt[3]{b^2}+\sqrt[3]{ba}+\sqrt[3]{a^2}\right)}+1-\frac{3\left(\sqrt[3]{b}+\sqrt[3]{a}\right)}{2\left(\sqrt[3]{b^2}+\sqrt[3]{ba}+\sqrt[3]{a^2}\right)}-\right.$$

$$\left.-\frac{6}{\sqrt[3]{b^2}+\sqrt[3]{ba}+\sqrt[3]{a^2}}+\frac{2}{b-a}arctg\frac{\sqrt[3]{b}-\sqrt[3]{a}}{1+\sqrt[3]{ab}}\right)$$

Proof. We start with Karamata‘s inequality

$$\ln x \le \frac{(x-1)\left(1+\sqrt[3]{x}\right)}{x+\sqrt[3]{x}} \text{ etc.}$$

Theorem 9. If $0 < a < b$, then

$$\exp\left(\frac{(b+n-1)^2\ln(b+n-1)-(a+n-1)^2\ln(a+n-1)}{2(b-a)}-\right.$$

$$\left.-\frac{b^2\ln b-a^2\ln a}{2(b-a)}-\frac{3(n-1)}{2}\right)\leq\prod_{k=1}^{n-1}I(a+k,b+k)\leq$$

$$\leq\exp\left(\frac{(b+n)^2\ln(b+n)-(a+n)^2\ln(a+n)}{2(b-a)}-\right.$$

$$\left.-\frac{(b+1)^2\ln(b+1)-(a+1)^2\ln(a+1)}{2(b-a)}-\frac{3(n-1)}{2}\right)$$

for all $n \geq 2$, $n \in N^*$.
Proof. We starting from

$$\int_a^b((x+n-1)\ln(x+n-1)-x\ln x)\,dx\leq\int_a^b(n-1)+\sum_{k=1}^{n-1}\ln(x+k)\,dx\leq$$

$$\leq\int_a^b((x+n)\ln(x+n)-(x+1)\ln(x+1))\,dx$$

(see [3], page 275.)

Theorem 10. If $\lambda > 0$ and $0 < a < b \leq 1$, then

$$I(a+\lambda,b+\lambda)\leq\lambda\left(\frac{\lambda+1}{\lambda}\right)^{\frac{b+a}{2}}\exp\left(\frac{3(b+a)-2\left(b^2+ba+a^2\right)}{12\lambda^2}\right)$$

Proof. We start with the inequality

$$0<\ln(\lambda+x)-\ln\lambda-x\ln\frac{\lambda+1}{\lambda}<\frac{x(1-x)}{2\lambda^2}$$

where $\lambda > 0$ and $0 < x \leq 1$ (see [3], page 274.)

Theorem 11. If $0 < r < 1$ and $0 \leq a < b$, $a \leq c < d$, then

$$I^r(a+1,b+1)\, I^{1-r}(c+1,d+1) \geq$$

$$\geq \exp\left(\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d \ln\left(1 + x^r y^{1-r}\right) dxdy\right)$$

Proof. If $x, y \geq 0$ and $0 < r < 1$, then

$$r \ln(1+x) + (1-r)\ln(1+y) \geq \ln\left(1 + x^r y^{1-r}\right)$$

(see [3], page 276), therefore

$$\int_a^b \int_c^d \ln(1+x)\, dxdy + (1-r) \int_a^b \int_c^d \ln(1+y)\, dxdy \geq$$

$$\geq \int_a^b \int_c^d \ln\left(1 + x^r y^{1-r}\right) dxdy \text{ etc.}$$

Theorem 12. If $0 < r < 1$ and $0 < a < b$, then

$$I^{1-r}(a+1,b+1) \leq \exp\left(\frac{1}{b-a} \int_a^b \ln\left(\frac{1-x}{1-x^r}\right) dx\right)$$

Proof. We have the inequality

$$\frac{1-x^r}{1-x}(x+1)^{1-r} \leq 0$$

(see [3], page 276), therefore

$$\int_a^b \ln \frac{1-x^r}{1-x} dx + (1-r) \int_a^b \ln(x+1)\, dx \leq 0$$

from which holds the result.

Theorem 13. If $0 < a < b$ and $c, d > 0$, then

$$\frac{I(c+a, c+b)}{I(d+a, d+b)} \geq \left(\frac{d+b}{d+a}\right)^{\frac{d}{b-a}\ln\frac{c}{d}}$$

Proof. Using the inequality

$$\left(\frac{c+x}{d+x}\right)^{d+x} \geq \left(\frac{c}{d}\right)^{d}$$

(see [3], page 279) we obtain

$$\ln(c+x) - \ln(d+x) \geq \frac{d\ln\frac{c}{d}}{d+x} \text{ and}$$

$$\int\limits_a^b \ln(c+x)\,dx - \int\limits_a^b \ln(d+x)\,dx \geq d\ln\frac{c}{d}\int\limits_a^b \frac{dx}{d+x} \text{ etc.}$$

Theorem 14. If $0 < c \leq d$ and $0 < a < b < 1$, then

$$I(1-a, 1-b) \geq$$

$$\geq \exp\left(\frac{c}{b-a}\int\limits_a^b \ln\left(1 - x^{\frac{c}{c+d}}\right)dx - \frac{d}{b-a}\int\limits_a^b \ln\left(1 - x^{\frac{d}{c+d}}\right)dx\right)$$

Proof. We start with inequality

$$\left(\frac{1-x^d}{1-x^{c+d}}\right)^d \geq \left(\frac{1-x^c}{1-x^{c+d}}\right)^c$$

(see [3], page 277) or

$$(1-x)^{d-c} \leq \left(1 - x^{\frac{d}{c+d}}\right)^d \left(1 - x^{\frac{c}{c+d}}\right)^{-c} \text{ or}$$

$$(d-c)\int\limits_a^b \ln(1-x)\,dx \leq d\int\limits_a^b \ln\left(1 - x^{\frac{d}{c+d}}\right)dx - c\int\limits_a^b \ln\left(1 - x^{\frac{c}{c+d}}\right)dx \text{ etc.}$$

Theorem 15. If $0 < a < b$, $0 < c \le d$, $0 < e \le f$ then

$$I(1-a, 1-b) \ge \exp\left(\frac{-f}{e(b-a)} \int_a^b \ln\left(1 - x^{\frac{c}{d}}\right) dx\right)$$

Proof. We have $\left(1 - x^d\right)^e \ge (1 - x^c)^f$ (see [3], page 278) or $(1-x)^e \ge \left(1 - x^{\frac{c}{d}}\right)^f$ and

$$e \int_a^b \ln(1-x)\, dx \ge f \int_a^b \ln\left(1 - x^{\frac{c}{d}}\right) dx \text{ etc.}$$

Theorem 16. If $0 < a < b$, then

$$I(a,b) =$$

$$= \exp\left(\frac{2}{b-a} \sum_{k=0}^{\infty} \sum_{p=0}^{2k+1} \frac{(-1)^{p+1} 2^p}{p-1} \binom{2k+1}{p} \left((b+1)^{-p+1} - (a+1)^{-p+1}\right)\right)$$

Proof. We have the identity:

$$\ln x = 2 \sum_{k=0}^{\infty} \frac{1}{2k+1} \left(\frac{x-1}{x+1}\right)^{2k+1} = 2 \sum_{k=0}^{\infty} \sum_{p=0}^{2k+1} (-1)^p 2^p \binom{2k+1}{p} \frac{1}{(x+1)^p}$$

therefore

$$(b-a) \ln I(a,b) = \int_a^b \ln x dx = 2 \int_a^b \sum_{k=0}^{\infty} \sum_{p=0}^{2k+1} (-1)^p 2^p \binom{2k+1}{p} \int_a^b \frac{dx}{(x+1)^p} \text{ etc.}$$

Theorem 17. If $0 < a < b \le 1$, then

$$I(a+1, b+1) = \exp\left(\frac{1}{b-a} \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \left(b^{k+1} - a^{k+1}\right)}{k(k+1)}\right)$$

Proof. We have

$$(b-a)\ln I\left(a+1,b+1\right)=\int\limits_a^b \ln\left(x+1\right)dx=\sum_{k=1}^{\infty}\frac{\left(-1\right)^{k-1}}{k}\int\limits_a^b x^k dx=$$

$$=\sum_{k=1}^{\infty}\frac{\left(-1\right)^{k-1}\left(b^{k+1}-a^{k+1}\right)}{k\left(k+1\right)}$$

Theorem 18. If $1\leq a<b$, then

$$\frac{I\left(a+1,b+1\right)}{I\left(a-1,b-1\right)}=\exp\left(\frac{2\ln\frac{b}{a}}{b-a}-\frac{1}{b-a}\sum_{k=1}^{\infty}\frac{1}{k\left(2k+1\right)}\left(b^{-2k}-a^{-2k}\right)\right)$$

Proof. We have

$$(b-a)\ln\frac{I\left(a+1,b+1\right)}{I\left(a-1,b-1\right)}=\int\limits_a^b \ln\frac{x+1}{x-1}dx=2\sum_{k=0}^{\infty}\frac{1}{2k+1}\int\limits_a^b\frac{dx}{x^{2k+1}}=$$

$$=2\ln\frac{b}{a}-\sum_{k=1}^{\infty}\frac{1}{k\left(2k+1\right)}\left(b^{-2k}-a^{-2k}\right)\text{ etc.}$$

Theorem 19. If $0<a<b<1$, then

$$I\left(1+a,1+b\right)I\left(1-a,1-b\right)=\exp\left(\frac{1}{b-a}\sum_{k=0}^{\infty}\frac{b^{2k+2}-a^{2k+2}}{\left(k+1\right)\left(2k+1\right)}\right)$$

Proof. We have

$$(b-a)\ln\left(I\left(1+a,1+b\right)I\left(1-a,1-b\right)\right)=\int\limits_a^b\ln\frac{1+x}{1-x}dx=$$

$$=2\int\limits_a^b\sum_{k=0}^{\infty}\frac{x^{2k+1}}{2k+1}dx=\sum_{k=0}^{\infty}\frac{b^{2k+2}-a^{2k+2}}{\left(k+1\right)\left(2k+1\right)}\text{ etc.}$$

Theorem 20. If $0 < a \leq c \leq d \leq b$, then

$$a^{b-\frac{c+d}{2}} \cdot b^{\frac{c+d}{2}-a} \leq (I(c,d))^{b-a}$$

Proof. From weighted AM-GM inequality we have:

$$x = \left(\frac{b-x}{b-a}\right) a + \left(\frac{x-b}{b-a}\right) b \geq a^{\frac{b-x}{b-a}} \cdot b^{\frac{x-a}{b-a}}$$

or $a^{b-x} b^{x-a} \leq x^{b-a}$ and

$$(b-x)\ln a + (x-a)\ln b \leq (b-a)\ln x.$$

Therefore

$$\int\limits_c^d (b-x)\ln a dx + \int\limits_c^d (x-a)\ln b dx \leq \int\limits_c^d (b-a)\ln x dx \text{ etc.}$$

If $a = c, b = d$ then we get $\sqrt{ab} \leq I(a,b)$.

Corollary 20.1. If $0 < a \leq c \leq d \leq b$, then

$$L(a,b) \leq \frac{(b-a)\left(d^{b-a+1} - c^{b-a+1}\right)}{(b-a+1)\left(b^{d-a}a^{b-d} - b^{c-a}a^{b-c}\right)}$$

where $L(a,b) = \frac{a-b}{\ln a - \ln b}$ denote the logarithmic mean.
Proof. Using the inequality $a^{b-x}b^{x-a} \leq x^{b-a}$ we get

$$\frac{L(a,b)\left(b^{d-a}a^{b-d} - b^{c-a}a^{b-c}\right)}{b-a} = \int\limits_c^d a^{b-x}b^{x-a}dx \leq \int\limits_c^d x^{b-a}dx =$$

$$= \frac{d^{b-a+1} - c^{b-a+1}}{b-a+1}$$

Corollary 20.2. If $x > 1$, then

$$\ln x > \frac{((x-1)e+1)\left(x^{(x-1)e} - 1\right)}{e\left(x^{(x-1)e} - 1\right)}$$

Proof. In Corollary 20.1 we take $c = a = e$, $d = b = xe$.

Corollary 20.3. If $1 < a < b$, then

$$I(a,b) > \exp\left(\frac{1}{b-a}\int\limits_a^b \frac{((x-1)e+1)\left(x^{(x-1)e}-1\right)}{e\left(x^{(x-1)e}-1\right)}dx\right)$$

Proof. We have

$$(b-a)\ln I(a,b) = \int\limits_a^b \ln x dx > \int\limits_a^b \frac{((x-1)e+1)\left(x^{(x-1)e}-1\right)}{e\left(x^{(x-1)e}-1\right)}dx \text{ etc.}$$

Corollary 20.4. If $0 < a_1 < a_2 < ... < a_{n+1}$ is an arithmetical progression with ratio r, then

$$\sum_{k=1}^{n} \frac{a_{k+1}^r - a_k^r}{a_{k+1}^{r+1} - a_k^{r+1}} < \frac{1}{r+1}\ln\frac{a_{n+1}}{a_1}$$

Proof. In Corollary 20.1 we take $c = a = a_k$, $d = b = a_{k+1}$ $(k = 1, 2, ..., n)$ and we obtain:

$$\sum_{k=1}^{n} \frac{a_{k+1}^r - a_k^r}{a_{k+1}^{r+1} - a_k^{r+1}} < \sum_{k=1}^{n} \frac{\ln a_{k+1} - \ln a_k}{a_{k+1} - a_k + 1} = \frac{1}{r+1}\sum_{k=1}^{n}(\ln a_{k+1} - \ln a_k) =$$

$$= \frac{1}{r+1}\ln\frac{a_{n+1}}{a_1}$$

Remark. *We conjectured, that*

$$\sum_{k=1}^{n} \frac{a_{k+1}^r - a_k^r}{a_{k+1}^{r+1} - a_k^r} > \frac{1}{r+1}\ln\frac{a_1}{a_{n+1}}$$

Theorem 21. If $0 < a \le c \le d \le b$, then

$$(I(d-a, c-a))^b (I(b-d, b-c))^{-a} \le \left(\left(\frac{b-a}{b+a}\right) I(c,d)\right)^{a+b}$$

Proof. From weighted AM-GM inequality we have

$$\left(\frac{b-a}{b+a}\right)x = \left(\frac{b}{a+b}\right)(x-a) + \left(\frac{a}{a+b}\right)(b-x) \ge (x-a)^{\frac{b}{a+b}}(b-x)^{\frac{a}{a+b}} \text{ or}$$

$$(x-a)^b (b-x)^a \le \left(\frac{b-a}{b+a}\right)^{a+b} x^{a+b}$$

Therefore

$$\frac{b}{a+b}\ln(x-a)+\frac{a}{a+b}\ln(b-x)\le \ln\frac{b-a}{b+a}+\ln x \text{ and}$$

$$\int_c^d \frac{b}{a+b}\ln(x-a)\,dx+\int_c^d \frac{a}{a+b}\ln(b-x)\,dx\le$$

$$\le \int_c^d \ln\frac{b-a}{b+a}dx+\int_c^d \ln x dx \text{ etc}$$

Corollary 21.1. If $0<a\le c\le d\le b$, then

$$\int_c^d (x-a)^b (b-x)^a\,dx \le \frac{(b-a)^{a+b}\left(d^{a+b+1}-c^{a+b+1}\right)}{(a+b+1)(b+a)^{a+b}}$$

Proof. Using the inequality

$$(x-b)^b (b-x)^a \le \left(\frac{b-a}{b+a}\right)^{a+b} x^{a+b}$$

we obtain

$$\int_c^d (x-b)^b (b-x)^a\,dx \le \int_c^d \left(\frac{b-a}{b+a}\right)^{a+b} x^{a+b}dx =$$

$$= \frac{(b-a)^{a+b}\left(d^{a+b+1}-c^{a+b+1}\right)}{(a+b+1)(b+a)^{a+b}}$$

Theorem 22. If $0<a<b$, then

$$\frac{a}{b}\left(I(a,b)\right)^{b-a} < \frac{\Gamma(b)}{\Gamma(a)} < \sqrt{\frac{a}{b}}\left(I(a,b)\right)^{b-a}$$

where Γ denote the Euler Gamma function.
Proof. We start from $\ln x-\frac{1}{x}<\Psi(x)<\ln x-\frac{1}{2x}$, therefore

$$(b-a)\ln I(a,b)-\ln\frac{b}{a}=\int\limits_a^b\left(\ln x-\frac{1}{x}\right)dx<\int\limits_a^b\Psi(x)\,dx=\ln\Gamma(x)\Big|_a^b=$$

$$=\ln\frac{\Gamma(b)}{\Gamma(a)}<\int\limits_a^b\left(\ln x-\frac{1}{2x}\right)dx=(b-a)\ln I(a,b)-\ln\sqrt{\frac{b}{a}}$$

Theorem 23. If $0<a<b$, then

$$\frac{1}{b-a}\int\limits_a^b\ln\Gamma(x)\,dx=$$

$$=\frac{-\gamma(b+a-2)}{2}+\sum_{k=1}^{\infty}\left(\frac{b+a-2}{2k}-\ln I(a+k,b+k)+\ln(k+1)\right)$$

where γ denote the Euler constant.

Proof. We have

$$\int\limits_a^b\ln\Gamma(x)\,dx=\int\limits_a^b\left(-\gamma(x-1)+\sum_{k=1}^{\infty}\frac{x-1}{k}-\ln(x+k)+\ln(x+1)\right)dx$$

Theorem 24. If $0<a<b$, then

$$\frac{I(a+1,b+1)}{I^4\left(\frac{a}{2}+1,\frac{b}{2}+1\right)}<$$

$$<\exp\left(\frac{1}{4(b-a)}\left(b^2\ln\left(\left(\frac{b}{2}+1\right)(b+1)\right)-a^2\ln\left(\left(\frac{a}{2}+1\right)(a+1)\right)\right)+\right.$$

$$\left.+\frac{1}{4}+\ln\frac{b+2}{a+2}+\frac{1}{4}\ln\frac{b+1}{a+1}\right)$$

Proof. We start from the inequality

$$2\ln(1+x)-4\ln\left(1+\frac{x}{2}\right)<x\ln\left(1+\frac{x}{2}\right)-x\ln(1+x)$$

for all $x > 0$, therefore

$$2\int_a^b \ln(1+x)\,dx - 4\int_a^b \ln\left(1+\frac{x}{2}\right)dx < \int_a^b x\ln\left(1+\frac{x}{2}\right)dx - \int_a^b x\ln(1+x)\,dx \text{ etc.}$$

Theorem 25. If $\frac{1}{2} < a < b$, then

$$I(a,b) = \exp\left(\frac{1}{b-a}\sum_{k=1}^{n}\frac{1}{k}\left(1 - k\ln\frac{b}{a} + \sum_{p=2}^{k}(-1)^p\frac{\binom{k}{p}}{p-1}\left(b^{-p+1} - a^{-p+1}\right)\right)\right)$$

Proof. We have

$$(b-a)\ln I(a,b) = \int_a^b \ln x dx = \sum_{k=1}^{\infty}\frac{1}{k}\int_a^b\left(1-\frac{1}{x}\right)^k dx \text{ etc.}$$

Theorem 26. If $0 < a < b \le 2$, then

$$I(a,b) = \exp\left(\frac{1}{b-a}\sum_{k=1}^{\infty}\frac{(-1)^k\left((b-1)^{k+1}-(a-1)^{k+1}\right)}{k(k+1)}\right)$$

Proof. We have

$$(b-a)\ln I(a,b) = \int_a^b \ln x dx = \sum_{k=1}^{\infty}(-1)^k\int_a^b\frac{(x-1)^k}{k}dx \text{ etc.}$$

Theorem 27. If $0 < a < b$, then

$$I(a,b) = \exp\left(\ln c + \frac{1}{b-a}\sum_{k=1}^{\infty}\frac{(b-c)^{k+1}-(a-c)^{k+1}}{k(k+1)c^k}\right)$$

where $c > 0$.
Proof. We have

$$(b-a)\ln I(a,b) = \int_a^b \ln x dx = \int_a^b\left(\ln c + \sum_{k=1}^{\infty}\frac{(x-c)^k}{kc^k}\right)dx \text{ etc.}$$

Theorem 28. If $0 < a < b$ and $c > 0$, then

$$I(a+c,b+c)=$$

$$=\exp\left(\ln c+\frac{2}{b-a}\sum_{k=0}^{\infty}\frac{1}{2k+1}\left(b-a-2c(2k+1)\ln\frac{b+2k}{a+2k}+\right.\right.$$

$$\left.\left.+\sum_{p=2}^{2k+1}\frac{(-1)^{p+1}(2c)^{2p+1}\binom{2k+1}{p}}{2p}\left((b+2c)^{-2p}-(a+2c)^{-2p}\right)\right)\right)$$

Proof. We have

$$(b-a)\ln I\left(a+c,b+c\right)=\int_a^b \ln(x+c)dx=$$

$$=\int_a^b\left(\ln c+2\sum_{k=0}^{\infty}\frac{1}{2k+1}\left(\frac{x}{2c+x}\right)^{2k+1}\right)dx \text{ etc.}$$

Theorem 29. If $0 < a \le b$, then

$$I\left(\frac{a+b}{2},b\right)-I\left(a,\frac{a+b}{2}\right)\ge\ln\frac{a+b}{2}$$

Proof. If $f:[a,b]\to R$ is convex, then

$$\int_a^{\frac{a+b}{2}} f(x)dx-\int_{\frac{a+b}{2}}^{b} f(x)dx\le\frac{b-a}{2}\left(f(a)-f\left(\frac{a+b}{2}\right)\right)$$

If f is concave, then holds the reverse inequality. We take $f(x)=\ln x$.

Theorem 30. If $0 < a \le c \le b$, then

$$I^2(a,b)\le cI(2a-c,2b-c)$$

Proof. If $f:R\to R$ is convex and $0 < a \le c \le b$, then

$$\int_a^b f(x)dx\le\frac{1}{2}(b-a)f(c)+\frac{1}{4}\int_{2a-c}^{2b-c} f(x)dx$$

In this we take $f(x) = -\ln x$.

Theorem 31. If $0 < a \leq b$, then exist $c \in (0,1)$, such that:

$$(I(a,b))^{b-a} = \frac{(bc)^b}{(ac)^a}$$

GENERALIZATIONS OF THE INDENTRIC MEAN

Defintion 1. Denote

$$I_n\left(a,b\right) = \left(\left(\frac{b^{b^{n+1}}}{a^{a^{n+1}}}\right)^{\frac{1}{n+1}} \cdot e^{-\frac{b^{n+1}-a^{n+1}}{(n+1)^2}}\right)^{\frac{1}{b-a}}$$

if $0 < a < b$ and $I_n\left(a,b\right) = a$ if $a = b$ the generalized identric mean. We have $I_0\left(a,b\right) = I\left(a,b\right)$ the classical identric mean.
Because the classical identric mean can be defined in following

$$(b-a)\ln I\left(a,b\right) = \int\limits_a^b \ln x dx,$$

we have

$$(b-a)\ln I_n\left(a,b\right) = \int\limits_a^b x^n \ln x dx = \left(\frac{x^{n+1}\ln x}{n+1} - \frac{x^{n+1}}{\left(n+1\right)^2}\right)\Big|_a^b =$$

$$= \frac{b^{n+1}\ln b - a^{n+1}\ln a}{n+1} - \frac{b^{n+1}-a^{n+1}}{\left(n+1\right)^2} \text{ etc.}$$

Defintion 2. Denote

$$I_n\left(a,b\right) = \left(\exp\left(\left(-1\right)^n n!\left(b\sum_{r=0}^{n}\frac{\left(-\ln b\right)^r}{r!} - a\sum_{r=0}^{n}\frac{\left(-\ln a\right)^r}{r!}\right)\right)\right)^{\frac{1}{b-a}}$$

if $0 < a < b$ and $I_n\left(a,b\right) = a$ if $a = b$ the generalized identric mean. We have $I_1\left(a,b\right) = I\left(a,b\right)$, the classical identric mean. We start from

$$(b-a)\ln I_n\left(a,b\right) = \int\limits_a^b \left(\ln x\right)^n dx = \left(\left(-1\right)^n n! x\sum_{r=0}^{n}\frac{\left(-\ln x\right)^r}{r!}\right)\Big|_a^b =$$

$$= (-1)^n n! \left(b \sum_{r=0}^{n} \frac{(-\ln b)^r}{r!} - a \sum_{r=0}^{n} \frac{(-\ln a)^r}{r!} \right) \text{ etc.}$$

Defintion 3. Denote

$$I_{m,n}(a,b) =$$

$$= \left(\exp \left((-1)^n \frac{n!}{m+1} \left(b^{m+1} \sum_{r=0}^{n} \frac{(-\ln b)^r}{r!\,(m+1)^{n-r}} - a^{m+1} \sum_{r=0}^{n} \frac{(-\ln b)^r}{r!\,(m+1)^{n-r}} \right) \right) \right)^{\frac{1}{b-a}}$$

if $0 < a < b$ and $I_{m,n}(a,b) = a$ if $a = b$ the generalized identric mean. We have $I_{0,1}(a,b) = I(a,b)$ the classical identric mean. We start from

$$(b-a) \ln I_{m,n}(a,b) = \int_a^b x^m (\ln x)^n \, dx =$$

$$= \left((-1)^n \frac{n!}{m+1} x^{m+1} \sum_{r=0}^{n} \frac{(-\ln x)^r}{r!\,(m+1)^{n-r}} \right) \Bigg|_a^b =$$

$$= (-1)^n \frac{n!}{m+1} \left(b^{m+1} \sum_{r=0}^{n} \frac{(-\ln b)^r}{r!\,(m+1)^{n-r}} - a^{m+1} \sum_{r=0}^{n} \frac{(-\ln a)^r}{r!\,(m+1)^{n-r}} \right) \text{ etc.}$$

Defintion 4. Denote

$$I_{m,c,d}(a,b) = \exp \left(\frac{1}{m+1} \left(b^{m+1} - \left(-\frac{d}{c} \right)^{m+1} \right) \ln (cb+d) - \right.$$

$$\left. - \frac{1}{m+1} \left(a^{m+1} - \left(-\frac{d}{c} \right)^{m+1} \right) \ln (ca+d) - \right.$$

$$\left. - \frac{1}{m+1} \left(-\frac{d}{c} \right)^{m+1} \sum_{r=1}^{m+1} \frac{1}{r} \left(\left(-\frac{cb}{d} \right)^r - \left(-\frac{ca}{d} \right)^r \right) \right)^{\frac{1}{b-a}}$$

if $0 < a < b$ and $I_{m,c,d}(a,b) = a$ if $a = b$, where $a, b \geq 0$.
We have $I_{0,1,0}(a,b) = I(a,b)$ the classical identric mean. We start from

$$(b-a)\ln I_{m,c,d}(a,b)=\int\limits_a^b x^m \ln(cx+d)\,dx=$$

$$=\left(\frac{1}{m+1}\left(x^{m+1}-\left(-\frac{d}{c}\right)^{m+1}\right)\ln(cx+d)-\frac{1}{m+1}\left(-\frac{d}{c}\right)^{m+1}\sum_{r=1}^{m+1}\frac{1}{r}\left(-\frac{cx}{d}\right)^r\right)\Big|_a^b=$$

$$=\frac{1}{m+1}\left(b^{m+1}-\left(-\frac{d}{c}\right)^{m+1}\right)\ln(cb+d)-\frac{1}{m+1}\left(a^{m+1}-\left(-\frac{d}{c}\right)^{m+1}\right)\ln(ca+d)-$$

$$-\frac{1}{m+1}\left(-\frac{d}{c}\right)^{m+1}\sum_{r=1}^{m+1}\frac{1}{r}\left(\left(-\frac{cb}{d}\right)^r-\left(-\frac{ca}{d}\right)^r\right)\ \text{etc.}$$

Defintion 5. Denote

$$I_{m,c,d}(a,b)=\Bigg(\Bigg(\exp\left(-\frac{1}{m-1}\left(\frac{\ln(cb+d)}{b^{m-1}}-\frac{\ln(ca+d)}{a^{m-1}}\right)+\right.$$

$$+\frac{1}{m-1}\left(-\frac{c}{d}\right)^{m-1}\left(\ln\frac{cb+d}{b}-\ln\frac{ca+d}{a}\right)+$$

$$+\frac{1}{m-1}\left(-\frac{c}{d}\right)^{m-1}\sum_{r=1}^{m-2}\frac{1}{r}\left(\left(-\frac{d}{cb}\right)^r-\left(-\frac{d}{ca}\right)^r\right)\Bigg)\Bigg)^{\frac{1}{b-a}}$$

if $0<a<b$ and $I_{m,c,d}(a,b)=a$ if $a=b$ the generalized identric mean, where $c,d\geq 0$, $m\geq 3$.
We start from

$$(b-a)\ln I_{m,c,d}(a,b)=\int\limits_a^b\frac{\ln(cx+d)\,dx}{x^m}=$$

$$=\left(-\frac{1}{m-1}\frac{\ln(cx+d)}{x^{m-1}}+\frac{1}{m-1}\left(-\frac{c}{d}\right)^{m-1}\ln\frac{cx+d}{x}+\right.$$

$$\left.+\frac{1}{m-1}\left(-\frac{c}{d}\right)^{m-1}\sum_{r=1}^{m-2}\frac{1}{r}\left(-\frac{d}{cx}\right)^r\right)\Big|_a^b\ \text{etc.}$$

In end of this paper we invite all mathematicians to study the properties of the previous introduced generalized identric means.

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A study of some properties of generalized groups

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1. INTRODUCTION

Generalized group is an algebraic structure which has a deep physical background in the unified gauge theory and has direct relation with isotopies.

1.1. THE IDEA OF UNIFIED THEORY

The unified theory has a direct relation with the geometry of space. It describes particles and their interactions in a quantum mechanical manner and the geometry of the space time through which they are moving. For instance, the result of experiment at very high energies in accelerators such as the Large Electron Position Collider [LEP] at CERN (the European Laboratory for Nuclear Research) in general has led to the discovery of a wealth of so called 'elementary' particles and that this variety of particles interact with each other via the weak electromagnetic and strong forces(the weak and electromagnetic having been unified). A simple application of Heisenberg's on certainty principle-the embodiment of quantum mechanics-at credibly small distances leads to the conclusion that at distance scale of around 10^{-35} metres(the plank length) there are huge fluctuation of energy that sufficiently big to create tiny black holes. Space is not empty but full of these tiny black holes fluctuations that come and go on time scale of around 10^{-35} seconds(the plank time). The vacuum is full of space-time foam. Currently, the most promising is super-string theory in which the so called 'elementary' particles are described as vibration of tiny(planck-length)closed loops of strings. In this theory the classical law of physics, such as electromagnetism and general relativity, are modified at time distances comparable to the length of the string. This notion of 'quantum space-time' is the goal of unified theory of physical forces.

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The unified theory offers a new insight into the structure, order and measures of the quantum world of the entire universe. It is in accordance with the quantum mechanics, quantum theory of space time and quantum gravity. It is in conformity with the proposition of Einsteins theory of relativity that the laws of nature and the geometry of space-time are independent. It was proposed that:
(i) There is conceptual unification and reconciliation of the theory of relativity with quantum theory and the mathematical concepts of continuum with the concept of the space-time.

(ii) The unification of all forces, field, matter and space-time into one unified force-gravity, that is pure geometry where all the constants of nature emerge form the theory itself.

(iii) The Grand unified MU-27 space-time coordinate supersedes the cartesian coordinate system which describes the world of unrelated.

(iv) The elucidation of the conceptual structure of space-time, planck constant and planck time and the planck scale often described as foamy through the primordial universal quantum.

(v) The rate at which the velocity of the galaxies increase with distance is determined by the system structure of the Grand unified theory MU-27 its principle of continuity and extensity of distance and volume criterion.

1.2 UNIFIED THEORY AND GENERALIZED GROUPS

The above notion of quantum space-time is the goal of any unified theory of the physical forces. For instance, from differential geometry, we know that metric can determine the geometry of space(Hawking and Ellis [5]). The idea of generalized groups are tools for constructing unified geometric theory. In this research work, the algebraic structure of generalized groups will be dealt with as follows.

Definition 1.1. ([9])
A generalized group G is a non-empty set admitting an operation called multiplication subject to the set of rules given below.

(i) $(xy)z = x(yz)$ for all $x, y, z \in G$.

(ii) For each $x \in G$ there exists a unique $e(x) \in G$ such that
$xe(x) = e(x)x = x$ (existence and uniqueness of identity element).
(iii) For each $x \in G$, there exists $x^{-1} \in G$ such that $xx^{-1} = x^{-1}x = e(x)$ (existence of inverse element).

Example 1.1. Let

$$G = \left\{ A = \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix}, \quad a, b \in \mathcal{R} \text{ and } b \neq 0 \right\}.$$

Then, G is a generalized group and for all $A \in G$,

$$e(A) = \begin{pmatrix} 0 & 0 \\ \frac{a}{b} & 1 \end{pmatrix} \text{ and } A^{-1} = \begin{pmatrix} 0 & 0 \\ \frac{a}{b^2} & \frac{1}{b} \end{pmatrix}$$

where $e(A)$ and A^{-1} are the identity and the inverse of matrix A respectively.

PROPERTIES OF GENERALIZED GROUPS

A Generalized group G exhibits the following properties:
(i) for each $x \in G$, there exists a unique $x^{-1} \in G$.
(ii) $e(e(x)) = e(x)$ and $e(x^{-1}) = e(x)$ where $x \in G$.
Then, $e(x)$ is a unique identity element of $x \in G$.

Definition 1.2. If $e(xy) = e(x)e(y)$, $\forall x, y \in G$, then G is called normal generalized group.

2. LITERATURE REVIEW

Mathematicians and physicists have been trying to construct a suitable unified theory, for example twistor theory, isotopies theory and so on. It was known that generalized groups are tools for constructions of a unified geometric theory, electroweak theories are essentially structured on Minkowskian axioms and gravitational theories are constructed on Riemannian axioms. Molaei [9] in his generalized groups established the uniqueness of the identity element of each element in a generalized group and where the identity element is not unique for each element, then a group is formed, some of his results are stated below.

Theorem 2.1. For each element x in a generalized group G, there exists a unique $x^{-1} \in G$.

Proof. Suppose $x \in G$, then

$$x = xe(x) = x(x^{-1}x) = x(x^{-1}e(x^{-1}))x = (e(x)e(x^{-1}))x$$

and similarly,

$$x = x(e(x^{-1}))e(x)$$

so the uniqueness of $e(x)$ shows that

$$e(x)e(x^{-1}) = e(x^{-1}) = e(x)$$

hence

$$e(x^{-1}) = e(x) \tag{1}$$

Now, let $zx = xz = e(x)$ and $yx = xy = e(x)$. Then $z(xy) = z(e(x))$, so $e(x)y = ze(x)$ which shows that $y = z$ from (1)
The next theorem shows that an abelian generalized group is a group.

Theorem 2.2. Let G be a generalized group and $xy = yx$ for all $x, y \in G$. Then G is a group.
Proof. We show that $e(x) = e(y)\ \forall\ x, y \in G$. Let $x, y \in G$ be given, then $(xy)e(y) = xy$ and $e(y)(xy) = e(y)(yx) = yx$. Thus, $e(xy) = e(y)$ and by the similar way $e(x) = e(xy)$. Hence, $e(x) = e(y)$.
M. Mehrabi, M. R. Molaei and A. Oloomi also considered the idea of generalized subgroups and homomorphisms of generalized groups. Some results on homomorphisms were proved as shown below

Theorem 2.3. A non-empty subset H of a generalized group G is a generalized subgroup of G if and only if for all $a, b \in H$, $ab^{-1} \in H$.
Proof. If H is a generalized subgroup and $a, b \in H$, then $b^{-1}, ab^{-1} \in H$.
Conversely, suppose $H \neq \emptyset$ and $a, b \in H$, then we have
$bb^{-1} = e(b) \in H, e(b)b^{-1} = b^{-1} \in H$ and $ab = a(b^{-1})^{-1}$.

Theorem 2.4. Let $\{H_i | i \in I\}$ be a family of generalized group G and $H_i \neq \emptyset$. Then H_i is a generalized subgroup of G.
Proof. If $a, b \in H$, then $a, b \in H_i, \forall\ i \in I$, thus$ab^{-1} \in H_i,\ \forall\ i \in I$. Hence $ab^{-1} \in H$ and H is a generalized subgroup of G since $a, b \in H, ab^{-1} \in H$.

Theorem 2.5. Let G be a generalized group such that $a \in G$. Then,

$$G_a = \{x \in G \ : \ e(x) = e(a)\}$$

is a generalized subgroup of G. Furthermore, G_a is a group.
Proof. For all $a, b \in G_a$ we have $(bc)e(a) = bce(c)$ and $e(a)bc = e(b)(bc) = bc$, so $e(bc) = e(a)$, we have that $e(c) = e(c^{-1})$. Hence $b, c^{-1} \in G_a$, and G_a is a generalized subgroup of G. Since the identity function is a constant function on G_a, it is also a group.

Definition 2.1. If G and H are two generalized groups and $f : g \to H$, then f is called a homomorphism if $f(ab) = f(a)f(b)$, $\forall a, b \in G$.
M. Mehrabi, M.R. Molaei and A. Oloomi [11] stated the following results on homomorphisms of generalized groups. These results are established in this work.

Theorem 2.6. Let $f \ : \ G \to H$ be a homomorphism where G and H are two distinct generalized groups. Then:
(i)] $f(e(a)) = e(f(a))$ is an identity element in H for all $a \in G$.
(ii)] $f(a^{-1}) = (f(a))^{-1}$
(iii) If K is a generalized subgroup of G, then $f(K)$ is a generalized subgroup of H.
(iv)] If G is a normal generalized group, then the set

$$\{(e(g), f(g)) \ : \ g \in G\}$$

with the product

$$(e(a), f(a))(e(b), f(b)) := (e(ab), f(ab))$$

is a generalized group denoted by $\cup f(G)$.

2.1. SOME OTHER GENERALIZED CONCEPTS

Topological Generalized Groups

Molaei [10] introduced topological generalized groups. According to him, a set G is called a topological generalized group if:
- G is a generalized group,
- G is a Hausdorff topological space and

the functions

$$m_1 \ : \ G \to G \qquad g \mapsto g^{-1}$$

and

$$m_2 \ : \ G \times G \to G \qquad (gh) \mapsto gh$$

are continuous maps.
Example 2.1. ([10]) Every non-empty Hausdorff topological space G with the operation:

$$m_2 \ : \ G \times G \to G \qquad (a, b) \mapsto a$$

is a topological generalized group.

GENERALIZED ACTIONS

Molaei [8] also introduced generalized actions to generalized groups as follows. Let G be a generalized group and let X be a set. A generalized action of G on X is a map $T \ : \ G \times X \to X$ with the properties
- $\hat{\imath}(a, \hat{\imath}(b, x)) = \hat{\imath}(ab, x)$ and
- $\hat{\imath}(e(g), x) = x \ \forall \ a, b, g \in G$ and $x \in X$.

X is otherwise called a G-set.

Theorem 2.7. Let a generalized group G on a set S, and let $f \ : \ G \to H(S)$ be a $1-1$ mapping defined by $f(g) = \phi_g$ where $H(S)$ is the set of all mapping $\phi_g \ : \ S \ \to \ S$ defined by $\phi_g(x) = g(x)$. Then $H(S)$ with the multiplication $\phi_g \phi_h = \phi_{gh}$ is a generalized group. Moreover, if G is normal then $H(S)$ is a normal generalized group.

SMOOTH GENERALIZED GROUP

Agboola [3] initiated the study of smooth generalized groups. A mapping $E_x \ : \ X \times X \to [0, 1]$ is called a fuzzy equality on X if and only if the following conditions are satisfied:
- $E_X(x, y) = 1 \Leftrightarrow x = y \ \forall \ x, y \in X$,
- $E_X(x, y) = E_X(y, x) \ \forall \ x, y \in X$,
- $\min[E_X(x, y), E_X(y, z)] \leq E_X(x, z) \ \forall \ x, y, z \in X$.

The real number $E_X(x, y)$ is called the degree of equality of x and y in X. The mapping $E_x \ : \ X \times X \to [0, 1]$ is said to be a unit fuzzy function on X if for all $x \in X$,

$$E_X(x, y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

and it is denoted by U_X that is $E_X = U_X$. U_X defined this way is indeed a fuzzy equality on X with respect to classical equality of the elements of X. Let E_X and E_Y be fuzzy equalities on X and Y respectively. The mapping $E^*_{X\times Y} \,:\, X \times Y \to [0,1]$ is said to be an induced fuzzy equality on $X \times Y$ with respect to(wrt) E_X and E_Y if for all $x, z \in X$ and for all $y, w \in Y$,

$$E^*_{X\times Y}\Big((x,y),(z,w)\Big) = E_X(x,z) \wedge E_y(y,w).$$

Let E_X and E_Y be fuzzy equalities on X and Y respectively. $f \,:\, X \to Y$ is called a fuzzy function wrt E_X and E_Y if and only if the characteristic function $\mu_f \,:\, X \times Y \to [0,1]$ of f satisfies the following conditions:

(i) for all $x \in X$, there exists $y \in Y$ such that $\mu_f(x,y) > 0$.

(ii) $\min\Big[\mu_f(x,z), \mu_f(y,w), E_X(x,y)\Big] \leq E_y(z,w)\ \forall\ x,y \in X$ and $\forall\ w,z \in Y$.
A fuzzy function $f \,:\, X \times X \to X$ wrt a fuzzy equality $E_{X\times X}$ on $X \times X$ and a fuzzy equality E_X on X is said to be a smooth binary operation on X wrt $E_{X\times X}$ and E_X if and only if

$$\min\Big[\mu_f(a,b,c), E_X(c,d)\Big] > 0 \Rightarrow \mu_f(a,b,d) > 0\ \forall\ a,b,c,d \in X.$$

A smooth binary operation $\circ$ on X wrt $E_{X\times X}$ and E_X is called a strong smooth binary operation on X if and only if

$$\min\Big[\mu_\circ(a,b,c), E_X(c,d)\Big] > 0 \Rightarrow \mu_\circ(a,b,d) > 0\ \forall\ a,b,c,d \in X.$$

Let $\circ$ be a smooth binary operation wrt a fuzzy equality $E_{X\times X}$ on $X \times X$ and a fuzzy equality E_X on X. Then

(i) $(X,\circ)$ is called a smooth semigroup if and only if the characteristic function $\mu_\circ \,:\, X \times X \times X \to [0,1]$ of $\circ$ satisfies the condition

$$\min\Big[\mu_\circ(b,c,d), \mu_\circ(a,d,m), \mu_\circ(a,b,q), \mu_\circ(q,c,w)\Big] > 0 \Rightarrow$$

$$\Rightarrow E_X(m,w) > 0\ \forall\ a,b,c,m,q,w \in X.$$

(ii) A smooth semigroup $(X,\circ)$ is a smooth monoid if and only if there exists an (two-sided) identity element $e \in X$ such that

$$\min\Big[\mu_\circ(e,a,a), \mu_\circ(a,e,a)\Big] > 0\ \forall\ a \in X.$$

(iii) A smooth monoid $(X, \circ)$ is a smooth group if and only if for each $a \in X$ there exists an (two-sided) inverse element $a^{-1} \in X$ such that

$$\min\Big[\mu_\circ(a^{-1}, a, e), \mu_\circ(a, a^{-1}, e)\Big] > 0 \ \forall \ a \in X.$$

(iv) A smooth semigroup $(X, \circ)$ is said to be abelian i.e commutative if and only if $\circ$ satisfies

$$\min\Big[\mu_\circ(a, b, m), \mu_\circ(b, a, w)\Big] > 0 \Rightarrow E_X(m, w) > 0 \ \forall \ a, b, m, w \in X.$$

Let $\circ$ be a closed binary operation on G wrt a fuzzy equality $E_{G\times G}$ on $G \times G$ and a fuzzy equality E_G on G. Then $(G, \circ)$ is said to be a smooth generalized group(SGG) if the following conditions hold:
(i)

$$\mu_\circ(y, z, a) \wedge \mu_\circ(x, a, u) \wedge \mu_\circ(x, y, b) \wedge \mu_\circ(b, z, v) > 0 \Rightarrow$$

$$\Rightarrow E_G(u, v) > 0 \ \forall \ a, b, u, v, x, y, z \in G.$$

(ii) For each $x \in G$ there exists a unique $e_x \in G$ such that

$$\mu_\circ(x, e_x, x) \wedge \mu_\circ(e_x, x, x) > 0.$$

(iii) For each $x \in G$ there exists $x^{-1} \in G$ such that

$$\mu_\circ(x, x^{-1}, e_x) \wedge \mu_\circ(x^{-1}, x, e_x) > 0.$$

It should be pointed out that if $(G, \circ)$ satisfies only (i), it is called a smooth semigroup and if only (i) and (ii) are satisfied, G is referred to as a smooth generalized monoid(SGM).

Definition 2.2. [3] Let (G, o) be a SGG wrt fuzzy equality$E_{G\times G}$ on a $G \times G$ and a fuzzy equality E_G on G. G is said to be abelian (commutative) if it additionally satisfies the condition

$$\mu_o(x, y, a) \wedge \mu_o(y, x, b) > 0 \Rightarrow E_G(a, b) > 0, \forall \ a, b, x, y \in G.$$

Definition 2.3. [3] Let (G, o) be a SGG wrt a fuzzy equality $E_{G\times G}$ on $G \times G$ and a fuzzy equality E_G on G. G is said to be normal if

$$\mu_o(x, y, a) > 0 \Rightarrow \mu_o(e_x, e_y, e_a) > 0, \forall \ a, x, y \in G$$

Γ GENERALIZED GROUP

According to Agboola [2], if G is a generalized group and Γ is a non-empty set , then Γ is said to act on G by endomorphisms if each $\alpha \in \Gamma$ defines an homomorphism of G i.e $\alpha : G \to G$ is an endomorphism with the property

$$\alpha(ab) = \alpha(a)\alpha(b) \ \forall \ a, b \in G.$$

The elements of Γ are called the operators and the pair (G, Γ) is called a Γ generalized group. Although each element of Γ defines an endomorphism of G, but different operators from Γ may define the same endomorphism of G.

Definition 2.4. Let G be a Γ-generalized group and let H be a generalized subgroup of G. H is said to be a Γ-generalized subgroup of a Γ-generalized group of G, if all members of Γ define endomorphism of H.

Definition 2.5. ([2])Let $f : G \to H$ be a generalized group homomorphism from a Γ-generalized group G into a Γ-generalized group H. f is said to be a Γ-generalized group homomorphism if

$$\alpha(f(a)) \ = \ f(\alpha(a)), \forall \ a \in G \ and \ \forall \ \alpha \in \Gamma$$

Remark 2.1. *In the special case of* $\Gamma \ = \ \emptyset$*, the notion of* Γ*-generalized group reduces naturally to the usual generalized group.*

PRODUCT STRUCTURE ON GENERALIZED GROUP

In the work of Agboola [2], if G is a generalized group and $a \in G$, then for all $n \in \mathcal{Z}$ he defined the power of a as follows:

$$a^0 = e_a$$

$$a^n = aaa \cdots a \qquad [n \text{ factors}]$$

$$a^{-n} = (a^{-1})^n$$

It is clear that each of these powers of a have a unique meaning in G and their natural extension of associative multiplicative binary operation of G.

Theorem 2.8. Let G be a finite generalized group and let $a \in G$. Then $\exists\ p \in \mathcal{Z}^+ \ni\ a^p\ =\ e_a$.
Proof. Suppose that a is any element of the finite generalized group G. Then, the infinite sequence

$$\cdots a^{-4}, a^{-3}, a^{-2}, a^{-1}, a^0 = e_a, a, a^2, a^3, a^4, \cdots$$

of power of a must contain repetitions. In this case, we can find integers m and n with $m > n \ni$, $a^m = a^n \Rightarrow a^{m-n} = a^0 = e_a$, which gives the required result if $p = m - n$

Definition 2.6. ([2]) Let G be a generalized group and let $a \in G$. The order of a in G denoted by $o(a)$ is the least positive integer p?????$a = e_a$.

Theorem 2.9. Let G be a generalized group. Then
(i) $o(e_a) = 1$ and (ii) $o(a^{-1}) = o(a), \forall\ a \in G$.
Proof.
(i) Suppose that $o(e_a) = k$. Then, $e_a^k = e_{ea} = e_a$, thus $k = 1$.
(ii) Suppose that $o(a)\ = m$. Then $a^m = e_a$, also
$(a^{-1})^m = (a^m)^{-1} = e_{a^{-1}} = e_a$. This therefore means that $o(a^{-1}) \leq o(a)$. And if a is replaced by a^{-1} we have $o(a) \leq o(a^{-1})$ and hence $o(a^{-1}) = o(a)$ as required.

Definition 2.7. Let G_i where $i = 1, 2, 3, ..., n$ be genertalized groups. the direct product of the G, denoted by $\prod\limits_{n}^{i=1} G_i$ is defined by

$$\prod_{n}^{i=1} G_i = \{(g_1, g_2, ..., g_n), g \in G_i, i = 1, 2, ..., n\}.$$

He got the result that follows.

Theorem 2.10. Let $\prod_{i=1}^{n} G_i$ be the direct product of generalized groups G_i and let $(g_1, g_2, \cdots, g_n)$ and $(h_1, h_2, \cdots, h_n)$ be in $\prod_{i=1}^{n} G_i$. Then $\left(\prod_{i=1}^{n} G_i, \otimes\right)$ is also a generalized group where $\otimes$ is a multiplicative binary operation defined by

$$(g_1, g_2, \cdots, g_n) \otimes (h_1, h_2, \cdots, h_n) = (g_1h_1, g_2h_2, \cdots, g_nh_n)$$

$$\forall\ g_i, h_i \in G_i,\ i = 1, 2, \cdots n.$$

Proof. It is clear that $\otimes$ is closed and associative over $\Pi_{i=1}^{n}G_i$. Now for each $g_i \in G_i, e_{g_i} \in G_i$, is the unique identity *wrt* g_i and so

$$\begin{aligned}(g_1, g_2, \cdots, g_n) \otimes (e_{g_1}, e_{g_2}, \cdots, e_{gn}) &= (e_{g_1}, e_{g_2}, \cdots, e_{gn}) \otimes (g_1, g_2, \cdots, g_n) \\ &= (g_1, g_2, \cdots, g_n)\end{aligned}$$

This shows that $(g_1, g_2, \cdots, g_n) \in \Pi_{i=1}^{n}G_i$ is the unique identity wrt $(g_1, g_2, \cdots, g_n) \in \Pi_{i=1}^{n}G_i$.
Lastly, for each $(g_1, g_2, \cdots, g_n) \in \Pi_{i=1}^{n}G_i$, we have

$$\begin{aligned}(g_1, g_2, \cdots, g_n) \otimes (g_1^{-1}, g_2^{-1}, \cdots, g_n^{-1}) &= (g_1g_1^{-1}, g_2g_2^{-1}, \cdots, g_ng_n^{-1}) \\ &= (e_{g_1}, e_{g_2}, \cdots, e_{gn})\end{aligned}$$

Using a similar argument we have
$(g_1^{-1}, g_2^{-1}, \cdots, g_n^{-1}) \otimes (g_1, g_2, \cdots, g_n) = (e_{g_1}, e_{g_2}, \cdots, e_{gn})$. This shows that $(g_1^{-1}, g_2^{-1}, \cdots, g_n^{-1})$ ia a unique inverse of $(g_1, g_2, \cdots, g_n) \in \Pi_{i=1}^{n}G_i$.
Accordingly, $(\Pi_{i=1}^{n}G_i, \otimes)$ is a generalized group.

Theorem 2.11. Let A_i be a generalized groups G_i where $i = 1, 2, \cdots, n$. Then $\Pi_{i=1}^{n}A_i$ is a generalized subgroup of group $\Pi_{i=1}^{n}G_i$

NORMAL GENERALIZED SUBGROUPS: FACTOR GENERALIZED GROUPS

Mehrabi, Molaei and Oloomi [11] studied normal generalized subgroups and introduced them into factor generalized groups.

Definition 2.8. A generalized subgroup N of a generalized group G is called a generalized normal subgroup of G if there exists a generalized group E and an homomorphism $f : G \to E$ such that $\forall a \in G$, $N_a = \phi$ or $N_a = Kerf_a$ where $N_a = N \cap G_a$, $f_a = f|_{G_a}$ and $Kerf_a = \{x \in G_a : f(x) = f(e(a))\}$. They established the result below.

Theorem 2.12. Let N be a generalized normal subgroup of the normal generalized group G. Then the set $G/N = \bigcup G_a/N_a$ is a normal generalized group with the multiplication

$$G/N \times G/N \to G/N \qquad (xN_a, yN_b) \mapsto (xy)N_{(ab)}.$$

Proof. In the definition of G/N, if $G_a = G_b$, then, $N_a = N_b$, so $G_a/N_a = G_b/N_b$. Thus the union of G/N can be taken over the distinct G_a. Now, we shall show that the multiplication is well defined.
Let $f : G \to E$ be a homomorphism that corresponds to N. Then $N_a = \ker f_a$ where $a \in G$ and $N_a \neq \emptyset$.
If $x_1N_a = x_2N_a$ and $y_1N_b = y_2N_b$, then $\exists n \in N_a$ and $n_b \in N_b \ni x_1 = x_2n_a$ and $y_1 = y_2n_b$. Because G is normal generalized group, $e(x_iy_i) = e(x_i)e(y_i) = e(a)e(b) = e(ab)$, for $i = 1, 2$. So $x_1, y_1, x_2, y_2 \in G_{ab}$ and we have

$$\begin{aligned} f_{ab}(x_1, y_1) &= f(x_2n_ay_2n_b) \\ &= f(x_2)f(n_a)f(y_2)f(n_b) \\ &= f(x_2)f(e(a))f(y_2)f(e(b)) \\ &= f(x_2e(a))f(y_2e(a)) \\ &= f(x_2)f(y_2) \\ &= f_{ab}(x_2y_2) \end{aligned}$$

Hence $x_1y_1 \ker f_{ab} = x_2y_2 \ker f_{ab}$ or $x_1y_1N_{ab} = x_2y_2N_{ab}$.
We now show that $e(xN_a) = e(x)N_a$. Let $(xN_a)(yN_b) = (yN_b)(xN_a) = xN_a$. Thus, $e(b)e(a) = e(ba) = e(a) = e(ab) = e(a)e(b)$. So $e(b) = e(e(a)) = e(a)$ and $N_a = N_b$. But G_a/N_b is a group with unique identity $e(x)N_a = e(a)N_a$, so $yN_b = yN_b = e(x)N_a$. The generalized group G/N is called the factor generalized group.

2.2. STATEMENT OF PROBLEMS

Since the generalized group is a new algebraic concept, in this research work generalized group is be studied so as to get analogous results just as they exist in classical group theory and the results would be extended to isomorphism theorems.

2.3. OBJECTIVES

The objectives of this study are as follows.
1. To investigate if some results that are true in classical group theory are as well true in generalized groups.
2. To find a way of constructing a Bol structure(i.e Bol loop or Bol

quasigroup or Bol groupoid) using a non-abelian generalized group.

·	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Table 1. Table of a group called the Klein 1-group, with a unique identity

3. ON CLASSICAL GROUP THEORY

Defintion 3.1. A group is a set G together with binary operation '×'such that $G \times G \to G$, satisfying the following conditions
(a) $a \cdot b \in G,\ \forall a, b \in G.$ (Closure)
(b)$a(b \cdot c) = (a \cdot b)c, \forall, a, b, c \in G.$ (Associativity)
(c) there exists $e \in G$ such that $a \cdot e = e \cdot a = a,\ \forall\ a \in G.$ (Existence of identity element)
(d) For each $a \in G,\ \exists\ b \in G$ such that $a \cdot b = b \cdot a = e.$ (Existence of inverse element)
Example 3.1.

Theorem 3.1. Let a, b be two elements of a group g. Then
(a) $(ab)^{-1} = b^{-1}a^{-1}$. (b) $(a^{-1})^{-1} = a$.
Proof.
(i) Since for each a in G there is $a^{-1} \in G$ with the property that $aa^{-1} = a^{-1}a = e$ then $(ab)^{-1}(ab) = e$ if the both sides of this is multiplied on the right by $b^{-1}a^{-1}$, we obtain

$$\begin{aligned} eb^{-1}a^{-1} &= [(ab)^{-1}(ab)]b^{-1}a^{-1} \\ b^{-1}a^{-1} &= (ab)^{-1}[(ab)b^{-1}a^{-1}] \\ &= (ab)^{-1}a(bb^{-1})a \\ &= (ab)^{-1}aea^{-1} \\ &= (ab)^{-1}aa^{-1} \\ &= (ab)^{-1}e \\ &= (ab)^{-1} \end{aligned}$$

(ii) Also, $(a^{-1})^{-1}a^{-1} = e$, if we multiply both sides of this expression on the right by a, we obtain

$$\begin{aligned} ea &= [(a^{-1})^{-1}a^{-1}]a \\ a &= (a^{-1})^{-1}(a^{-1}a) \\ &= (a^{-1})^{-1}e \\ &= (a^{-1})^{-1} \end{aligned}$$

3.1. PROPERTIES OF GROUPS

The followings are the elementary properties of group.

Theorem 3.2. Let G be a group, then
(1) The identity element $e \in G$ is unique.
(i) The inverse $a^{-1} of a \in G$ is unique.
(ii) $(a^{-1})^{-1} = a, \ \forall\ a \in G.$ and $(ab)^{-1} = b^{-1}a^{-1}, \ \forall a, b \in G.$
(iii)If $a, b, c \in G$, then $ab = ac \Rightarrow b = c$ and $ac = bc \Rightarrow a = c, \ \forall\ a, b, c \in G.$

3.2. SUBGROUPS, COSET, NORMAL SUBGROUPS AND QUOTIENT GROUPS

Defintion 3.2. A subset H of a group g is called a subgroup of g if H is also a group under the same binary operation as that of G. When H is a subgroup of g, we write $H \leq G$.

Theorem 3.3. Let H be a subset of a group G. The following are equivalent.
(i) H is a subgroup of G. (ii) For all $a, b \in H, \ ab^{-1} \in H.$ (iii) For all $a, b \in H, \ ab \in H$ and $a^{-1} \in H.$
Proof.
(i)$\Rightarrow$ (ii) By the definition of a group, H is a group and given $b \in H$ its inverse $b^{-1} \in H$, and so by closure property $ab^{-1} \in H, \forall\ a \in H.$
(ii)$\Rightarrow$ (iii) Let $a \in H$ then by (b), $aa^{-1} = e \in H$, e is the identity of element of G. If b is any element of H, then by (b) again, $e \cdot b^{-1} = b^{-1} \in H.$ So the first part of (c) is satisfied. Also, given $a, b \in H$, we have $b^{-1} \in H$ and $a(b^{-1} \in H$ by (b), i.e $ab \in H$, and (c) is satisfied.
(iii)$\Rightarrow$ (i) Suppose (c) is satisfied. Then it is only necessary to show that associative law holds and that the identity exists in H. Now for all $a, b, c \in H$, $(ab)c, a(bc) \in H$, and by repeated application of the first part of (c), $(ab)c = a(bc)$ in H, since they are equal in G. Thus, the associative law

holds in H. Given $a \in H$, then by the second part of (c), $a^{-1} \in H$ and by the first part, $aa^{-1} = e \in H$, where e is the identity element of G. e obviously is the identity element of H.

Theorem 3.4. Let $\{H_\alpha\}_{\alpha\in\Omega}$ be a family of subgroup of a group G. Then the intersection $\bigcup_{\alpha\in\Omega} H_\alpha$ is a subgroup of group G.
Proof. Let $a, b \in \bigcap_{\alpha\in\Omega} H_\alpha$. Then $a, b \in H_\alpha$ for each $\alpha \in \Omega$. But since $H_\alpha, \alpha \in \Omega$ is subgroup(given). It follows that (i) $a, b \in H_\alpha, \alpha \in \Omega$ and (ii) $a^{-1} \in H_\alpha, \alpha \in \Omega$. (i) and $(ii) \Rightarrow a, b \in \bigcap_{\alpha\in\Omega} H_\alpha a^{-1} \in \bigcap_{\alpha\in\Omega} H_\alpha$. Thus $\bigcap_{\alpha\in\Omega} H_\alpha$ is a subgroup of G.

3.3. COSETS

Let $H \leq G$, aH is called a left coset of H determined by $a \in G$ and Ha is a right coset of H determined by $a \in G$.

Definition 3.3. The set of all right cosets of H in G is called the right quotient set of G by H and denoted by $HG\backslash G$. The left quotient set of G by H is defined analogously and denoted by G/H.

Definition 3.4. Let G be a group, and H a subgroup of G. H is said to be normal if $Ha = aH,\ \forall\ a \in G$. This is written as $H \lhd G$.

Theorem 3.5. Let G be a group and H a subgroup of G. The following are equivalent:
(i) H is normal in G.
(ii) $a^{-1}Ha = H,\ \forall\ a \in G$.
(iii) for any $h \in H,\ a \in G,\ a^{-1}ha \in H$.
Proof.
(i)$\Rightarrow$ (ii) Suppose $Ha = aH$, pre-multiplying both sides by a^{-1}, we obtain $a^{-1}H = H,\ \forall\ a \in G$.
(ii)$\Rightarrow$ (iii) Trivially, given any $h \in H, a^{-1}ha \in G$ and $a^{-1}ha \in a^{-1}Ha = H$.
(iii)$\Rightarrow$ (i) Since (c) holds, $a^{-1}ha = h$, say for some $h \in H$ i.e $ha = ah$. Now if $x \in Ha$, then $x = ha$, say for some $h \in H$. But $ha = ah$, for some $h \in H$, so $x = ah \in aH$. That is $Ha \subseteq aH$. By a similar argument, one can show that $aH \subseteq Ha$. Thus $aH = Ha$.

Theorem 3.6. Let G be a group and H is normal subgroup of G. Then G/H is a group called the quotient group G by H.

Proof. It is necessary to check the axioms of a group on G/H. Let $a, b, c \in G$ and $aH, bH, cH \in G/H$. Then $aH(bH \cdot cH) = (aH \cdot bH) \cdot cH$.
Closure and Associative law
If $aH \cdot eH = aH$, since e is the identity of G, eH is the identity element of G/H. Thus $eH \cdot aH = eaH = aH$.
Existence of identity element
$aH \cdot a^{-1}H = aa^{-1}H = eH$, which implies that given any $aH \in G/H$ its inverse is given by $a^{-1}H$, since $a^{-1}aH = eH = H$.
Existence of inverse element

3.4. HOMOMORPHISM AND ISOMORPHISM THEOREMS

Definition 3.5. Let $(G, \cdot)$ and $(H, \cdot)$ be two groups. A homomorphism from G to H is a mapping $f : G \to H, \ni f(a \cdot b) = f(a) \cdot f(b)$.

Definition 3.6. Let G and H be two groups. Then, the function $f : G \to H$ is said to be $1 - 1$ or mono or injective if $f(a) = f(b) \Rightarrow a = b$ or
$f(a) \neq f(b) \Rightarrow a \neq b$

Theorem 3.7. For any set S of groups, $\cong$ is an equivalent relation on S.
Proof.
(i) $\forall G \in S$, $G \cong G$, because ι_G is an isomorphism from G to G, which establishes the reflexive law.
(ii) For $G_1, G_2 \in S$, $G_1 \cong G_2 \Rightarrow, \exists$ an isomorphism $f : G_1 \to G_2$. Since f is a bijection, $\exists$ an isomorphism, $f^{-1} : G_2 \to G_1, \Rightarrow G_2 \cong G_1$. Thus the symmetric law holds.
(iii) For $G_1, G_2, G_3 \in S, G_1 \cong G_2$ and $G_2 \cong G_3 \Rightarrow, \exists$ isomorphisms $f_1 : G_1 \to G_2$ and $f_2 : G_2 \to G_3$ so also , $\exists$ an isomorphism $(f_2 \cdot f_1) : G_1 \to G_3, \Rightarrow G_1 \cong G_3$, which finally shows the transitive law, and the theorem is proved.

Definition 3.7. Let G and H be two groups. Then, the function $f : G \to H$ is said to be 1 - 1 or mono or onjective if $f(a) = f(b) \to a = b$ or
$f(a) \neq f(b) \to a \neq b$.

Definition 3.8. A mapping $f : G \to H$ is said to be onto or surjective or epi if the range of $f = H$, i.e given $d \in H$, $\exists a \in G \ni, f(a) = d$.

Definition 3.9. A homomorphism f which maps G onto itself is called an endomorphism. A bijective endomorphism is called an automorphism.

Definition 3.10. Let f be a homomorphism from a group G to a group H. The kernel of f denoted by $\ker f = \{a \in G : f(a) = e_H\}$.

Theorem 3.8. Let f be a homomorphism from a group G to a group H. Then
(i) $\ker f$ is a subgroup of G.
(ii) $\ker f$ is normal in $G(\ker f \lhd G)$.
(iii) f is a monomorphism if and only if $\ker f = \{e_G\}$.
Proof.
(i) For $a, b \in \ker f$,

$$\begin{aligned} f(ab^{-1}) &= f(a)f(b^{-1}) \text{ Since f is a homomorphism} \\ &= e_H \cdot e_H \text{ by definition} \\ &= e_H \end{aligned}$$

So $ab^{-1} \in \ker f$, so $\ker f$ is a subgroup of G.
(ii) For any $k \in \ker f, b \in G$. Thus

$$\begin{aligned} f(b^{-1}kb) &= f(b^{-1})f(k)f(b) \\ &= f(b^{-1})e_H f(b) \\ &= f(b^{-1})f(b) \\ &= e_H \end{aligned}$$

Hence, $b^{-1}kb \in \ker f$, so $\ker f \lhd G$.
(iii) Suppose that $\ker f = e_H$ and $f(b) = f(c)$, then $(f(c))^{-1}f(b) = e_H \Rightarrow f(c^{-1}b) = e_H \Rightarrow c^{-1}b \in \ker f = \{e_H\}$. Hence $b = c$ and so f is a monomorphism. Suppose that f is a monomorphism and $k \in \ker f$, $k \neq e_G$. Then, $f(k) = e_H = f(e_G)$, which leads to contradiction since f is a monomorphism. So $k = e_G$.

Theorem 3.9. Let $f : G \to H$ be a group homomorphism with kernel K. Then $f(G)$ is a subgroup of H, and there is a canonical isomorphism of $f(G)$ with G/K. If f is surjective then $G/K \cong H$.
Proof. $a, b \in G$ if and only if $f(a), f(b) \in f(G)$. Now

$$f(a) \cdot f(b^{-1}) = f(ab^{-1}) \in f(G)$$

showing that $ab^{-1} \in G$, thus $f(G)$ is a subgroup of H. For the second part, define

$$\phi : G/K \rightarrow \phi(G) \text{ by } \phi(gK) = \theta(g).$$

For ϕ to be an isomorphism, we need to show that
(i) ϕ is well defined, i.e. the definition is independent of the choice of the representation $g \in G$,
(ii) ϕ is a homomorphism,
(iii) ϕ is injective,
(iv) ϕ is surjective.
To show (i) - (iv), we have the following:
(i) To prove that ϕ is well defined, let $g_1 \in gK$, then we show that $\phi(g_1K) = \phi(gK)$. Now

$$g_1 \in gK \Rightarrow \exists k \in K \ni g_1 = gk \Rightarrow g^{-1}g_1 = k.$$

Then

$$e' = \theta(k) = \theta(g^{-1}g_1) = \theta(g^{-1})\theta(g_1) \Rightarrow \theta(g_1) = \theta(g) \Rightarrow \phi(g_1K) = \phi(gK)$$

where $e' = \theta(e)$.
(ii) Let $g, g_1 \in G$, then

$$\phi[(gK)(g_1K)] = \phi[(gg_1)K] = \theta(gg_1) = \theta(g)\theta(g_1) = \phi(gK)\phi(g_1K).$$

Thus, ϕ is a homomorphism.
(iii) Suppose $\phi(g_1K) = \phi(gK)$, then

$$\theta(g) = \theta(g_1) \Rightarrow \theta(gg_1) = \theta(e) = e' \Rightarrow gg_1^{-1} \in \ker\theta = K \Rightarrow g_1 \in gK \Rightarrow g_1K = gK.$$

Thus, ϕ is injective.
(iv) The surjection follows from the definition thus completing proof of the second part.
For the last part, θ is onto hence $\theta(G) = H$ and the proof is complete.

Theorem 3.10. Any two infinite cyclic group of the same order are isomorphic to each other.
Proof Let G and H be two infinite cyclic groups of order k each. Let g be a generator of G and h be a generator of H. Then both g and h have order k.

Moreover, $1, g, g^2 \cdots, g^{k-1}$ is a complete repetition free list of the elements of G and $1, h, h^2, \cdots, h^{k-1}$ is a free list of the elements of H.
An obvious bijection from G to H is the θ specified by

$$\theta(g^n) = h^n,\ n = 0, 1, 2, \cdots, k-1. \tag{2}$$

We shall prove that θ is an isomorphism. First, to prove that (2) holds for all integral values of n(not just in the range 0 to $k-1$). For this purpose, let n be an arbitrary integer and let r be its principal remainder on division by K. $g^n = g^r$ and $h^n = h^r$. Hence,

$$\theta(g^n) = \theta(g^r) = h^r = h^n \text{ by definition of } \theta,\ \text{ since } 0 \leq r \leq k-1.$$

Now let x, y be any arbitrary elements of G, since $G = [g]$, $x = g^s$ and $y = g^t$, for some s, t, and hence

$$\begin{aligned} \theta(xy) &= \theta(g^s \cdot g^t) \\ &= \theta(g^{s+t}) \\ &= h^{s+t} \\ &= h^s \cdot h^t \\ &= \theta(g^s)\theta(g^t) \\ &= \theta(x)\theta(y) \end{aligned}$$

since x are arbitrary and θ is bijective. It follows that θ is an isomorphism from $G \to H$. Hence, $G \cong H$.

Theorem 3.11. If G is the internal direct product of subgroup N and H, then $G \cong N \times H$.
Proof. Let $g \in G$. Then $g = n_1h_1$ for some $n_1 \in H, h_1 \in H$, suppose we may also $g = n_2h_2$ for some $n_2 \in N, h_2 \in H$. Then $n_1h_1 = n_2h_2$ so that $n_1^{-1}n_2 = h_2h_1^{-1} \in N \bigcap H$. Therefore, $n_1 = n_2$ and $h_1 = h_2$ so that the factorization $g = n_1h_1$ is unique.
Define $f : G \to N \times H$ by $f(g) = (n_1, h_1)$ where $g = n_1h_1$. This function is well defined the previous paragraph, which also shows that f is one-to-one correspondence. It remains to check that f is a group homomorphism. Suppose that $g_1 = n_1h_1$ and $g_2 = n_2h_2$. Then $g_1g_2 = n_1h_1n_2h_2$. We claim that $h_2n_1 = n_1h_2, \forall\ n_1 \in N, h_2 \in H$. Indeed, $(h_2n_1h_2^{-1})n_1^{-1} \in N$, since N is normal in G and $h_2(n_1h_2^{-1})n_1^{-1}) \in H$, since H is also normal in G. But $N \bigcap H = \{e\}$, so $h_2n_1h_2^{-1}n_1^{-1} \in N \bigcap H = \{e\}$. Thus, $h_2n_1 = n_1h_2$.
Therefore,

$$f(g_1g_2) = f(n_1h_1n_2h_2) = f(n_1n_2h_1h_2) = (n_1n_2, h_1h_2) =$$

$$= (n_1, h_1)(n_2, h_2) = f(g_1)f(g_2),$$

so that f is a group homomorphism and hence, a group isomorphism since it is one-to-one correspondence.

Theorem 3.12. Let G be a group and N_1, N_1 be two normal subgroups of G such that $N_1 \bigcap N_2 = \{e\}$, then $N_1N_2 \cong N_1 \times N_2$

Proof. Define a mapping $\phi : N_1N_2 \to N_1 \times N_2$ by $\phi(a_1, a_2) = (a_1, a_2)$. We need to show that ϕ is well defined. Suppose that $a_1a_2 = b_1b_2$, where $a_1, b_1 \in N_1$ and $a_2, b_2 \in N_2$. Then, $a_2b_2^{-1} = a_1^{-1}b_1 \in N_1 \bigcap N_2$, since $N_1 \bigcap N_2 = e$, we have $a_2b_2^{-1} = e = a_1^{-1}b_1$ i.e. $a_1 = b_1, a_2 = b_2$, so $(a_1, b_1) = (a_2, b_2)$.

Now suppose $a_1, a_2, b_1, b_2 \in N_1N_2$, to prove that ϕ is a homomorphism, we need to show that

$$\phi(a_1, a_2, b_1, b_2) = \phi(a_1, a_2)\phi(b_1, b_2) = (a_1, a_2, b_1, b_2).$$

This will be so if and only if $a_2b_1 = b_1a_2$, i.e if and only if $(b_1, a_2)^{-1} \cdot a_2b_1 = e \Rightarrow$ $(a_2^{-1}b_1^{-1} \cdot a_2b_1 = e$. But $b_1^{-1}N_2b_1 = N_2$. So $b_1^{-1}a_2b_1 \in N_2$. Hence $a_2^{-1}b_1^{-1}a_2b_1 \in N_1$. Thus $a_2^{-1}b_1^{-1}a_2b_1 \in N_1 \bigcap N_2 = \{e\}$, i.e $a_2^{-1}b_1^{-1}a_2b_1$ as required. So ϕ is a homomorphism.

ϕ is mono since $(a_1, a_2) = (b_1, b_2) \Rightarrow a_1 = b_1, a_2 = b_2$. Also ϕ is an epimorphism since given $(a_1, a_2) \in N_1 \times N_2$, $\exists a_1, a_2 \in N_1N_2 \ni \phi(a_1, a_2) = (a_1, a_2)$. Therefore, ϕ is an isomorphism.

4. RESULTS ON GENERALIZED GROUPS AND HOMOMORPHISMS

Results

Theorem 4.1. Let G be a generalized group. For all $a \in G$, $(a^{-1})^{-1} = a$.

Proof. $(a^{-1})^{-1}a^{-1} = e(a^{-1}) = e(a)$. Post multiplying by a, we obtain

$$[(a^{-1})^{-1}a^{-1}]a = e(a)a. \tag{3}$$

From the L. H. S.,

$$(a^{-1})^{-1}(a^{-1}a) = (a^{-1})^{-1}e(a) = (a^{-1})^{-1}e(a^{-1}) =$$

$$= (a^{-1})^{-1}e((a^{-1})^{-1}) = (a^{-1})^{-1}. \tag{4}$$

Hence from (3) and (4), $(a^{-1})^{-1} = a$.

Remark 4.1. *The result obtained in Theorem 4.1 are possible through the application of definition 1.2 which defines a normal generalized group as follows. If $e(xy) = e(x)e(y), \forall\ x, y \in G$. Then G is called a normal generalized group. The results thus obtained are similar to those in [?]*

Definition 4.1. Let H be a non-empty subset of a generalized group G such that it is generalized group with the multiplication inherited from G. Then H is called a generalized subgroup of G denoted by $H \leq G$.

Theorem 4.2. Let G be a generalized group in which the left cancellation law holds. G is a idempotent generalized group if and only if $e(a)b^{-1} = b^{-1}e(a)\ \forall\ a, b \in G$.
Proof. $e(a)b^{-1} = b^{-1}e(a) \Leftrightarrow (ae(a))b^{-1} = ab^{-1}e(a) \Leftrightarrow ab^{-1} = ab^{-1}e(a) \Leftrightarrow e(a) = e(ab^{-1}) \Leftrightarrow ab^{-1} = a \Leftrightarrow ab^{-1}b = ab \Leftrightarrow ae(b) = ab \Leftrightarrow a^{-1}ae(b) = a^{-1}ab \Leftrightarrow e(a)e(b) = e(a)b \Leftrightarrow e(b) = b \Leftrightarrow b = bb$.

Theorem 4.3. Let G be a normal generalized group in which $e(a)b^{-1} = b^{-1}e(a)\ \forall\ a, b \in G$. Then, $(ab)^{-1} = b^{-1}a^{-1}\ \forall\ a, b \in G$.
Proof. Since $(ab)^{-1}(ab) = e(ab)$, then by multiplying both sides of the equation on the right by $b^{-1}a^{-1}$ we obtain

$$[(ab)^{-1}ab]b^{-1}a^{-1} = e(ab)b^{-1}a^{-1}. \tag{5}$$

So,

$$[(ab)^{-1}ab]b^{-1}a^{-1} = (ab)^{-1}a(bb^{-1})a^{-1} = (ab)^{-1}a(e(b)a^{-1}) = (ab)^{-1}(aa^{-1})e(b) =$$

$$= (ab)^{-1}(e(a)e(b)) = (ab)^{-1}e(ab) = (ab)^{-1}e((ab)^{-1}) = (ab)^{-1} \tag{6}$$

Using (5) and (6), we get
$[(ab)^{-1}ab]b^{-1}a^{-1} = (ab)^{-1} \Rightarrow e(ab)(b^{-1}a^{-1}) = (ab)^{-1} \Rightarrow (ab)^{-1} = b^{-1}a^{-1}$.

Theorem 4.4. Let H be a non-empty subset of a generalized group G. The following are equivalent.
(i) H is a generalized subgroup of G.

(ii) For $a, b \in H$, $ab^{-1} \in H$.
(iii) For $a, b \in H$, $ab \in H$ and for any $a \in H$, $a^{-1} \in H$.
Proof.
(i) $\Rightarrow$ (ii) If H is a generalized subgroup of G and $b \in G$, then $b^{-1} \in H$. So by closure property, $ab^{-1} \in H \ \forall \ a \in H$.
(ii) $\Rightarrow$ (iii) If $H \neq \phi$, and $a, b \in H$, then we have $bb^{-1} = e(b) \in H$, $e(b)b^{-1} = b^{-1} \in H$ and $ab = a(b^{-1})^{-1} \in H$ i.e $ab \in H$.
(iii) $\Rightarrow$ (i) $H \subseteq G$ so H is associative since G is associative. Obviously, for any $a \in H$, $a^{-1} \in H$. Let $a \in H$, then $a^{-1} \in H$. So, $aa^{-1} = a^{-1}a = e(a) \in H$. Thus, H is a generalized subgroup of G.

Remark 4.2. *In the above Theorem, H is a generalized subgroup of generalized group G as in Theorem[?] where H is a subgroup of G. The axioms of generalized group G holds.*

Theorem 4.5. Let $a \in G$ and $f \ : \ G \to H$ be an homomorphism. If $Ker \ f$ at a is denoted by

$$Ker \ f_a = \{x \in G \ : \ f(x) = f(e(a)).$$

Then,
(i) $Ker \ f_a \lhd G$.
(ii) f is a monomorphism if and only if $Ker \ f_a = \{e(a) \ : \ \forall \ a \in G\}$.
Proof.
(i) It is necessary to show that $Ker \ f_a \leq G$. Let $x, y \in Ker \ f_a \leq G$, then $f(xy^{-1}) = f(x)f(y^{-1}) = f(e(a))(f(e(a)))^{-1} = f(e(a))f(e(a)^{-1}) = f(e(a))f(e(a)) = f(e(a))$. So, $xy^{-1} \in Ker \ f_a$. Thus, $Ker \ f_a \leq G$. To show that $Ker \ f_a \lhd G$, since $y \in Ker \ f_a$, then by the definition of $Ker \ f_a$, $f(xyx^{-1}) = f(x)f(y)f(x^{-1}) = f(e(a))f(e(a))f(e(a))^{-1} = f(e(a))f(e(a))f(e(a)) = f(e(a)) \Rightarrow xyx^{-1}Ker \ f_a$. So, $Ker \ f_a \lhd G$.
(ii) $f \ : \ G \to H$. Let $Ker \ f_a = \{e(a) \ : \ \forall \ a \in G\}$ and $f(x) = f(y)$, this implies that $f(x)f(y)^{-1} = f(y)f(y)^{-1} \Rightarrow f(xy^{-1}) = e(f(y)) = f(e(y) \Rightarrow xy^{-1} \in Ker \ f_y \Rightarrow$

$$xy^{-1} = e(y) \tag{7}$$

and $f(x)f(y)^{-1} = f(x)f(x)^{-1} \Rightarrow f(xy^{-1}) = e(f(x)) = f(e(x) \Rightarrow xy^{-1} \in Ker \ f_x \Rightarrow$

$$xy^{-1} = e(x). \tag{8}$$

Using (7) and (8), $xy^{-1} = e(y) = e(x) \Leftrightarrow x = y$. So, f is a monomorphism. Conversely, if f is mono, then $f(y) = f(x) \Rightarrow y = x$. Let $k \in Ker\ f_a\ \forall\ a \in G$. Then, $f(k) = f(e(a)) \Rightarrow k = e(a)$. So, $Ker\ f_a = \{e(a)\ :\ \forall\ a \in G\}$.

Remark 4.3. *The results obtained here is similar to those obtained in the classical group theory in Theorem 9.8.*

Theorem 4.6. Let G be a generalized group and H a generalized subgroup of G. Then G/H is a generalized group called the quotient or factor generalized group of G by H.

Proof. It is necessary to check the axioms of generalized group on G/H.

Associativity. Let $a, b, c \in G$ and $aH, bH, cH \in G/H$. Then $aH(bH \cdot cH) = (aH \cdot bH)cH$, so associativity law holds.

Identity. If $e(a)$ is the identity element for each $a \in G$, then $e(a)H$ is the identity element of aH in G/H since $e(a)H \cdot aH = e(a) \cdot aH = aH \cdot e(a) = aH$. Therefore identity element exists and is unique for each elements aH in G/H.

Inverse. $(aH)(a^{-1}H) = (aa^{-1})H = e(a)H = (a^{-1}a)H = (a^{-1}H)(aH)$ shows that $a^{-1}H$ is the inverse of aH in G/H.

So the axioms of generalized group are satisfied in G/H.

Remark 4.4. *This result is a particular case of Theorem 3.6 in classical group theory.*

Theorem 4.7. Let G and H be two generalized groups. The direct product of G and H denoted by

$$G \times H = \{(g, h)\ :\ g \in G \text{ and } h \in H\}$$

is a generalized group under the binary operation $\circ$ such that

$$(g_1, h_1) \circ (g_2, h_2) = (g_1 g_2, h_1 h_2).$$

Proof. This is achieved by investigating the axioms of generalized group for the pair $(G \times H, \circ)$.

Theorem 4.8. Let G be a generalized group with two abelian generalized subgroups N and H of G such $G = NH$. If $N \subseteq COM(H)$ or $H \subseteq COM(N)$ where $COM(N)$ and $COM(H)$ represent the commutators of N and H respectively, then $G \cong N \times H$.

Proof. Let $a \in G$. Then $a = nh$ for some $n \in N$ and $h \in H$. Also, let $a = n_1h_1$ for some $n_1 \in N$ and $h_1 \in H$. Then $nh = n_1h_1$ so that $e(nh) = e(n_1h_1)$, therefore $n = n_1$ and $h = h_1$. So that $a = nh$ is unique. Define $f : G \to H$ by $f(a) = (n, h)$ where $a = nh$. This function is well defined in the previous paragraph which also shows that f is one-one correspondence. It remains to check that f is a group homomorphism. Suppose that $a = nh$ and $b = n_1h_1$, then $ab = nhn_1h_1$ and $hn_1 = n_1h$. Therefore,
$f(ab) = f(nhn_1h_1) = f(nn_1hh_1) = (nn_1, hh_1) = (n, h)(n_1, h_1) = f(a)f(b)$.
So, f is a group homomorphism. Hence a group isomorphism since it is a bijection.

4.4. CONSTRUCTION OF A BOL QUASIGROUP WITH ONE SIDED IDENTITY USING A GENERALIZED GROUP

Theorem 4.9. Let H be a subgroup of a non-abelian generalized group G and let $A = H \times G$. For $(h_1, g_1), (h_2, g_2) \in A$, define

$$(h_1, g_1) \circ (h_2, g_2) = (h_1h_2, h_2g_1h_2^{-1}g_2)$$

then $(A, \circ)$ is a Bol groupoid.
Proof. Let $x, y, z \in A$. By checking, it is true that $x \circ (y \circ z) \neq (x \circ y) \circ z$. So, $(A, \circ)$ is non-associative. H is a quasigroup and a loop(groups are quasigroups and loops) but G is neither a quasigroup nor a loop(generalized groups are neither quasigroups nor a loops) so A is neither a quasigroup nor a loop but is a groupoid because H and G are groupoids.
Let us now verify the Bol identity:

$$((x \circ y) \circ z) \circ y = x \circ ((y \circ z) \circ y)$$

$$\text{L. H. S.} \qquad = ((x \circ y) \circ z) \circ y = (h_1h_2h_3h_2, h_2h_3h_2g_1h_2^{-1}g_2h_3^{-1}g_3h_2^{-1}g_2).$$

$$\text{R. H. S.} = x \circ ((y \circ z) \circ y) = (h_1h_2h_3h_2, h_2h_3h_2g_1h_2^{-1}(h_3^{-1}h_2^{-1}h_2h_3)g_2h_3^{-1}g_3h_2^{-1}g_2) =$$

$$= (h_1h_2h_3h_2, h_2h_3h_2g_1h_2^{-1}g_2h_3^{-1}g_3h_2^{-1}g_2).$$

So, L. H. S.=R. H. S.. Hence, $(A, \circ)$ is a Bol groupoid.

Corollary 4.1. Let H be a abelian generalized subgroup of a non-abelian generalized group G and let $A = H \times G$. For $(h_1, g_1), (h_2, g_2) \in A$, define

$$(h_1, g_1) \circ (h_2, g_2) = (h_1 h_2, h_2 g_1 h_2^{-1} g_2)$$

then $(A, \circ)$ is a Bol quasigroup with a left identity element.
Proof. By Theorem 2.2, an abelian generalized group is a group, so H is a group. The rest of the claim follows from Theorem 4.9.

Corollary 4.2. Let H be a subgroup of a non-abelian generalized group G such that G has the cancellation law and let $A = H \times G$. For $(h_1, g_1), (h_2, g_2) \in A$, define

$$(h_1, g_1) \circ (h_2, g_2) = (h_1 h_2, h_2 g_1 h_2^{-1} g_2)$$

then $(A, \circ)$ is a Bol quasigroup with a left identity element.
Proof. The proof of this goes in line with Theorem 4.9. A groupoid which has the cancellation law is a quasigroup, so G is a quasigroup hence A is a quasigroup. Thus, $(A, \circ)$ is a Bol quasigroup with a left identity element since by Kunen [7], every quasigroup satisfying the right Bol identity has a left identity.

Corollary 4.3. Let H be a abelian generalized subgroup of a non-abelian generalized group G such that G has the cancellation law and let $A = H \times G$. For $(h_1, g_1), (h_2, g_2) \in A$, define

$$(h_1, g_1) \circ (h_2, g_2) = (h_1 h_2, h_2 g_1 h_2^{-1} g_2)$$

then $(A, \circ)$ is a Bol quasigroup with a left identity element.
Proof. By Theorem 2.2, an abelian generalized group is a group, so H is a group. The rest of the claim follows from Theorem 4.2.

5. CONTRIBUTIONS TO KNOWLEDGE

1. It is shown that in a generalized group G, $(a^{-1})^{-1} = a \; \forall \; a \in G$. But in a normal generalized group, it is shown that the anti-automorphic inverse property holds under a necessary condition.
A necessary and sufficient condition for a generalized group(which obeys the cancellation law) to be idempotent is established.
The basic theorem used in classical groups to define the subgroup of a group is shown to be true for generalized groups.

2. The kernel of any homomorphism(at a fixed point) mapping a generalized group to another generalized group is shown to be a normal subgroup. Furthermore, the homomorphism is found to be an injection if and only if its kernel is the set of the identity element at the fixed point.
3. Given a generalized group G with a generalized subgroup H, it is shown that the factor set G/H is a generalized group.
The direct product of two generalized group is shown to be a generalized group. Furthermore, necessary condition for a generalized group G to be isomorphic to the direct product of any two abelian generalized subgroups is shown.
4. It is shown that a Bol groupoid can be constructed using a non-abelian generalized group with an abelian generalized subgroup. Furthermore, if is established that if the non-abelian generalized group obeys the cancellation, then a Bol quasigroup with a left identity element can be constructed.

5.1. FURTHER STUDY

There is need to investigate the Cayley theorem in generalized groups. That is to check if every generalized group is isomorphic to a permutation group or if there exist a special group which every generalized groups are isomorphic to.
Contribution are still been made to the study of finite groups. And so, there is urgent need to study finite generalized groups. In particular, the Lagrange's theorem, Sylow theorems and Cauchy theorem should be given attention.

5.2. RECOMMENDATION AND CONCLUSION

Recommendation. Since it has been established in Theorem 4.5 that the factor set formed by a generalized group and a generalized subgroup is a generalized group then it is recommended that this theorem should be used as a starting point to prove the Lagrange's theorem for generalized groups. Theorem 4.9 and Corollary 4.1 show that a Bol groupoid and a Bol quasigroup can be constructed using a non-abelian generalized group. It is recommended that these should be improved on to construct a Bol loop and adjusted to construct some other loops of Bol-Moufang type e.g central loops.
Conclusion. So far in this study, some results that are true in classical groups have been investigated in generalized groups and have been found to be either true in generalized groups or true in some types of generalized groups. Also, this work has only been able to show how a Bol groupoid and

a Bol quasigroup can be constructed using a non-abelian generalized group. There could be a class of non-abelian generalized group with which a Bol loop can be constructed.

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On $f-$ amicable pairs

József Sándor and Mihály Bencze[10]

ABSTRACT. For an arithmetical function $f : N \to R$ define $F : N \to R$ by $F(n) = \sum\limits_{d|n} f(d)$, $n \in N$. We say that the numbers a and b are $f-$ amicable, if $F(a) = F(b) = f(a) + f(b)$. We will study this notion for various particular arithmetical functions f.

1. INTRODUCTION

It is well-known that two positive integers a and b are called amicable, if one has:

$$\sigma(a) = \sigma(b) = a + b, \tag{1}$$

where $\sigma(n) = \sum\limits_{d|n} d$ denotes the sum of distinct positive divisors of n. For $a = b$ we reobtain from (1) the perfect numbers, as

$$\sigma(a) = 2a \tag{2}$$

The least pair of amicable numbers $a \neq b$ are $a = 220$ and $b = 284$, known also from the Bible.

For the history and survey of old or recent results on amicable numbers, we quote the monograph [10].

The aim of this paper is to introduce a generalization of the notion of amicable numbers; and to study some particular cases. As we will see, in certain cases we will be able to settle completely the problem; in other cases

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only particular solutions will be obtained, along with many open problems-which is similar in certain sense with the classical case.

Let $f : N \rightarrow R$ be an arithmetical function, and define its summatory function by $F : N \rightarrow R$,

$$F(n) = \sum_{d|n} f(d), n \in N \tag{3}$$

where the sum runs through all distinct positive divisors d of n.
In what follows, we shall say that the pair of numbers a and b are $f-$ amicable, if

$$F(a) = F(b) = f(a) + f(b) \tag{4}$$

When $a = b$, then we reobtain a notion of f - perfect number a, given by

$$F(a) = 2f(a) \tag{5}$$

studied in certain papers in the References (along with similar or variations of notions)
We note then we could define, in place of (4) a and b to be almost-amicable, if

$$F(a) = F(b) = f(a) + f(b) - 1 \tag{6}$$

or quasi-amicable, if

$$F(a) = F(b) = f(a) + f(b) + 1 \tag{7}$$

but these will be considered in another places.

Clearly, when $f(n) = E(n) = n$, $(n \in N)$, from (3) we get $F(n) = \sigma(n)$, so (4) contains the classical case of amicable numbers, in the sense of (1).
Similarly, from (6) and (7) we get the classical cases of almost-amicable and quasi-amicable numbers (see [10]).

2. MAIN RESULTS

In this section we will consider separately some particular cases of (4).

1. I-amicable pairs. Let $I(n) = 1$, $\forall n \in N$. The following result is true.

Theorem 1. The only I- amicable numbers a, b are given by $a = p$ and $b = q$; where p, q are arbitrary prime numbers.
Proof. By (3) we get $F(n) = \sum\limits_{d|n} 1 = d(n) =$ number of divisors of n. Then equation (4) becomes

$$d(a) = d(b) = 2$$

It is well-known that $d(n) = 2$ if and only if $n =$ prime, so the result follows.

2. φ-amicable pairs. Let $\varphi(n)$ be Euler's totient. We have:

Theorem 2. All φ- amicable numbers a and b are given by

$$a = b = 2^k \ (k \geq 1, \text{ integer})$$

Proof. As $F(n) = \sum\limits_{d|n} \varphi(d) = n$ (by Gauss' identity), we get from (4) that

$$a = b = \varphi(a) + \varphi(b),$$

so $2\varphi(a) = a$; i.e.

$$\varphi(a) = \frac{a}{2} \tag{8}$$

It is well-known (see e.g. [11]) all solutions of (8) are $a = 2^k$, $k \geq 1$. This finishes the proof of Theorem 2.

3. $1/E$ - AMICABLE PAIRS

Let $E : N \to N$, $E(n) = n$. Then $f(n) = \frac{1}{n}$, and
$F(n) = \sum\limits_{d|n} \frac{1}{d} = \frac{1}{n} \sum\limits_{d|n} \frac{n}{d} = \frac{1}{n} \sum\limits_{d|n} d = \frac{\sigma(n)}{n}$.
Therefore (4) gives

$$\frac{\sigma(a)}{a} = \frac{\sigma(b)}{b} = \frac{1}{a} + \frac{1}{b} \tag{9}$$

As from (9), at one part $\sigma(a) = 1 + \frac{a}{b}$, so b divides a; and from another part, $\sigma(b) = 1 + \frac{b}{a}$, so a divides b, we get $b = a$. This implies by (9) that $\frac{\sigma(a)}{a} = \frac{2}{a}$, so

$$\sigma(a) = 2 \tag{10}$$

As for $a \geq 2$ one has $\sigma(a) \geq a + 1 \geq 3$; and $\sigma(1) = 1$, equation (10) has no solutions. We have obtained:

Theorem 3. There are no $\frac{1}{E}-$ amicable pairs.

4. k - AMICABLE PAIRS

Let in what follows introduce the arithmetical function

$$k(n) = \begin{cases} n, & \text{if} \quad n = \text{ prime} \\ 1, & \text{if} \quad n = \text{ composite or } n = 1 \end{cases} \tag{11}$$

Then $F(1) = 1$ and for $n > 1$ one has

$$F(n) = \sum_{\substack{d|n \\ d \text{ prime}}} d + \sum_{\substack{d|n \\ d \text{ composite}}} 1 = \sum_{p|n} p + \left(\sum_{d|n} 1 - \sum_{\substack{d|n \\ d \text{ prime}}} 1 \right) = B(n) + d(n) - \omega(n),$$

where $\omega(n)$ denotes the number of distinct prime divisors of n, while B is a much studied arithmetical function (see [10])

$$B(n) = \sum_{p|n} p, \tag{12}$$

where p runs through the prime divisors of n. Let $a, b > 1$. Then (4) leads to

$$B(a) + d(a) - \omega(a) = B(b) + d(b) - \omega(b) = k(a) + k(b)$$

As $k(a) + k(b) = a + b$ if a and b are prime; $a + 1$ if $a =$ prime, $b =$ composite; 2 if a, b are composite; and since $B(a) = a$ if $a =$ prime; $> a$ if $a =$ composite; clearly only the case $a =$ prime; $b =$ composite can be allowed. As $\omega(a) = 1$, $d(a) = 2$, $a + 1 = a + 1$ but
$B(b) + d(b) - \omega(b) = a + 1$ only if

$$B(b) + d(b) - \omega(b) - 1 = a \text{ is a prime number} \tag{13}$$

Theorem 4. All k- amicable numbers $a > 1$ and $b > 1$ are given by the equation (13), where B is given by (12).

Remark. *There are infinitely many solutions to (13). Put e.g.* $b = p^k$*, where* p *is a prime. Then (13) gives* $p + k - 1$ *= prime. Thus is true for any* $k = q - p + 1$*, where* $q > p$ *are primes.*
For another example, let $b = p \cdot p'$, where $p \cdot p'$ are primes. Then we have that $p + p' + 1 = q$ is a prime. This is possible, e.g. for $p = 3$, $p' = 7$, where $q = 11$; or $p = 5$, $p' = 11$, when $q = 17$; etc.

5. δ_+ - AMICABLE PAIRS

Let $d_+(n)$ denote the nuber of even divisors of n. (This function has been studied e.g. in [6], [8]).
Clearly, $d_+(n) = 0$ if n is odd; while for $n$even, written in the for $n = 2^k N$, one has (with $N \geq 1$, odd)

$$d_+(n) = kd(N) \tag{14}$$

(see e.g. [6]). let $\delta_+(n) = \begin{cases} 1, & n \quad \text{even} \\ 0, & n \quad \text{odd} \end{cases}$.

Theorem 5. If a and b are δ_+- amicable numbers, then $a = 4$ or $a = 2p$ and $b = 4$ or $b = 2q$; where p and q are prime numbers.
Proof. Since $F(n) = \sum\limits_{d|n} \delta_+(d) = \sum\limits_{d|n,\text{ even}} 1 = d_+(n)$, we get

$$F(n) = d_+(n) \tag{15}$$

Now, equation (4) gives

$$d_+(a) = d_+(b) = \delta_+(a) + \delta_+(b) = \begin{cases} 2, & \text{if} \quad a, b \text{ are even} \\ 1, & \text{if} \quad a \text{ and } b \text{ are of distinct parity} \end{cases}$$

As $d_+(a) = 2$ by (14) only if $kd(N) = 2$, and $d(N) = 1 \Leftrightarrow N = 1$ we get that $d_+(a) = 2$ only if $a = 2^m A$ ($m \geq 1$, A odd) has the form $a = 2^2 \cdot 1$ or $a = 2^1 \cdot p$. Similarly for b, so Theorem 5 follows.

6. ON + AMICABLE PAIRS

Let

$$f(n)=\begin{cases} n, & \text{if } \; n=\text{even} \\ 0, & \text{if } \; n=\text{odd} \end{cases}$$

Then $F(n)=\sum_{\substack{d|n \\ d \text{ even}}} d=\sigma_+(n)=$ sum of even divisors of n. Therefore, equation (4) leads in this case to:

$$\sigma_+(a)=\sigma_+(b)=a+b;\; a,b \text{ even} \tag{16}$$

We will call the numbers a and b as + amicable numbers.

Theorem 6. All + amicable numbers $a=2^kN$ and $b=2^sM$ $(N,M$ odd) must satisfies the equations

$$\left(2^k-1\right)\sigma(N)=(2^s-1)\sigma(M)=2^{k-1}N+2^{s-1}M \tag{17}$$

Proof. As it is shown in [6], one has

$$\sigma_+(a)=\sigma_+\left(2^kN\right)=2\left(2^k-1\right)\sigma(N) \tag{18}$$

By using (18), and defintion (4), relation (17) follows
Remarks. *1). When $a=b=2^kN$, then (17) leads to the + perfect - numbers, which all were determined in [6]. For + superperfect or related numbers, see [8].*
2). The determination of all dolutions to (17) is however, an open problem.

7. $\varphi - I-$ AMICABLE PAIRS

Let $I(n)=1$, so $f(n)=\varphi(n)-1$. Then
$F(n)=\sum_{d|n}(\varphi(d)-1)=\sum_{d|n}\varphi(d)-\sum_{d|n}1=n-d(n)$; so by (4) we get the equations:

$$a-d(a)=b-d(b)=\varphi(a)+\varphi(b)-2 \tag{19}$$

Theorem 7. All $\varphi - I-$ amicable numbers are given by equation (19). If a is prime, then $a=2$ and $b\in\{1,2\}$. There are no $\varphi - I-$ amicable numbers a,b such that $a,b\geq 3$ and $a-d(a)$ is odd.

Proof. If a is prime, then $d(a)=2$ and $\varphi(a)=a-1$, so (19) implies $\varphi(b)=1$; giving $b\in\{1,2\}$. But then $b-d(b)=0$, so we get $a=2$.
Since for $n\geq 3$, $\varphi(n)$ is even; the right-side of (19) being even, and $a-d(a)$ being odd, such solutions are not available.
Remark. *The determination of all solutions to (19) is still open.*

8. $\mu-$ AMICABLE PAIRS

Let μ denote the classical Möbius function, defined by

$$\mu(n)=\begin{cases}1, & n=1\\ 0, \quad \text{if} & a^2|n \text{ for some } a>1\\ (-1)^r, \quad if & n=p_1...p_r,\ \text{with } p_i \text{ distinct primes}\end{cases}$$

It is well-known from classical textbooks of number theory that,

$$\sum_{d|n}\mu(d)=e(n)=\begin{cases}1, \quad \text{if} & n=1\\ 0, & n\geq 2\end{cases}$$

Therefore, by letting $f(n)=\mu(n)$, relation (4) gives

$$\varphi(a)=e(b)=\mu(a)+\mu(b)$$

This is true, if $a,b\geq 2$ and $\mu(a)+\mu(b)=0$. This may happen only when $\mu(a)=\mu(b)=0$, or when $\mu(a)=-\mu(b)$, when $\mu(b)\neq 0$, $\mu(a)\neq 0$. Thus we have:
Theorem 8. If a,b are $\mu-$amicable numbers, then we have
i). a and b are squarefull numbers;
ii). a and b are squarefree, a having r and b having $r+1$ (or reciprocally) prime factors

9. $\frac{\mu}{E}-$ AMICABLE NUMBERS

From Gauss‘identity we have $\sum\limits_{d|n}\varphi(d)=n$; so by applying the Möbius inversion formula, we get

$$\varphi(n)=\sum_{d|n}\mu(t)\cdot\frac{n}{d};\ \text{implying}\ \frac{\varphi(n)}{n}=\sum_{d|n}\frac{\mu(d)}{d}$$

Let $f(n)=\frac{\mu(n)}{n}$; when $F(n)=\frac{\varphi(n)}{n}$. Relation (4) implies

$$\frac{\varphi(a)}{a}=\frac{\varphi(b)}{b}=\frac{\mu(a)}{a}+\frac{\mu(b)}{b} \tag{$*$}$$

It is immediate that (*) implies a divides b and b divides a, so $a = b$. Therefore, we get $\frac{\varphi(a)}{a} = 2\frac{\mu(a)}{a}$, or

$$\varphi(a) = 2\mu(a)$$

Since $\mu(1) = 1$, $\varphi(1) = 1$ and $\mu(a) \neq 0$ only when a is squarefree, we get $\varphi(a) = 2$, so $a = 3, 6$. But $\mu(3) = -1$, $\mu(6) = 1$; so the single solution is $a = 6$.

Theorem 9. If a and b are $\frac{\mu}{E}-$ amicable numbers, then $a = b = 6$.

Remark. *Thus, in this case we have obtained not only a finite number of solutions, but only a single one.*

10. $\Lambda-$ AMICABLE NUMBERS

Let Λ be the classical von Mangoldt function, given by

$$\Lambda(n) = \begin{cases} \log p, & \text{if} \quad n = p^m \ \ (p \text{ prime}, n \geq 1) \\ 0, & \qquad \text{otherwise} \end{cases}$$

It is well-known that

$$\sum_{d|n} \Lambda(d) = \log n$$

Letting $f(n) = \Lambda(n)$, we get $F(n) = \log n$. By (4) we can write

$$\log a = \log b = \Lambda(a) + \Lambda(b)$$

Thus $a = b$, and $\log a = 2\Lambda(a)$. This is true for $a = 1$.
Let $a = p^m > 1$. Then $\log a = m \log p$ and $\Lambda(a) = \log p$, so $m \log p = 2 \log p \Leftrightarrow m = 2$. Therefore:

Theorem 10. If a and b are $\Lambda-$ amicable pairs, then $a = b = 1$ or $a = b = p^2$, where p is an arbitrary prime number.

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Some new inequalities for means

Mihály Bencze and Zhao Changjian[11]

ABSTRACT. In this paper using the Mac Laurian‘s inequality we present some new inequalities for classical means

MAIN RESULTS

Theorem 1. (Mac Laurian) If $x_i > 0$ $(i = 1, 2, ..., n)$ and $k \in \{1, 2, ..., n\}$, then

$$\sum_{cyclic} x_1 x_2 ... x_k \le \frac{\binom{n}{k}}{n^k} \left(\sum_{i=1}^{n} x_i \right)^k$$

(see [1]).

In following we use the notations $A(x_1, x_2, ..., x_n) = \frac{1}{n} \sum\limits_{i=1}^{n} x_i$ for the arithmetic mean, $G(x_1, x_2, ..., x_n) = \sqrt[n]{\prod\limits_{i=1}^{n} x_i}$ for the geometric mean, and $H(x_1, x_2, ..., x_n) = \frac{n}{\sum\limits_{i=1}^{n} \frac{1}{x_i}}$ for the harmonic mean.

Theorem 2. If $x_i > 0$ $(i = 1, 2, ..., n)$ and $k \in \{1, 2, ..., n\}$, then

$$A^{n-k}(x_1, x_2, ..., x_n) H(x_1 x_2 ... x_k, x_2 x_3 ... x_{k+1}, ..., x_n x_1 ... x_{k-1}) \ge$$

$$\ge \frac{n}{\binom{n}{k}} G^n(x_1, x_2, ..., x_n)$$

Proof. Using the Mac Laurian‘s inequality we have:

$$\left(\sum_{i=1}^{n} x_i \right)^k \ge \frac{n^k}{\binom{n}{k}} \sum_{cyclic} x_1 ... x_k \text{ or}$$

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$$\left(\sum_{i=1}^{n} x_i\right)^k \cdot \frac{n}{\sum\limits_{cyclic} \frac{1}{x_{k+1}...x_n}} \geq \frac{n^{k+1}}{\binom{n}{k}} x_1 x_2 ... x_n$$

If in this we change k with $n-k$, then we obtain

$$\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^{n-k} \cdot \frac{n}{\sum\limits_{cyclic} \frac{1}{x_1 x_2 ... x_k}} \geq \frac{n}{\binom{n}{k}} x_1 x_2 ... x_n \text{ or}$$

$$A^{n-k}\left(x_1, x_2, ..., x_n\right) H\left(x_1 x_2 ... x_k, x_2 x_3 ... x_{k+1}, ..., x_n x_1 ... x_{k-1}\right) \geq$$

$$\geq \frac{n}{\binom{n}{k}} G^n\left(x_1, x_2, ..., x_n\right)$$

Corollary 2.1. If $x_i > 0$ $(i = 1, 2, ..., n)$, then

$$\left(\sum_{i=1}^{n} x_i\right)^{n-1} \geq n^{n-2} \sum_{cyclic} x_1 x_2 ... x_{n-1}$$

Proof. In Theorem 2 we take $k = 1$. If $n = 3$, then we reobtain the classical inequality $\left(x_1 + x_2 + x_3\right)^2 \geq 3\left(x_1 x_2 + x_2 x_3 + x_3 x_1\right).$

If $k = n - 1$, then we obtain identity.

Theorem 3. If $x_i > 0$ $(i = 1, 2, ..., n)$ and $k \in \{1, 2, ..., n\}$, then

$$\left(\frac{(n-k) A\left(x_1, x_2, ..., x_n\right) + H\left(x_1 x_2 ... x_k, x_2 x_3 ... x_{k+1}, ..., x_n x_1 ... x_{k-1}\right)}{n-k+1}\right)^{n-k+1} \geq$$

$$\geq \frac{n}{\binom{n}{k}} G^n\left(x_1, x_2, ..., x_n\right)$$

Proof. Using the weighted AM-GM inequality, we get:

$$\left(\frac{(n-k) A\left(x_1, x_2, ..., x_n\right) + H\left(x_1 x_2 ... x_k, ..., x_n x_1 ... x_{k-1}\right)}{n-k+1}\right)^{n-k+1} \geq$$

$$\geq A^{n-k}\left(x_1, x_2, ..., x_n\right) H\left(x_1 x_2 ... x_k, ..., x_n x_1 ... x_{k-1}\right) \geq$$

$$\geq \frac{n}{\binom{n}{k}} G^n\left(x_1, x_2, ..., x_n\right)$$

Corollary 3.1. In all triangle ABC holds:
1). $\frac{s^2+r^2+10Rr}{s^2+r^2+4Rr} \geq 4\sqrt[3]{\frac{4Rr}{s^2}}$
2). $\frac{R+r}{4R+r} \geq \sqrt[3]{\left(\frac{r}{s}\right)^2}$
3). $s^2+r^2+10Rr \geq 8\sqrt[3]{2s^2R^2r^2}$
4). $R+r \geq \sqrt[3]{s^2r}$
5). $\frac{2R-r}{2R} + \frac{3r^2}{s^2+r^2-8Rr} \geq \sqrt[3]{\left(\frac{2r}{R}\right)^2}$
6). $\frac{4R+r}{2R} + \frac{3s^2}{s^2+(4R+r)^2} \geq \sqrt[3]{\left(\frac{2s}{R}\right)^2}$
Proof. From Theorem 3 we get

$$x_1+x_2+x_3+\frac{3}{\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}} \geq 4\sqrt[3]{x_1x_2x_3}$$

and in this we take

$$(x_1,x_2,x_3) \in \{(a,b,c), (s-a,s-b,s-c); (h_a,h_b,h_c); (r_a,r_b,r_c);$$

$$\left.\left(\sin^2\frac{A}{2},\sin^2\frac{B}{2},\sin^2\frac{C}{2}\right); \left(\cos^2\frac{A}{2},\cos^2\frac{B}{2},\cos^2\frac{C}{2}\right)\right\}$$

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$e^{i\pi}=-1$

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Extensions of a category of inequalities

D.M. Bătineţu-Giurgiu and Mihály Bencze[12]

ABSTRACT. In this paper we present extensions of a category of inequalities, which have applications in fundamental theory of inequalities, and evidently in mathematical contests.

MAIN RESULTS

Theorem 1. If $a_1, ..., a_k$, $b_1, ..., b_p$, $x_t, y_t > 0$, $\alpha, \gamma \in R$, $\beta \in (-\infty, 0] \cup [2, +\infty)$ and $i_1, ..., i_k, j_1, ..., j_p \in \{1, 2, ..., n\}$ are different indexes, then

$$\sum_{cyclic} \frac{\left(a_1 x_{i_1}^{\alpha} + ... + a_k x_{i_k}^{\alpha}\right)^{\beta}}{b_1 y_{j_1}^{\gamma} + ... + b_p y_{j_p}^{\gamma}} \geq \frac{n^{2-\beta} \left(a_1 + ... + a_k\right)^{p} \left(\sum\limits_{t=1}^{n} x_t^{\alpha}\right)^{\beta}}{\left(b_1 + ... + b_p\right) \sum\limits_{t=1}^{n} y_t^{\gamma}}$$

Proof. Using first the Cauchy-Schwarz‘s and after them the Jensen‘s inequality we obtain:

$$\left(b_1 + ... + b_p\right) \sum_{t=1}^{n} y_t^{\gamma} \sum_{cyclic} \frac{\left(a_1 x_{i_1}^{\alpha} + ... + a_k x_{i_k}^{\alpha}\right)^{\beta}}{b_1 y_{j_1}^{\gamma} + ... + b_p y_{j_p}^{\gamma}} =$$

$$= \left(\sum_{cyclic} \left(b_1 y_{j_1}^{\gamma} + ... + b_p y_{j_p}^{\gamma}\right)\right) \sum_{cyclic} \frac{\left(a_1 x_{i_1}^{\alpha} + ... + a_k x_{i_k}^{\alpha}\right)^{\beta}}{b_1 y_{j_1}^{\gamma} + ... + b_p y_{j_p}^{\gamma}} \geq$$

$$\geq \left(\sum_{cyclic} \left(a_1 x_{i_1}^{\alpha} + ... + a_k x_{i_k}^{\alpha}\right)^{\frac{\beta}{2}}\right)^{2} \geq$$

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$$\geq \left(\frac{1}{n^{\frac{\beta}{2}-1}}\sum_{cyclic}\left(a_1x_{i_1}^{\alpha}+...+a_kx_{i_k}^{\alpha}\right)^{\frac{\beta}{2}}\right)^2 = n^{2-\beta}\left(a_1+...+a_k\right)^{\beta}\left(\sum_{t=1}^{n}x_t^{\alpha}\right)^{\beta}$$

Corollary 1.1. If $a, b, x_t > 0$ $(t = 1, 2, ..., n)$ and $\beta \in (-\infty, 0] \cup [2, +\infty)$, then

$$\sum_{cyclic}\frac{x_1^{\beta}}{ax_2+bx_3} \geq \frac{n^{2-\beta}\left(\sum\limits_{t=1}^{n}x_t^{\alpha}\right)^{\beta}}{a+b}$$

Proof. In Theorem 1 we take $\alpha = 1$, $a_1 = 1$, $a_2 = a_3 = ... = a_k = 0$, $b_1 = a$, $b_2 = b$, $y_1 = x_2$, $y_2 = x_3$, $b_3 = b_4 = ... = b_p = 0$, $\gamma = 1$.

If $a = b = 1$ then we obtain the Corollary 2.1 from [2].

If $\prod\limits_{t=1}^{n} x_t = 1$, then $\sum\limits_{t=1}^{n} x_t \geq n$ and we get

$$\sum_{cyclic}\frac{x_1^{\beta}}{ax_2+bx_3} \geq \frac{n}{a+b}$$

which is the Remark 2.1.2 from [2].

If $x_t \to \frac{1}{x_t}$ $(t = 1, 2, ..., n)$, then we get

$$\sum_{cyclic}\frac{x_2x_3}{x_1^{\beta}\left(ax_3+bx_2\right)} \geq \frac{n^{2-\beta}\left(\sum\limits_{t=1}^{n}\frac{1}{x_t}\right)^{\beta}}{a+b}$$

If $\prod\limits_{t=1}^{n} x_t = 1$, then from the previous inequality we obtain

$$\sum_{cyclic}\frac{x_2x_3}{x_1^{\beta}\left(ax_2+bx_3\right)} \geq \frac{n}{a+b}$$

If $n = 3$, $\beta = 2$ and $x_1x_2x_3 = 1$, then

$$\sum\frac{1}{x_1^3\left(ax_2+bx_3\right)} \geq \frac{3}{a+b}$$

is a problem proposed by Russia in 1995 and presented at IMO.

Corollary 1.2. If $a, b, x_t > 0$ $(t = 1, 2, ..., n)$, $(a \geq b)$ and $\beta \in (-\infty, 0] \cup [2, +\infty)$, then

$$\sum_{cyclic} \frac{x_1^{\beta}}{a(x_1 + ... + x_{k-1}) + (a-b)x_k + a(x_{k+1} + ... + x_n)} \geq \frac{n^{2-\beta}\left(\sum\limits_{t=1}^{n} x_t\right)^{\beta-1}}{an - b}$$

Proof. In Theorem 1 we take $a_1 = 1$, $a_2 = ... = a_k = 0$, $x_t = y_t$ $(t = 1, 2, ..., n)$, $b_1 = ... = b_{k-1} = b_{k+1} = ... = b_n = a$, $b_k = a - b$, $\alpha = \gamma = 1$.

Corollary 1.3. If $a, b, c, d, x_t > 0$ $(t = 1, 2, ..., n)$, $\beta \in (-\infty, 0] \cup [2, +\infty)$, then

$$\sum_{cyclic} \frac{(ax_1 + bx_2)^{\beta}}{xc_3 + dx_4} \geq \frac{n^{2-\beta}(a+b)^{\beta}\left(\sum\limits_{t=1}^{n} x_t\right)^{\beta}}{c+d}$$

from this we obtain

$$\sum_{cyclic} \frac{(x_1 + x_2)^{\beta}}{x_3 + u} \geq \frac{2^{\beta} n^{2-\beta}}{nu + 1}$$

which is Corollary 2.4.1 from [2].

If $\beta = 2$, then we get

$$\sum_{cyclic} \frac{(x_1 + x_2)^{\beta}}{x_3 + u} \geq \frac{4}{1 + nu}$$

If $u = 1$ and $n = 3$ then we obtain the problem 24380 from Gazeta Matematica 10/2000, author Mihai Opincariu, namely if $a + b + c = 1$, then

$$\sum \frac{(a+b)^2}{c+1} \geq 1$$

After some modification we get the following: If $a, b, c > 0$ then

$$\sum \frac{(a+b)^2}{a+b+2c} \geq a + b + c$$

In same way we obtain the following:

$$\sum_{cyclic} \frac{x_1^\beta}{x_2+x_3} \geq \frac{1}{2} n^{2-\beta} \left(\sum_{t=1}^{n} x_t \right)^{\beta-1}$$

which is Corollary 2.6 from [2].

If $\beta = 2$, then we get the following: If $a, b, c, d > 0$ and $a+b+c+d = 1$, then

$$\sum \frac{a^2}{a+b} \geq \frac{1}{2}$$

which is a problem presented by Ireland at IMO 1999.

Corollary 1.4. If $b_1, ..., b_p, x_t > 0$ $(t = 1, 2, ..., n)$, $\beta \in (-\infty, 0] \cup [2, +\infty)$, then

$$\sum_{cyclic} \frac{(x_1+x_2+...+x_k)^\beta}{b_1x_1+b_2x_2+...+b_px_p} \geq \frac{k^\beta \left(\sum\limits_{t=1}^{n} x_t \right)^{\beta-1}}{(b_1+...+b_p)\, n^{\beta-2}}$$

which is Corollary 2.7 from [2].

If $k = 2, \beta = 2, b_1 = b_2 = 1, b_3 = b, b_4 = ... = b_p = 0$, then we get the following: If $x, y, z, b > 0$, then

$$\sum \frac{(x+y)^2}{x+y+bz} \geq \frac{4(x+y+z)}{b+2}$$

which is problem C.2403 from Gazeta Matematica 5-6/2001, author Titu Zvonaru.

After some substitutions we get:

$$\sum \frac{1}{x_1+x_2+...+x_k} \geq \frac{n^2}{k \sum\limits_{t=1}^{n} x_t}$$

If $n = 3, k = 2$ then we obtain the following: If $a, b, c > 0$, then

$$\sum \frac{1}{a+b} \geq \frac{9}{2(a+b+c)}$$

which was a problem in Ireland at a Mathematical Contest in 1999.
After another specification we obtain the following:
Corollary 1.5. If $x_t > u > 0$ $(t = 1, 2, ..., n)$ and $\beta \in (-\infty, 0] \cup [2, +\infty)$, then

$$\sum \frac{x_1^{\beta}}{x_2 - u} \geq \frac{\left(\sum\limits_{t=1}^{n} x_t\right)^{\beta}}{n^{\beta-2}\left(\sum\limits_{t=1}^{n} x_t - nu\right)}$$

If $\beta = 2, a, b \geq u = 1$, $n = 2$, then we obtain the following: If $a, b \geq 1$, then

$$\frac{a^2}{b-1} + \frac{b^2}{a-1} \geq 8$$

which was a problem in Russian Mathematical Olympiad 1992. From Corollary 1.5 we get:

$$\frac{a^2}{b-1} + \frac{b^2}{a-1} \geq \frac{(a+b)^2}{a+b-2}$$

but $(a+b)^2 \geq 8(a+b-2) \Leftrightarrow (a+b-4)^2 \geq 0$.
In same way we obtain the following problem: If $a, b \geq 2$, then

$$\frac{a^2}{b-2} + \frac{b^2}{a-2} \geq 16$$

which was presented at Mathematical Olympiad in Moldavia. From Corollary 1.5 we get

$$\frac{a^2}{b-2} + \frac{b^2}{a-2} \geq \frac{(a+b)^2}{a+b-4}$$

but

$$(a+b)^2 \geq 16(a+b-4) \Leftrightarrow (a+b-8)^2 \geq 0$$

Corollary 1.6. If $a, b, x_t > 0$ $(t = 1, 2, ..., n)$, then

$$\sum_{1 \leq i < j \leq n} \frac{1}{ax_ix_j + b} \geq \frac{\binom{n}{2}^2}{a \sum\limits_{1 \leq i < j \leq n} x_ix_j + \binom{n}{2}b} \geq \frac{\binom{n}{2}^2}{\frac{a(n-1)}{2} \sum\limits_{t=1}^{n} x_t^2 + \binom{n}{2}b}$$

which is Corollary 2.10.1 from [2].

If $\sum\limits_{t=1}^{n} x_t^2 = n$, then

$$\sum_{1\le i<j\le n} \frac{1}{ax_ix_j+b} \ge \frac{n(n-1)}{2(a+b)}$$

If $a=1$ then we obtain corollary 2.10.2 from [2].
If $n=3$, $a=b=1$ then we obtain the following: If $a,b,c>0$ and $a^2+b^2+c^2=3$, then

$$\sum \frac{1}{ab+1} \ge \frac{3}{2}$$

which is a problem presented at Mathematical Olympiad in Belarus 1999.

The result of corollary 1.6 can be generalized in following way: If $a,b,x_t>0$ $(t=1,2,...,n)$ then

$$\sum_{1\le i_1<...<i_k\le n} \frac{1}{ax_{i_1}x_{i_2}...x_{i_k}+b} \ge \frac{\binom{n}{k}^2}{a\sum\limits_{1\le i_1<...<i_k\le n} x_{i_1}...x_{i_k}+\binom{n}{k}b}$$

Corollary 1.7. If $b_t,x_t>0$ $(t=1,2,...,n)$ and $\beta\in(-\infty,0]\cup[2,+\infty)$, then

$$\frac{x_1^\beta}{\frac{b_1}{x_1}+\frac{b_2}{x_2}+...+\frac{b_n}{x_n}}+\frac{x_2^\beta}{\frac{b_1}{x_2}+\frac{b_2}{x_3}+...+\frac{b_n}{x_n}}+...+\frac{x_n^\beta}{\frac{b_1}{x_n}+\frac{b_2}{x_1}+...+\frac{b_n}{x_n}} \ge$$

$$\ge \frac{\left(\sum\limits_{t=1}^{n} x_t\right)^\beta}{n^{\beta-1}\left(\sum\limits_{t=1}^{n}\frac{1}{x_t}\right)\left(\sum\limits_{t=1}^{n} b_t\right)}$$

Proof. In Theorem 1 we take $a_1=1, a_2=a_3=...=a_k=0, p=n, y_t=\frac{1}{x_t}$ $(t=1,2,...,n)$.

After some particularization we get the following problem: If $a,b,c,d,e>0$ and $\beta\in(-\infty,0]\cup[2,+\infty)$, then

$$\sum \frac{a^\beta}{b+c} \ge \frac{\left(\sum a\right)^{\beta-1}}{2\cdot 5^{\beta-2}}$$

If $m=1$ then we get

$$\sum \frac{a}{b+c} \geq \frac{5}{2}$$

which is a problem of selection test in USA, for IMO 2001, author Titu Andreescu.

Corollary 1.8. If $a, b, c > 0$ and $\beta \in (-\infty, 0] \cup [2, +\infty)$, then

$$\sum \frac{a^\beta}{a^2 - bc + 1} \geq \frac{3^{2-\beta} \left(\sum a\right)^\beta}{\left(\sum a\right)^2 - 3\sum ab + 3}$$

If $\sum ab = \frac{1}{3}$, then

$$\sum \frac{a^\beta}{a^2 - bc + 1} \geq 3^{1-\beta} \left(\sum a\right)^{\beta-2}$$

If $\beta = 1$, then

$$\sum \frac{a}{a^2 - bc + 1} \geq \frac{1}{\sum a}$$

which is problem O:976 Gazeta Matematica 9-10/2001, author Titu Andreescu.

Corollary 1.9. If $a, x, y, z, t > 0$ and $\beta \in (-\infty, 0] \cup [2, +\infty)$, then

$$\sum \frac{x^\beta}{x^3 - 3yzt + ax} \geq \frac{4^{2-\beta} \left(\sum x\right)^{\beta-1}}{\sum x^2 - \sum xy + a}$$

If $\beta = 2$, then

$$\sum \frac{x^2}{x^3 - 3yzt + ax} \geq \frac{\sum x}{\sum x^2 - \sum xy + a}$$

If $\sum xy = \frac{a}{3}$, then

$$\sum \frac{x^\beta}{x^3 - 3yzt + a} \geq 4^{2-\beta} \left(\sum x\right)^{\beta-3}$$

Corollary 1.10. If $x_t, y_t > 0$ $(t = 1, 2, ..., n)$ and $k \in N$, then

$$ky_1^{k+1} + \sum_{t=2}^{n} \frac{y_t^{k+1}}{x_{t-1}} \geq \frac{(k+1)\, y_1^k \sum\limits_{t=2}^{n} y_t}{\sqrt[k+1]{n^{k-1} \sum\limits_{t=2}^{n} x_{t-1}}}$$

Proof. Using the Theorem 1 we get

$$ky_1^{k+1}+\sum_{t=2}^{n}\frac{y_t^{k+1}}{x_{t-1}}\geq ky_1^{k+1}+\frac{\left(\sum\limits_{t=2}^{n}y_t\right)^{k+1}}{(n-1)^{k-1}\sum\limits_{t=2}^{n}x_{t-1}}=\underbrace{y_1^{k+1}+...+y_1^{k+1}}_{k-time}+$$

$$+\frac{\left(\sum\limits_{t=2}^{n}y_t\right)^{k+1}}{(n-1)^{k-1}\sum\limits_{t=2}^{n}x_{t-1}}\geq\frac{(k+1)\,y_1^k\sum\limits_{t=2}^{n}y_t}{\sqrt[k+1]{(n-1)^{k-1}\sum\limits_{t=2}^{n}x_{t-1}}}$$

If $\sum\limits_{t=2}^{n}x_{t-1}=1$, then we get

$$ky_1^{k+1}+\sum_{t=2}^{n}\frac{y_t^{k+1}}{x_{t-1}}\geq\frac{(k+1)\,y_1^k\sum\limits_{t=2}^{n}y_t}{\sqrt[k+1]{(n-1)^{k-1}}}$$

which is Corollary 2.11.1 from [2].
If $k=1$ then we get

$$y_1^2+\frac{y_2^2}{x_1}+\frac{y_3^2}{x_2}+...+\frac{y_n^2}{x_{n-1}}\geq 2y_1\sum_{t=2}^{n}y_t$$

which is a problem from O.B.M.J, 2000 author D.Acu.

Corollary 1.11. If $x_t>0$ $(k=1,2,...,n)$, and $y_0>y_1>...>y_n$, then

$$ky_0+\sum_{t=1}^{n}\frac{x_t^{k+1}}{y_{k-1}-y_k}\geq(k+1)\left(\sum_{t=1}^{n}x_t\right)\sqrt[k+1]{\left(\frac{y_0-y_n}{n}\right)^{k-1}}+ky_n$$

If $x_t=1$ $(t=1,2,...,n)$ and $k=1$ then we obtain:

$$y_0+\sum_{t=1}^{n}\frac{1}{y_{t-1}-y_t}\geq y_n+2n$$

which is a problem presented at Mathematical Olympiad Sankt Petersburg 1999.

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The sums of some power series

Adrian Corduneanu and Gheorghe Costovici[13]

ABSTRACT. The sums of certain power series are found.

MAIN RESULTS

We are going to find the sums of series of the form $\sum\limits_{n=0}^{\infty} a_n x^n$ assuming that $\frac{a_{n+1}}{a_n} = \frac{n+b}{n+c}$ for all $n = 0, 1, 2, \ldots$, and for some b, c and x real numbers. It is understand that $c \neq -n$ for all $n = 0, 1, 2, \ldots$, and suppose $b \neq 0$.

The radius of convergence of the series being 1, we put

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + \\ + a_n x^n + a_{n+1} x^{n+1} + \ldots \quad \text{for } x \in (-1, 1)$$

and we have

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + \\ + n a_n x^{n-1} + (n+1) a_{n+1} x^n + \ldots \quad \text{for } x \in (-1, 1).$$

From the hypotheses we have

$$(n+c) a_{n+1} = (n+b) a_n$$

and then

$$(n+1-1+c) a_{n+1} = n a_n + b a_n,$$

that is

$$(n+1) a_{n+1} + (c-1) a_{n+1} = n a_n + b a_n.$$

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Multiplying this equation by x^n for $x \in (-1,1)$ and summing, we obtain

$$\sum_{n=0}^{\infty}(n+1)\,a_{n+1}x^n+\sum_{n=0}^{\infty}(c-1)\,a_{n+1}x^n=\sum_{n=0}^{\infty}na_nx^n+\sum_{n=0}^{\infty}ba_nx^n$$

for $x \in (-1,1)$ because all the series have the same radius of convergence $= 1$. With regard to f and for $x \neq 0$, this means that

$$f'(x)+\frac{c-1}{x}\left(f(x)-a_0\right)=xf'(x)+bf(x),$$

thus

$$(1-x)\,f'(x)=\left(b-\frac{c-1}{x}\right)f\,(x)+\frac{a_0\,(c-1)}{x},$$

and, finally,

$$f'\,(x)=\frac{bx-c+1}{x\,(1-x)}f\,(x)+\frac{a_0\,(c-1)}{x\,(1-x)}$$

for $x \in (-1,0)\cup(0,1)$, that is a first-order linear differential equation is got. Putting $P\,(x)=\dfrac{bx-c+1}{x\,(1-x)}=\dfrac{1-c}{x}+\dfrac{b+1-c}{1-x}$ and $Q\,(x)=\dfrac{a_0\,(c-1)}{x\,(1-x)}$, we have

$$f\,(x)=e^{\int P(x)dx}\left(k+\int dxQ\,(x)\,e^{-\int P(x)dx}\right)$$

with $k \in R$.

Now

$$\int P\,(x)\,dx=(1-c)\ln|x|-(b+1-c)\ln(1-x)=$$

$$=-\,(c-1)\ln|x|-(b+1-c)\ln(1-x)=\ln\left[|x|^{c-1}\,(1-x)^{b-c+1}\right]^{-1}$$

and

$$\int Q\,(x)e^{-\int P(x)dx}dx=\int\frac{a_0\,(c-1)}{x\,(1-x)}\,|x|^{c-1}\,(1-x)^{b-c+1}\,dx=$$

$$=\begin{cases} a_0\,(c-1)\int|x|^{c-2}\,(1-x)^{b-c}\,dx \text{ for } x\in(0,1)\,,\\ -a_0\,(c-1)\int|x|^{c-2}\,(1-x)^{b-c}\,dx \text{ for } x\in(-1,0)\,.\end{cases}$$

Then

$$f\,(x)=\begin{cases}\dfrac{1}{|x|^{c-1}\,(1-x)^{b-c+1}}\left[k+a_0\,(c-1)\int|x|^{c-2}\,(1-x)^{b-c}\,dx\right],\text{for } x\in(0,1)\,,\\ a_0 \text{ for } x=0,\\ \dfrac{1}{|x|^{c-1}\,(1-x)^{b-c+1}}\left[k-a_0\,(c-1)\int|x|^{c-2}\,(1-x)^{b-c}\,dx\right],\text{for } x\in(-1,0)\,.\end{cases}$$

We have thus obtained The general solution under integral form.
Computing the integrals of the binomial differentials in the expression of $f(x)$ for some b and c – computations being more or less laborious –, and takeing into account the continuity of f in $x = 0$
for finding k, we outline some

Peculiar cases.

If $b = c$ one finds $f(x) = a_0 \frac{1}{1-x}$. This case can be obtained directly (not involving integrals).

If $c = 1$ one finds $f(x) = \frac{a_0}{(1-x)^b}$.

If $c = 2$ and $b = 1$ one finds

$$f(x) = \begin{cases} -\frac{a_0 \ln(1-x)}{x} \text{ for } x \in (-1,0) \cup (0,1), \\ a_0 \text{ for } x = 0. \end{cases}$$

If $c = 2$ and $b \neq 1$ one finds

$$f(x) = \begin{cases} \frac{1}{x(1-x)^{b-1}} \left[\frac{a_0}{b-1} - a_0 \frac{(1-x)^{b-1}}{b-1} \right] \text{ for } x \in (-1,0) \cup (0,1), \\ a_0 \text{ for } x = 0. \end{cases}$$

If $c \in \{3, 4, 5, \dots\}$ and $b \neq 1, 2, \dots c-1$, one finds

$$f(x) = \begin{cases} \frac{1}{x^{c-1}(1-x)^{b-c+1}} \left[a_0 (c-1) \sum\limits_{i=\overline{0,c-2}} (-1)^i C_{c-2}^i \frac{1}{i+b-c+1} - \right. \\ \left. -a_0 (c-1) \sum\limits_{i=\overline{0,c-2}} (-1)^i C_{c-2}^i \frac{(1-x)^{i+b-c+1}}{i+b-c+1} \right], \text{for } x \in (-1,0) \cup (0,1), \\ a_0 \text{ for } x = 0. \end{cases}$$

If $c \in \{3, 4, 5, \dots\}$ and $b = 1$ one finds

$$f(x) = \begin{cases} \frac{1}{x^{c-1}(1-x)^{2-c}} \left[a_0 (c-1) \sum\limits_{i=\overline{0,c-3}} (-1)^i \frac{C_{c-2}^i}{i-c+2} - \right. \\ \left. -a_0 (c-1) \left(\sum\limits_{i=\overline{0,c-3}} (-1)^i C_{c-2}^i \frac{(1-x)^{i-c+2}}{i-c+2} + (-1)^{c-2} \ln(1-x) \right) \right] \\ \qquad \qquad \text{for } x \in (-1,0) \cup (0,1), \\ a_0 \text{ for } x = 0. \end{cases}$$

If $c \in \{3, 4.5. \dots\}$ and $b = c - 1$ one finds

$$f(x) = \begin{cases} \dfrac{1}{x^{c-1}} \Big[a_0 (c-1) \displaystyle\sum_{i=\overline{1,c-2}} (-1)^i \dfrac{C_{c-2}^i}{i} - \\ -a_0 (c-1) \left(\ln(1-x) + \displaystyle\sum_{i=\overline{1,c-2}} (-1)^i C_{c-2}^i \dfrac{(1-x)^i}{i} \right) \Big] \\ \qquad \qquad \qquad \qquad \text{for } x \in (-1,0) \cup (0,1), \\ a_0 \text{ for } x = 0. \end{cases}$$

If $c \in \{4, 5, 6, \dots\}$ and $b \in \{2, 3, \dots, c-2\}$ one finds

$$f(x) = \begin{cases} \dfrac{1}{x^{c-1} (1-x)^{b-c+1}} \Big[a_0 (c-1) \Big(\displaystyle\sum_{i=\overline{0,c-b-2}} (-1)^i \dfrac{C_{c-2}^i}{i+b-c+1} + \\ \quad + \displaystyle\sum_{i=\overline{c-b,c-2}} (-1)^i \dfrac{C_{c-2}^i}{i+b-c+1} \Big) - \\ \quad -a_0 (c-1) \Big(\displaystyle\sum_{i=\overline{0,c-b-2}} (-1)^i \dfrac{C_{c-2}^i (1-x)^{i+b-c+1}}{i+b-c+1} + \\ \quad + (-1)^{c-b+1} C_{c-2}^{c-b-1} \ln(1-x) + \\ + \displaystyle\sum_{i=\overline{c-b,c-2}} (-1)^i \dfrac{C_{c-2}^i (1-x)^{i+b-c+1}}{i+b-c+1} \Big) \Big] \text{ for } x \in (-1,0) \cup (0,1), \\ a_0 \text{ for } x = 0. \end{cases}$$

If $c \in Q \setminus Z$, $b \in \{-1, -2, \dots\}$ and $c > 1$ one finds

$$f(x) = \begin{cases} \dfrac{-a_0 (c-1)}{x^{c-1} (1-x)^{b-c+1}} \displaystyle\sum_{i=\overline{0,-b}} C_{-b}^i \dfrac{\left(\frac{1-x}{x} \right)^{i+1+b-c}}{i+1+b-c} \text{ for } x \in (0,1), \\ a_0 \text{ for } x = 0, \\ \dfrac{-a_0 (c-1)}{(-x)^{c-1} (1-x)^{b-c+1}} \displaystyle\sum_{i=\overline{0,-b}} (-1)^{b+i} C_{-b}^i \dfrac{\left(\frac{1-x}{-x} \right)^{i+1+b-c}}{i+1+b-c} \\ \qquad \qquad \qquad \qquad \text{for } x \in (-1,0). \end{cases}$$

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New identities and inequalities in triangle

Mihály Bencze[14]

ABSTRACT. In this paper we present some new identities and inequalities in triangle

MAIN RESULTS

Theorem 1. If $x, y, z \in C$, $x+y \neq 0$, $y+z \neq 0$, $z+x \neq 0$, then

$$\frac{x-y}{x+y}+\frac{y-z}{y+z}+\frac{z-x}{z+x}=\frac{(x-y)(y-z)(z-x)}{(x+y)(y+z)(z+x)}$$

and

$$\left|\frac{x-y}{x+y}\right|+\left|\frac{y-z}{y+z}\right|+\left|\frac{z-x}{z+x}\right| \geq \left|\frac{(x-y)(y-z)(z-x)}{(x+y)(y+z)(z+x)}\right|$$

Proof. We have the followings:

$$\frac{x-y}{x+y}+\frac{y-z}{y+z}+\frac{z-x}{z+x}=\frac{x-y}{x+y}+\frac{y-z}{y+z}-\left(\frac{x-y}{z+x}+\frac{y-z}{z+x}\right)=$$

$$=\left(\frac{x-y}{x+y}-\frac{x-y}{z+x}\right)+\left(\frac{y-z}{y+z}-\frac{y-z}{z+x}\right)=\frac{(x-y)(z-y)}{(x+y)(z+x)}+\frac{(x-y)(y-z)}{(y+z)(z+x)}=$$

$$=\frac{(x-y)(y-z)(z-x)}{(x+y)(y+z)(z+x)} \text{ and}$$

$$\left|\frac{(x-y)(y-z)(z-x)}{(x+y)(y+z)(z+x)}\right|=\left|\frac{x-y}{x+y}+\frac{y-z}{y+z}+\frac{z-x}{z+x}\right| \leq$$

$$\leq\left|\frac{x-y}{x+y}\right|+\left|\frac{y-z}{y+z}\right|+\left|\frac{z-x}{z+x}\right|$$

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Corollary 1.1. In all triangle ABC holds

1). $\sum \frac{a-b}{a+b} = \frac{\prod(a-b)}{2s(s^2+r^2+2Rr)}$
2). $\sum \frac{b-a}{c} = \frac{\prod(b-a)}{4sRr}$
3). $\sum \frac{h_a-h_b}{h_a+h_b} = \frac{R^2\prod(h_a-h_b)}{s^2r(s^2+r^2+2Rr)}$
4). $\sum \frac{r_a-r_b}{r_a+r_b} = \frac{\prod(r_a-r_b)}{4s^2R}$
5). $\sum tg\frac{A-B}{2}tg\frac{C}{2} = \frac{r}{s}\prod tg\frac{A-B}{2}$
6). $\sum tg\frac{A-B}{2}ctg\frac{C}{2} = \frac{s}{r}\prod tg\frac{A-B}{2}$
7). $\sum \frac{\sin\frac{A-B}{2}}{\cos\frac{C}{2}} = \frac{4R}{s}\prod \sin\frac{A-B}{2}$
8). $\sum \frac{\sin\frac{A-B}{2}\cos\frac{C}{2}}{1-\cos\frac{A-B}{2}\sin\frac{C}{2}} = \frac{8sR^2\prod\sin\frac{A-B}{2}}{(2R-r)(s^2+r^2-8Rr)-2Rr^2}$
9). $\sum \frac{\sin\frac{A-B}{2}\cos\frac{C}{2}}{1+\cos\frac{A-B}{2}\sin\frac{C}{2}} = \frac{8sR^2\prod\sin\frac{A-B}{2}}{(4R+r)^3+s^2(2R+r)}$

Proof. In Theorem 1 we take
$(x,y,z) \in \{(a,b,c);(s-a,s-b,s-c);(h_a,h_b,h_c);(r_a,r_b,r_c);$
$(\sin A,\sin B,\sin C);(\cos A,\cos B,\cos C);\left(tg\frac{A}{2},tg\frac{B}{2},tg\frac{C}{2}\right);$
$\left(\sin^2\frac{A}{2},\sin^2\frac{B}{2},\sin^2\frac{C}{2}\right);\left(\cos^2\frac{A}{2},\cos^2\frac{B}{2},\cos^2\frac{C}{2}\right)\}$

Corollary 1.2. In all triangle ABC holds

1). $\sum \frac{|a-b|}{a+b} \geq \frac{\prod|a-b|}{2s(s^2+r^2+2Rr)}$
2). $\sum \frac{|b-a|}{c} \geq \frac{\prod|b-a|}{4sRr}$
3). $\sum \frac{|h_a-h_b|}{h_a+h_b} \geq \frac{R^2\prod|h_a-h_b|}{s^2r(s^2+r^2+2Rr)}$
4). $\sum \frac{|r_a-r_b|}{r_a+r_b} \geq \frac{\prod|r_a-r_b|}{4s^2R}$
5). $\sum \left|tg\frac{A-B}{2}\right| tg\frac{C}{2} \geq \frac{r}{s}\prod\left|tg\frac{A-B}{2}\right|$
6). $\sum \left|tg\frac{A-B}{2}\right| ctg\frac{C}{2} \geq \frac{s}{r}\prod\left|tg\frac{A-B}{2}\right|$
7). $\sum \frac{\left|\sin\frac{A-B}{2}\right|}{\cos\frac{C}{2}} \geq \frac{4R}{s}\prod\left|\sin\frac{A-B}{2}\right|$
8). $\sum \frac{\left|\sin\frac{A-B}{2}\right|\cos\frac{C}{2}}{1-\cos\frac{A-B}{2}\sin\frac{C}{2}} \geq \frac{8sR^2\prod\left|\sin\frac{A-B}{2}\right|}{(2R-r)(s^2+r^2-8Rr)-2Rr^2}$
9). $\sum \frac{\left|\sin\frac{A-B}{2}\right|\cos\frac{C}{2}}{1+\cos\frac{A-B}{2}\sin\frac{C}{2}} \geq \frac{8sR^2\prod\left|\sin\frac{A-B}{2}\right|}{(4R+r)^3+s^2(2R+r)}$

Corollary 1.3. Denote K, L, M the midpoints of sides AB, BC, CA in triangle ABC, then for all points P in the plane of triangle, holds

$$\frac{AB}{PK}+\frac{BC}{PL}+\frac{CA}{PM}\geq\frac{AB\cdot BC\cdot CA}{4PK\cdot PL\cdot PM}$$

Proof. If $A\left(a\right), B\left(b\right), C\left(c\right), P\left(t\right), K\left(\frac{a+b}{2}\right), L\left(\frac{b+c}{2}\right), M\left(\frac{c+a}{2}\right)$, $x=t-a$, $y=t-b$, $z=t-c$, then from Theorem 1 we get

$$\sum\left|\frac{x-y}{x+y}\right|=\frac{1}{2}\sum\left|\frac{a-b}{z-\frac{a+b}{2}}\right|\geq\prod\left|\frac{x-y}{x+y}\right|=\frac{1}{8}\prod\left|\frac{a-b}{z-\frac{a+b}{2}}\right| \text{ or}$$

$$\sum\frac{AB}{PK}\geq\frac{1}{4}\prod\frac{AB}{PK}$$

Because $AB\cdot BC\cdot CA=4sRr$, then we have the following:

Remark.

$$\sum\frac{AB}{PK}\geq\frac{sRr}{\prod PK}$$

Corollary 1.4. In all triangle ABC holds
1). $\frac{2c}{a}+\frac{a}{m_a}+2\geq\frac{c}{2m_a}$
2). $\frac{2a}{b}+\frac{b}{m_b}+2\geq\frac{a}{2m_b}$
3). $\frac{2b}{c}+\frac{c}{m_c}+2\geq\frac{b}{2m_c}$
Proof. In Corollary 1.3 we take $P\equiv A$, $P\equiv B, P\equiv C$.

Corollary 1.5. In all triangle ABC holds
1). $\frac{2(b+c)}{a}+\frac{a}{m_a}\geq\frac{16sRr}{a^2m_a}$
2). $\frac{2(c+a)}{b}+\frac{b}{m_b}\geq\frac{16sRr}{b^2m_b}$
3). $\frac{2(a+b)}{c}+\frac{c}{m_c}\geq\frac{16sRr}{c^2m_c}$
Proof. In Corollary 1.3 denote P the midpoint of KM.

Corollary 1.6. In all triangle ABC holds

$$\sum\left|tgA\right|\geq\frac{srR^2}{2\left|s^2-(4R+r)^2\right|}$$

Proof. In Corollary 1.3 we take $P\equiv O$.

Corollary 1.7. In all triangle ABC holds

$$\sum \frac{a\cos\frac{C}{2}}{(s-c)\sqrt{2a^2+2b^2-c^2-4ab+bc+ca}} \geq$$

$$\geq \frac{8s}{r\prod\sqrt{2a^2+2b^2-c^2-4ab+bc+ca}}$$

Proof. In Corollary 1.3 we take $P\equiv I$.

Corollary 1.8. In all triangle ABC holds

$$\sum\frac{a}{m_a}\geq\frac{9sRr}{m_am_bm_c}$$

Proof. In Corollary 1.3 we take $P\equiv G$.

Corollary 1.9. Denote K, L, M the midpoints of sides AB, BC, CA and M_1, K_1, L_1 the midpoints of sides KL, LM, MK in triangle ABC, then for all P in the plane of the triangle we get:

$$\frac{AC}{PM_1}+\frac{BA}{PK_1}+\frac{CB}{PL_1}\geq\frac{AB\cdot BC\cdot CA}{16PM_1\cdot PK_1\cdot PL_1}$$

Proof. If $A(a)$, $B(b)$, $C(c)$, $K\left(\frac{a+b}{2}\right)$, $L\left(\frac{b+c}{2}\right)$, $M\left(\frac{c+a}{2}\right)$, $M_1\left(\frac{a+2b+c}{4}\right)$, $K_1\left(\frac{b+2c+a}{4}\right)$, $L_1\left(\frac{c+2a+b}{4}\right)$, $P(t)$, $x=t-\frac{a+b}{2}$, $y=t-\frac{b+c}{2}$, $z=t-\frac{c+a}{2}$, then from Theorem 1 we get the desired result.

Problem. Determine all triangles ABC in which
$\sum|(a-b)(b-c)(c+a)|\geq$
$6s\left(s^2+r^2+2Rr\right)+\sqrt{2}\sum(b+c)(c+a)\sqrt{a^2+b^2}$.

Theorem 2. If $x, y, z\in C$, $x+y\neq0$, $y+z\neq0$, $z+x\neq0$, then

$$\left(\frac{(x-y)(y-z)(z-x)}{(x+y)(y+z)(z+x)}\right)^3-\left(\frac{x-y}{x+y}\right)^3-\left(\frac{y-z}{y+z}\right)^3-\left(\frac{z-x}{z+x}\right)^3=$$

$$=\frac{24xyz(x-z)(y-x)(z-y)}{(x+y)^2(y+z)^2(z+x)^2}\text{ and}$$

$$\left|\frac{(x-y)(y-z)(z-x)}{(x+y)(y+z)(z+x)}\right|^3+\left|\frac{x-y}{x+y}\right|^3+\left|\frac{y-z}{y+z}\right|^3+\left|\frac{z-x}{z+x}\right|^3\geq$$

$$\geq 24\left|\frac{xyz\,(x-z)\,(y-x)\,(z-y)}{(x+y)^2\,(y+z)^2\,(z+x)^2}\right|$$

Proof. If $u=\frac{x-y}{x+y}, v=\frac{y-z}{y+z}, w=\frac{z-x}{z+x}$, then $u+v+w=uvw$ and $(u+v+w)^3-\left(u^3+v^3+w^3\right)=3\,(u+v)\,(v+w)\,(w+u)$ etc.

Corollary 2.1. In all triangles ABC holds

1). $\left(\prod \frac{a-b}{a+b}\right)^3-\sum\left(\frac{a-b}{a+b}\right)^3=\frac{24Rr\prod(a-b)}{s(s^2+r^2+2Rr)^2}$

2). $\left(\prod \frac{b-a}{c}\right)^3-\sum\left(\frac{b-a}{c}\right)^3=\frac{3\prod(b-a)}{2sR^2}$

3). $\left(\prod \frac{h_a-h_b}{h_a+h_b}\right)^3-\sum\left(\frac{h_a-h_b}{h_a+h_b}\right)^3=\frac{48R^3\prod(h_a-h_b)}{s^2(s^2+r^2+2Rr)^2}$

4). $\left(\prod \frac{r_a-r_b}{r_a+r_b}\right)^3-\sum\left(\frac{r_a-r_b}{r_a+r_b}\right)^3=\frac{3r\prod(r_a-r_b)}{2s^2R^2}$

5). $\left(\frac{r}{s}\prod tg\frac{A-B}{2}\right)^3-\sum\left(tg\frac{A-B}{2}tg\frac{C}{2}\right)^3=\frac{384R^3r^2\prod\sin\frac{A-B}{2}}{s(s^2+r^2+2Rr)^2}$

6). $\left(\frac{s}{r}\prod tg\frac{A-B}{2}\right)^3-\sum\left(tg\frac{A-B}{2}ctg\frac{C}{2}\right)^3=\frac{192sR^3\left(s^2-(2R+r)^2\right)\prod\sin\frac{A-B}{2}}{r^2(s^2+r^2+2Rr)^2}$

7). $\left(\frac{4R}{s}\prod\sin\frac{A-B}{2}\right)^3-\sum\left(\frac{\sin\frac{A-B}{2}}{\cos\frac{C}{2}}\right)^3=\frac{24r\prod\sin\frac{A-B}{2}}{s}$

8).
$$\left(\frac{s}{4R}\prod\frac{\sin\frac{A-B}{2}}{1-\cos\frac{A-B}{2}\sin\frac{C}{2}}\right)^3-\sum\left(\frac{\sin\frac{A-B}{2}\cos\frac{C}{2}}{1-\cos\frac{A-B}{2}\sin\frac{C}{2}}\right)^3=\frac{3072sr^2R^3\prod\sin\frac{A-B}{2}}{((2R-r)(s^2+r^2-8Rr)-2Rr^2)^3}$$

9). $\left(\frac{s}{4R}\prod\frac{\sin\frac{A-B}{2}}{1+\cos\frac{A-B}{2}\sin\frac{C}{2}}\right)^3-\sum\left(\frac{\sin\frac{A-B}{2}\cos\frac{C}{2}}{1+\cos\frac{A-B}{2}\sin\frac{C}{2}}\right)^3=\frac{3072s^3R^3\prod\sin\frac{A-B}{2}}{\left((4R+r)^3+s^2(2R+r)\right)^2}$

Proof. In Theorem 2 we take
$(x,y,z)\in\{(a,b,c)\,;(s-a,s-b,s-c)\,;(h_a,h_b,h_c)\,;(r_a,r_b,r_c)\,;$
$(\sin A,\sin B,\sin C)\,;(\cos A,\cos B,\cos C)\,;\left(tg\frac{A}{2},tg\frac{B}{2},tg\frac{C}{2}\right);$
$\left(\sin^2\frac{A}{2},\sin^2\frac{B}{2},\sin^2\frac{C}{2}\right);\left(\cos^2\frac{A}{2},\cos^2\frac{B}{2},\cos^2\frac{C}{2}\right)\}$.

Corollary 2.2. Denote K,L,M the midpoints of sides AB,BC,CA in triangle ABC, then for all points P in the plane of triangle we get:

$$\left(\frac{sRr}{PK\cdot PL\cdot PM}\right)^3+\left(\frac{c}{PK}\right)^3+\left(\frac{a}{PL}\right)^3+\left(\frac{b}{PM}\right)^3\geq\frac{12sRr\cdot PA\cdot PB\cdot PC}{PK^2\cdot PL^2\cdot PM^2}$$

Proof. If $A\,(a,)\,B\,(b)\,,C\,(c)\,,P\,(t)\,,\;K\left(\frac{a+b}{2}\right),\;L\left(\frac{b+c}{2}\right),\;M\left(\frac{c+a}{2}\right),\;x=t-a$, $y=t-b$, $z=t-c$, then from the inequality of Theorem 2 we get the desired result.

Theorem 3. If $x, y, z \in C$, $x+y \neq 0$, $y+z \neq 0$, $z+x \neq 0$, then

$$\sum \left(\frac{x-y}{x+y}\right)^3 = \left(\prod \frac{x-y}{x+y}\right)\left(3+2\sum\left(\frac{xz-y^2}{(x+y)(y+z)}\right)^2\right)$$

Proof. If $u=\frac{x-y}{x+y}$, $v=\frac{y-z}{y+z}$, $w=\frac{z-x}{z+x}$, then $u+v+w=uvw$ and
$u^3+v^3+w^3=3uvw+\frac{1}{2}(u+v+w)\left((u-v)^2+(v-w)^2+(w-u)^2\right)$

Corollary 3.1. In all triangle ABC holds

1). $\sum\left(\frac{a-b}{a+b}\right)^3=\left(\prod\frac{a-b}{a+b}\right)\left(3+2\sum\left(\frac{ac-b^2}{(a+b)(b+c)}\right)^2\right)$

2). $\sum\left(\frac{b-a}{c}\right)^3=\left(\prod\frac{b-a}{c}\right)\left(3+2\sum\left(\frac{(a+c)b-\left(a^2+c^2\right)}{ac}\right)\right)^2$

3). $\sum\left(tg\frac{A-B}{2}tg\frac{C}{2}\right)^3=\left(\frac{r}{s}\prod tg\frac{A-B}{2}\right)\left(3+\frac{1}{8}\sum\left(\frac{\sin A\sin C-\sin^2 B}{\cos\frac{A}{2}\cos\frac{C}{2}\cos\frac{A-B}{2}\cos\frac{B-C}{2}}\right)^2\right)$

4). $\sum\left(tg\frac{A-B}{2}ctg\frac{C}{2}\right)^3=\left(\frac{s}{r}\prod tg\frac{A-B}{2}\right)\left(3+\frac{1}{8}\sum\left(\frac{\cos A\cos C-\cos^2 B}{\sin\frac{A}{2}\sin\frac{C}{2}\cos\frac{A-B}{2}\cos\frac{B-C}{2}}\right)^2\right)$

5). $\sum\left(\frac{\sin\frac{A-B}{2}}{\cos\frac{C}{2}}\right)^3=\left(\frac{4R}{s}\prod\sin\frac{A-B}{2}\right)\left(3+\sum\left(tg\frac{A}{2}tg\frac{C}{2}-tg^2\frac{B}{2}\right)^2\cos^4\frac{B}{2}\right)$

6). $\sum\left(\frac{\sin\frac{A-B}{2}\cos\frac{C}{2}}{1-\cos\frac{A-B}{2}\sin\frac{C}{2}}\right)^3=$

$=\left(\frac{s}{4R}\prod\frac{\sin\frac{A-B}{2}}{1-\cos\frac{A-B}{2}\sin\frac{C}{2}}\right)\left(3+2\sum\left(\frac{\sin^2\frac{A}{2}\sin^2\frac{C}{2}-\sin^4\frac{B}{2}}{\left(1-\cos\frac{A-B}{2}\sin\frac{C}{2}\right)\left(1-\cos\frac{B-C}{2}\sin\frac{A}{2}\right)}\right)^2\right)$

7). $\sum\left(\frac{\sin\frac{A-B}{2}\cos\frac{C}{2}}{1+\cos\frac{A-B}{2}\sin\frac{C}{2}}\right)^3=$

$=\left(\frac{s}{4R}\prod\frac{\sin\frac{A-B}{2}}{1+\cos\frac{A-B}{2}\sin\frac{C}{2}}\right)\left(3+2\sum\left(\frac{\cos^2\frac{A}{2}\cos^2\frac{C}{2}-\cos^4\frac{B}{2}}{\left(1+\cos\frac{A-B}{2}\sin\frac{C}{2}\right)\left(1+\cos\frac{B-C}{2}\sin\frac{A}{2}\right)}\right)^2\right)$

Proof. In Theorem 3 we take $(x,y,z)\in\{(a,b,c);(s-a,s-b,s-c);$
$(\sin A,\sin B,\sin C);(\cos A,\cos B,\cos C);\left(tg\frac{A}{2},tg\frac{B}{2},tg\frac{C}{2}\right);$
$\left(\sin^2\frac{A}{2},\sin^2\frac{B}{2},\sin^2\frac{C}{2}\right);\left(\cos^2\frac{A}{2},\cos^2\frac{B}{2},\cos^2\frac{C}{2}\right)\}.$

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About some elementary inequalities

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ABSTRACT. In this paper we present some inequalities connected to inf and sup of elementary functions.

MAIN RESULTS

Theorem 1. (is due to Sándor, J. and Szabó, V.E.S., On an inequality for the sum of infimum of functions, J. Math. Anal. Appl. 204(1996), pp. 646-654) If $a_k, b_k > 0$ $(k = 1, 2, ..., n)$, then

$$\prod_{k=1}^{n} \left(\frac{b_k}{a_k}\right)^{a_k} \le \left(\frac{\sum\limits_{k=1}^{n} b_k}{\sum\limits_{k=1}^{n} a_k}\right)^{\sum\limits_{k=1}^{n} a_k}$$

Proof. If $f_k(x) = a_k \ln x + \frac{b_k}{x^\alpha}$, where $x > 0$, $\alpha > 0$ then $f_k'(x) = \frac{a_k x^\alpha - b_k^\alpha}{x^{\alpha+1}}$ and $f_k'(x) = 0$ for $x = \left(\frac{\alpha b_k}{a_k}\right)^{\frac{1}{\alpha}}$, therefore
$\inf\limits_{x>0} f_k(x) = \min\limits_{x>0} f_k(x) = \ln\left(\frac{\alpha b_k}{a_k}\right)^{\frac{a_k}{\alpha}} + \frac{a_k}{\alpha}$.
Using the definition of infimum we get:

$$\sum_{k=1}^{n} \inf_{x>0} f_k(x) = \sum_{k=1}^{n} \min_{x>0} f_k(x) = \sum_{k=1}^{n} \left(\ln\left(\frac{\alpha b_k}{a_k}\right)^{\frac{a_k}{\alpha}} + \frac{a_k}{\alpha}\right) = \ln \prod_{k=1}^{n} \left(\frac{\alpha b_k}{a_k}\right)^{\frac{a_k}{\alpha}} +$$

$$+\frac{1}{\alpha}\sum_{k=1}^{n} a_k \le \inf_{x>0} \sum_{k=1}^{n} f_k(x) = \min_{x>0} \sum_{k=1}^{n} f_k(x) = \ln\left(\frac{\alpha \sum\limits_{k=1}^{n} b_k}{\sum\limits_{k=1}^{n} a_k}\right)^{\frac{1}{\alpha}\sum\limits_{k=1}^{n} a_k} + \frac{1}{\alpha}\sum_{k=1}^{n} a_k$$

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which implies the desired inequality.

Corollary 1.1. If $b_k > 0$ $(k = 1, 2, ..., n)$, then

$$\frac{1}{n}\sum_{k=1}^{n} b_k \geq \sqrt[n]{\prod_{k=1}^{n} b_k}$$

Proof. In Theorem 1 we take $a_k = 1$ $(k = 1, 2, ..., n)$ and this is a new and elementary proof of AM-GM inequality.

Corollary 1.2. If $a_k > 0$ $(k = 1, 2, ..., n)$, then

$$\prod_{k=1}^{n} a_k^{a_k} \geq \left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)^{\sum\limits_{k=1}^{n} a_k}$$

Proof. In Theorem 1 we take $b_k = 1$ $(k = 1, 2, ..., n)$.

Corollary 1.3. In all triangle ABC holds
1). $\prod a^a \geq \left(\frac{2s}{3}\right)^{2s}$
2). $\prod (s-a)^{s-a} \geq \left(\frac{s}{3}\right)^{s}$
3). $\prod h_a^{h_a} \geq \left(\frac{s^2+r^2+4Rr}{6R}\right)^{\frac{s^2+r^2+4Rr}{2R}}$
4). $\prod r_a^{r_a} \geq \left(\frac{4R+r}{3}\right)^{4R+r}$
5). $\prod \left(\sin\frac{A}{2}\right)^{2\sin^2\frac{A}{2}} \geq \left(\frac{2R-r}{6R}\right)^{\frac{2R-r}{2R}}$
6). $\prod \left(\cos\frac{A}{2}\right)^{2\cos^2\frac{A}{2}} \geq \left(\frac{4R+r}{6R}\right)^{\frac{4R+r}{2R}}$
7). $\prod A^A \geq \left(\frac{\pi}{3}\right)^{\pi}$
Proof. In Corollary 1.2 we take $(a_1, a_2, a_3) \in$
$\left\{(a,b,c); (s-a, s-b, s-c); (h_a, h_b, h_c); (r_a, r_b, r_c); \left(\sin^2\frac{A}{2}, \sin^2\frac{B}{2}, \sin^2\frac{C}{2}\right);\right.$
$\left.\left(\cos^2\frac{A}{2}, \cos^2\frac{B}{2}, \cos^2\frac{C}{2}\right)\right\}$.

Corollary 1.4. In all tetrahedron $ABCD$ holds
1). $\prod \left(\frac{1}{h_a}\right)^{\frac{1}{h_a}} \geq \left(\frac{1}{4r}\right)^{\frac{1}{r}}$
2). $\prod \left(\frac{1}{r_a}\right)^{\frac{1}{r_a}} \geq \left(\frac{1}{2r}\right)^{\frac{2}{r}}$
3). $\prod A^A \geq \left(\frac{\pi}{3}\right)^{2\pi}$

Corollary 1.5. In all convex polygon $A_1A_2...A_n$ holds

$$\prod_{k=1}^{n} A_k^{A_k} \geq \left(\frac{(n-2)\pi}{n}\right)^{(n-2)\pi}$$

Remark 1. *If* $f_k(x) = a_k c^x - \alpha x_k$ $(k = 1, 2, ..., n)$ *where* $x > 0$, $c \in (0, 1)$, $\alpha > 0$, *then we obtain the inequality from Theorem 1.*

Theorem 2. If $a_k, b_k > 0$ $(k = 1, 2, ..., n)$, $\alpha \geq \beta > 0$ then

$$\sum_{k=1}^{n} \left(a_k \left(\frac{(\beta+1) b_k}{(\alpha+1) a_k} \right)^{\frac{\alpha+1}{\alpha-\beta}} - b_k \left(\frac{(\beta+1) b_k}{(\alpha+1) a_k} \right)^{\frac{\beta+1}{\alpha-\beta}} \right) \leq$$

$$\leq \left(\sum_{k=1}^{n} a_k \right) \left(\frac{(\beta+1) \sum_{k=1}^{n} b_k}{(\alpha+1) \sum_{k=1}^{n} a_k} \right)^{\frac{\alpha+1}{\alpha-\beta}} - \left(\sum_{k=1}^{n} b_k \right) \left(\frac{(\beta+1) \sum_{k=1}^{n} b_k}{(\alpha+1) \sum_{k=1}^{n} a_k} \right)^{\frac{\beta+1}{\alpha-\beta}}$$

Proof. We consider the functions

$$f_k(x) = a_k x^{\alpha+1} - b_k x^{\beta+1} \ (k = 1, 2, ..., n), \ x > 0,$$

and after then we apply the method of Theorem 1.

Theorem 3. If $a_k, b_k > 0$ $(k = 1, 2, ..., n)$, then

$$\sum_{k=1}^{n} \sqrt{a_k b_k} \leq \sqrt{\left(\sum_{k=1}^{n} a_k \right) \left(\sum_{k=1}^{n} b_k \right)}$$

Proof. We use the method presented in Theorem 1 for the functions

$$f_k(x) = a_k e^x - b_k e^{-x} \ (k = 1, 2, ..., n), \ x \in R$$

Theorem 4. If $a_k, b_k > 0$ $(k = 1, 2, ..., n)$, then

$$\prod_{k=1}^{n} \frac{a_k^{a_k} b_k^{b_k}}{(a_k + b_k)^{a_k + b_k}} \geq \frac{\left(\sum_{k=1}^{n} a_k \right)^{\sum_{k=1}^{n} a_k} \left(\sum_{k=1}^{n} b_k \right)^{\sum_{k=1}^{n} b_k}}{\left(\sum_{k=1}^{n} (a_k + b_k) \right)^{\sum_{k=1}^{n} (a_k + b_k)}}$$

Proof. If $f_k(x) = (\sin x)^{a_k} (\cos x)^{b_k}$, where $x \in \left[0, \frac{\pi}{2}\right]$ and $a_k, b_k > 0$ $(k = 1, 2, ..., n)$, then $f_k'(x) = f_k(x)\left(a_k \cos^2 x - b_k \sin^2 x\right) = 0$ if and only if $x_k = arctg\sqrt{\frac{a_k}{b_k}}$, therefore

$$\sup_{x\in\left[0,\frac{\pi}{2}\right]} f_k(x) = \max_{x\in\left[0,\frac{\pi}{2}\right]} f_k(x) = \max_{x\in\left[0,\frac{\pi}{2}\right]} (\sin x)^{a_k} (\cos x)^{b_k} \leq$$

$$\leq \left(\sin\left(arctg\sqrt{\frac{a_k}{b_k}}\right)\right)^{a_k} \left(\cos\left(arctg\sqrt{\frac{a_k}{b_k}}\right)\right)^{b_k} = \sqrt{\frac{a_k^{a_k} b_k^{b_k}}{(a_k+b_k)^{a_k+b_k}}}$$

$(k = 1, 2, ..., n)$, therefore

$$\prod_{k=1}^{n} \sup_{x\in\left[0,\frac{\pi}{2}\right]} f_k(x) = \prod_{k=1}^{n} \max_{x\in\left[0,\frac{\pi}{2}\right]} f_k(x) = \prod_{k=1}^{n} \sqrt{\frac{a_k^{a_k} b_k^{b_k}}{(a_k+b_k)^{a_k+b_k}}} \geq$$

$$\geq \sup_{x\in\left[0,\frac{\pi}{2}\right]} \prod_{k=1}^{n} f_k(x) = \max_{x\in\left[0,\frac{\pi}{2}\right]} \prod_{k=1}^{n} f_k(x) = \sqrt{\frac{\left(\sum\limits_{k=1}^{n} a_k\right)^{\sum\limits_{k=1}^{n} a_k} \left(\sum\limits_{k=1}^{n} b_k\right)^{\sum\limits_{k=1}^{n} b_k}}{\left(\sum\limits_{k=1}^{n} (a_k+b_k)\right)^{\sum\limits_{k=1}^{n} (a_k+b_k)}}}$$

Open Question 1. If $a_{ij} > 0$ $(i = 1, 2, ..., n; j = 1, 2, ..., m)$, then

$$\prod_{i=1}^{n} \frac{\prod\limits_{j=1}^{m} a_{ij}^{a_{ij}}}{\left(\sum\limits_{j=1}^{m} a_{ij}\right)^{\sum\limits_{j=1}^{m} a_{ij}}} \geq \frac{\prod\limits_{j=1}^{m} \left(\sum\limits_{i=1}^{n} a_{ij}\right)^{\sum\limits_{i=1}^{n} a_{ij}}}{\left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{m} a_{ij}\right)\right)^{\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{m} a_{ij}\right)}}$$

Corollary 4.1. If $a_k > 0$ $(k = 1, 2, ..., n)$, then

$$\prod_{k=1}^{n} a_k^{a_k} \geq \prod_{cyclic} \left(\frac{a_1+a_2}{2}\right)^{\frac{a_1+a_2}{2}} \geq \prod_{cyclic} a_1^{\frac{a_1+a_2+a_n}{3}}$$

Proof. In Theorem 4 we take $b_k = a_{k+1}$ $(k = 1, 2, ..., n)$, therefore

$$\prod_{k=1}^{n} a_k^{a_k} \geq \prod_{cyclic} \left(\frac{a_1+a_2}{2}\right)^{\frac{a_1+a_2}{2}}$$

Using the AM-GM inequality we get

$$\prod_{cyclic}\left(\frac{a_1+a_2}{2}\right)^{\frac{a_1+a_2}{2}} \geq \prod_{cyclic}\left(\sqrt{a_1a_2}\right)^{\frac{a_1+a_2}{2}} = \prod_{cyclic} a_1^{\frac{a_1+a_2+a_n}{4}} \prod_{k=1}^{n} a_k^{\frac{a_k}{4}}$$

therefore

$$\prod_{cyclic}\left(\frac{a_1+a_2}{2}\right)^{\frac{a_1+a_2}{2}} \geq \prod_{cyclic} a_1^{\frac{a_1+a_2+a_n}{4}} \prod_{cyclic}\left(a_k^{a_k}\right)^{\frac{1}{4}} \geq$$

$$\geq \prod_{cyclic} a_1^{\frac{a_1+a_2+a_n}{4}} \left(\prod_{cyclic}\left(\frac{a_1+a_2}{2}\right)^{\frac{a_1+a_2}{2}}\right)^{\frac{1}{4}}$$

therefore

$$\prod_{cyclic}\left(\frac{a_1+a_2}{2}\right)^{\frac{a_1+a_2}{2}} \geq \prod_{cyclic} a_1^{\frac{a_1+a_2+a_n}{3}}$$

Corollary 4.2. If $a,b,c>0$, then

$$a^a b^b c^c \geq \left(\frac{a+b}{2}\right)^{\frac{a+b}{2}} \left(\frac{b+c}{2}\right)^{\frac{b+c}{2}} \left(\frac{c+a}{2}\right)^{\frac{c+a}{2}} \geq (abc)^{\frac{a+b+c}{3}}$$

Proof. In Corollary 4.1 we take $n=3$, this was a problem in USAMO 1974.

Open Question 2. If $a_i>0$ $(i=1,2,...,n)$ and $k\in\{1,2,...,n\}$, then

$$\prod_{i=1}^{n} a_i^{a_i} \geq \prod_{cyclic}\left(\frac{a_1+a_2+...+a_k}{k}\right)^{\frac{a_1+a_2+...+a_k}{k}} \geq \left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}\sum\limits_{i=1}^{n} a_i}$$

Remark 2. *If* $f_k(x)=x^{a_k}(1-x^{\alpha})^{\frac{b_k}{\alpha}}$, $x\in[0,1]$, $a_k,b_k>0$ *using the method presented in the proof of Theorem 4 we obtain the same inequality like in Theorem 4.*

Theorem 5. If $b_k > a_k > 0$ $(k = 1, 2, ..., n)$, then

$$\prod_{k=1}^{n} \frac{a_k^{a_k} b_k^{-b_k}}{(b_k - a_k)^{b_k - a_k}} \le \frac{\left(\sum\limits_{k=1}^{n} a_k\right)^{\sum\limits_{k=1}^{n} a_k} \left(\sum\limits_{k=1}^{n} b_k\right)^{-\sum\limits_{k=1}^{n} b_k}}{\left(\sum\limits_{k=1}^{n} (b_k - a_k)\right)^{\sum\limits_{k=1}^{n} (b_k - a_k)}}$$

Proof. We apply the method described in Theorem 4 for the functions

$f_k(x) = (shx)^{a_k} (chx)^{-b_k}$, $x > 0$, $b_k > a_k > 0$ $(k = 1, 2, ..., n)$. We have

$$(shx)^{a_k} (chx)^{-b_k} \le \sqrt{\frac{a_k^{a_k} b_k^{-b_k}}{(b_k - a_k)^{b_k - a_k}}}$$

$(k = 1, 2, ..., n)$ etc.

Corollary 5.1. If $0 < a_1 < a_2 < ... < a_n < a_{n+1}$, then

$$\prod_{k=1}^{n} \frac{a_k^{a_k} a_{k+1}^{-a_{k+1}}}{(a_{k+1} - a_k)^{a_{k+1} - a_k}} \le \frac{\left(\sum\limits_{k=1}^{n} a_k\right) \left(\sum\limits_{k=1}^{n} a_{k+1}\right)}{(a_{n+1} - a_1)^{a_{n+1} - a_1}}$$

Proof. In Theorem 5 we take $b_k = a_{k+1}$ $(k = 1, 2, ..., n)$.

Corollary 5.2. If $0 < a_1 < a_2 < ... < a_n < a_{n+1}$ is an

arithmetical progression with ratio $r > 0$, then

$$\prod_{k=1}^{n} \frac{a_k^{a_k}}{a_{k+1}^{a_{k+1}}} \le \left(\frac{\left(\frac{n(a_1+a_2)}{2}\right)^{\frac{a_1+a_2}{2}} \left(\frac{n(a_2+a_{n+1})}{2}\right)^{\frac{a_2+a_{n+1}}{2}}}{n^r} \right)^n$$

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Improved exponential estimator for population variance using two auxiliary variables

Rajesh Singh, Pankaj Chauhan, Nirmala Sawan and Florentin Smarandache [16]

ABSTRACT. In this paper exponential ratio and exponential product type estimators using two auxiliary variables are proposed for estimating unknown popoulation variance S_y^2. Problem is extended to the case of two-phase sampling. Theoretical results are supported by an empirical study

1. INTRODUCTION

It is common practice to use the auxiliary variable for improving the precision of the estimate of a parameter. Out of many ratio and product methods are good examples in this context. When the correlation between the study and the auxiliary variate is positive (high) ratio method of estimation is quite effective. On the other hand, when this correlation is negative (high) product method of estimation can be employed effectively. Let y and (x, y) denotes the study variate and auxiliary variates taking the values y_i and (x_i, z_i) respectively, on the unit U_1 $(i = 1, 2, ..., N)$, where x is positively correlated with y and z is negatively correlated with y. To estimate

$$S_y^2 = \frac{1}{(N-1)} \sum_{i=1}^{N} (y_i - \overline{y})^2,$$

it is assumed that

$$S_x^2 = \frac{1}{(N-1)} \sum_{i=1}^{N} \left(x_i - \overline{X}\right)^2 \text{ and } S_z^2 = \frac{1}{(N-1)} \sum_{i=1}^{N} \left(z_i - \overline{Z}\right)^2$$

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are known. Assume that population size N is large so that the finite population correction terms are ignored.

Assume that a simple random samle of size n is drawn without replacement (SRSWOR) from U. The usual unbiased estimator of S_y^2 is

$$s_y^2 = \frac{1}{(n-1)} \sum_{i=1}^{n} (y_i - \overline{y})^2 \tag{1.1}$$

where $\overline{y} = \frac{1}{n} \sum\limits_{i=1}^{n} y_i$ is the sample mean of y.

When the population variance $S_x^2 = \frac{1}{(N-1)} \sum\limits_{i=1}^{n} \left(x_i - \overline{X}\right)^2$ is known, Isaki (1983) proposed a ratio estimator for S_y^2 as

$$t_k = s_y^2 \frac{S_x^2}{s_x^2} \tag{1.2}$$

where $s_x^2 = \frac{1}{(n-1)} \sum\limits_{i=1}^{n} \left(x_i - \overline{X}\right)^2$ is an unbiased estimator of S_x^2.

Upto the first order of approximation, the variance of S_y^2 and MSE of t_k (ignoring the finite population correction (fpc) term) are respectively given by

$$var\left(s_y^2\right) = \left(\frac{S_y^4}{n}\right)\left[\theta_{400} - 1\right] \tag{1.3}$$

$$MSE\left(t_k\right) = \left(\frac{S_y^4}{n}\right)\left[\theta_{400} + \theta_{040} - 2\theta_{220}\right] \tag{1.4}$$

where $\delta_{pqr} = \frac{\mu_{pqr}}{\left(\mu_{200}^{p/2}\mu_{020}^{q/2}\mu_{002}^{r/2}\right)}$, $\mu_{pqr} = \frac{1}{N} \sum\limits_{i=1}^{n} \left(y_i - \overline{Y}\right)^p \left(x_i - \overline{X}\right)^q \left(z_i - \overline{Z}\right)^r$;
p, q, r being the non-negative integers.
Following Bahl and Tuteja (1991), we propose exponential ratio type and exponential product type estimators for estimating population variance S_y^2 as

$$t_1 = s_y^2 \exp\left[\frac{S_x^2 - s_x^2}{S_x^2 + s_x^2}\right] \tag{1.5}$$

$$t_2 = s_y^2 \exp\left[\frac{s_z^2 - S_z^2}{s_z^2 + S_z^2}\right] \tag{1.6}$$

2. BIAS AND MSE OF PROPOSED ESTIMATORS

To obtain the bias and MSE of t_1, we write

$$s_y^2 = S_y^2\left(1+e_0\right), s_x^2 = S_x^2\left(1+e_1\right)$$

Such that $E\left(e_0\right) = E\left(e_1\right) = 0$ and

$$E\left(e_0^2\right) = \frac{1}{n}\left(\theta_{400}-1\right), E\left(e_1^2\right) = \frac{1}{n}\left(\theta_{040}-1\right), E\left(e_0 e_1\right) = \frac{1}{n}\left(\theta_{220}-1\right).$$

After simplification we get the bias and MSE of t_1 as

$$B\left(t_1\right) \cong \frac{S_y^2}{n}\left[\frac{\theta_{040}}{8} - \frac{\theta_{220}}{2} + \frac{3}{8}\right] \tag{2.1}$$

$$MSE\left(t_1\right) \cong \frac{S_y^2}{n}\left[\theta_{400} + \frac{\theta_{040}}{4} - \theta_{220} + \frac{1}{4}\right] \tag{2.2}$$

To obtain the bias and MSE of t_2, we write

$$s_y^2 = S_y^2\left(1+e_0\right), s_z^2 = S_Z^2\left(1+e_2\right)$$

Such that $E\left(e_0\right) = E\left(e_2\right) = 0$

$$E\left(e_2^2\right) = \frac{1}{n}\left(\theta_{004}-1\right), E\left(e_0 e_2\right) = \frac{1}{n}\left(\theta_{202}-1\right)$$

After simplification we get the bias and MSE of t_2 as

$$B\left(t_2\right) \cong \frac{S_y^2}{n}\left[\frac{\theta_{004}}{8} - \frac{\theta_{202}}{2} - \frac{5}{8}\right] \tag{2.3}$$

$$MSE\left(t_2\right) \cong \frac{S_y^2}{n}\left[\theta_{400} + \frac{\theta_{004}}{4} + \theta_{202} - \frac{9}{4}\right] \tag{2.4}$$

3. IMPROVED ESTIMATOR

Following Kadilar and Cingi (2006) and Singh et. al (2007), we propose an improved estimator for estimating population variance S_y^2 as -

$$t = s_y^2 \left[\alpha \exp\left\{ \frac{S_x^2 - s_x^2}{S_x^2 + s_x^2} \right\} + (1-\alpha)\exp\left\{ \frac{s_z^2 - S_z^2}{s_z^2 + S_z^2} \right\} \right] \tag{3.1}$$

where α is a real constant to be determined such that the MSE of t is minimum.
Expressing t in terms of e‘s, we have

$$t = S_y^2 (1 + e_0) \cdot$$

$$\cdot \left[\alpha \exp\left\{ -\frac{e_1}{2}\left(1+\frac{e_1}{2}\right)^{-1} \right\} + (1-\alpha)\exp\left\{ \frac{e_2}{2}\left(1+\frac{e_2}{2}\right)^{-1} \right\} \right] \tag{3.2}$$

Expanding the right hand side of (3.2) and retaining terms upto second power of e‘s, we have

$$t \cong S_y^2 \left[1 + e_0 + \frac{e_2}{2} + \frac{e_2^2}{8} + \frac{e_0 e_2}{2} + \alpha\left(-\frac{e_1}{2} + \frac{e_1^2}{8} \right) - \alpha\left(-\frac{e_2}{2} + \frac{e_2^2}{8} \right) + \right.$$

$$\left. + e_0\alpha\left(-\frac{e_1}{2} + \frac{e_1^2}{8} \right) - \alpha e_0 \left(\frac{e_2}{2} + \frac{e_2^2}{8} \right) \right] \tag{3.3}$$

Taking expectations of both sides of (3.30 and then substracting S_y^2 from both sides, we get the bias of the estimator t, upto the first order of approximation, as

$$B(t) = \frac{S_y^2}{n} \left[\frac{\alpha}{8}(\theta_{040} - 1) + \frac{(1-\alpha)}{8}(\theta_{004} - 1) + \frac{(1-\alpha)}{2}(\theta_{202} - 1) - \right.$$

$$\left. - \frac{\alpha}{2}(\theta_{220} - 1) \right] \tag{3.4}$$

From (3.4) we have

$$\left(t - S_y^2\right) \cong S_y^2 \left[e_0 - \frac{\alpha e_1}{2} + \frac{(1-\alpha)}{2} e_2 \right] \tag{3.5}$$

Squaring both the sides of (3.5) and then taking expectation, we get MSE of the estimator t, up to the first order of approximation, as

$$MSE \cong \frac{S_y^4}{n}\left[(\theta_{400}-1)+\frac{\alpha^2}{4}(\theta_{040}-1)+\frac{(1-\alpha^2)}{4}(\theta_{004}-1)-\right.$$

$$\left.-\alpha(\theta_{220}-1)+(1-\alpha)(\theta_{202}-1)-\frac{\alpha(1-\alpha)}{2}(\theta_{022}-1)\right] \tag{3.6}$$

Minimization of (3,6) with respect to α yields its optimum value as

$$\alpha=\frac{\{\theta_{004}+2(\theta_{220}+\theta_{202})+\theta_{022}-6\}}{(\theta_{040}+\theta_{004}+2\theta_{022}-4)}=\alpha_0\,(say) \tag{3.7}$$

Substitution of α_0 from (3.70 into (3.6) gives minimum value of MSE of t.

4. PROPOSED ESTIMATORS IN TWO_PHASE SAMPLING

In certain practical situations when S_x^2 is not known a priori, the technique of twophase or double sampling is used. This scheme requires collection of information on x and z the first phase sample s' of size n' $(n'<N)$ and on y for the second phase sample s of size n $(n<n')$ from the first phase sample. The estimators t_1, t_2 and t in two-phase sampling will take the following form, respectively

$$t_{1d}=s_y^2\exp\left[\frac{s_x'^2-s_x^2}{s_x'^2+s_x^2}\right] \tag{4.1}$$

$$t_{2d}=s_z^2\exp\left[\frac{s_z'^2-s_z^2}{s_z'^2+s_z^2}\right] \tag{4.2}$$

$$t_d=s_y^2\left[k\exp\left\{\frac{s_x'^2-s_x^2}{s_x'^2+s_x^2}\right\}+(1-k)\exp\left\{\frac{s_z'^2-s_z^2}{s_z'^2+s_z^2}\right\}\right] \tag{4.3}$$

To obtain the bias and MSE of t_{1d}, t_{2d}, t_d, we write

$$s_y^2=S_y^2(1+e_0),\, s_x^2=S_x^2(1+e_1),\, s_x'^2=S_x^2\left(1+e_1'\right)$$

$$s_z^2=S_z^2(1+e_2),\, s_z'^2=S_z^2\left(1+e_2'\right)$$

where

$$s_x'^2=\frac{1}{(n'-1)}\sum_{i=1}^{n'}\left(x_i-\overline{x'}\right)^2,\, s_z^2=\frac{1}{(n'-1)}\sum_{i=1}^{n'}\left(z_i-\overline{z'}\right)^2$$

$$\overline{x'} = \frac{1}{n'}\sum_{i=1}^{n'} x_i, \overline{z'} = \frac{1}{n'}\sum_{i=1}^{n'} z_i$$

Also,

$$E\left(e_1'\right) = E\left(e_2'\right) = 0,$$

$$E\left(e_1'^2\right) = \frac{1}{n'}\left(\theta_{040} - 1\right), E\left(e_2'^2\right) = \frac{1}{n}\left(\theta_{004} - 1\right),$$

$$E\left(e_1'e_2'\right) = \frac{1}{n'}\left(\theta_{220} - 1\right)$$

Expressing t_{1d}, t_{2d}, and t_d in terms of e's and following the procedure explained in section 2 and section 3 we get the MSE of these estimators, respectively as -

$$MSE\left(t_{1d}\right) \cong S_y^4\left[\frac{1}{n}\left(\theta_{400} - 1\right) + \frac{1}{4}\left(\frac{1}{n} - \frac{1}{n'}\right)\left(\theta_{040} - 1\right) + \right.$$
$$\left. + \left(\frac{1}{n'} - \frac{1}{n}\right)\left(\theta_{220} - 1\right)\right] \tag{4.4}$$

$$MSE\left(t_{2d}\right) \cong$$

$$\cong S_y^4\left[\frac{1}{n}\left(\theta_{400} - 1\right) + \frac{1}{4}\left(\frac{1}{n} - \frac{1}{n'}\right)\left(\theta_{004} - 1\right) - \left(\frac{1}{n'} - \frac{1}{n}\right)\left(\theta_{202} - 1\right)\right] \tag{4.5}$$

$$MSE\left(t_d\right) \cong S_y^4\left[\frac{1}{n}\left(\theta_{400} - 1\right) + \frac{k^2}{4}\left(\frac{1}{n} - \frac{1}{n'}\right)\left(\theta_{040} - 1\right) + \right.$$
$$+ \frac{\left(k^2 - 1\right)}{4}\left(\frac{1}{n} - \frac{1}{n'}\right)\left(\theta_{004} - 1\right) + k\left(\frac{1}{n} - \frac{1}{n'}\right)\left(\theta_{220} - 1\right) +$$
$$\left. + \left(k - 1\right)\left(\frac{1}{n'} - \frac{1}{n}\right)\left(\theta_{202} - 1\right) - \frac{k\left(k - 1\right)}{2}\left(\frac{1}{n'} - \frac{1}{n}\right)\left(\theta_{022} - 1\right)\right] \tag{4.6}$$

Minimization of (4.6) with respect to k yields its optimum value as

$$k = \frac{\left\{\theta_{004} + 2\left(\theta_{220} - 1\right) + \theta_{022} - 6\right\}}{\left(\theta_{040} + \theta_{004} + 2\theta_{022} - 4\right)} = k_0 \,(\text{say}) \tag{4.7}$$

Substitution of k_0 from (4.7) to (4.6) gives minimum value of MSE of t_d.

5. EMPIRICAL STUDY

To illustrate the performance of various estimators of S_y^2, we consider the data given in Murthy (1967, p.-226). The variates are:
y : output, x: number of workers, z: fixed capital,
$N=80$, $n'=25$, $n=10$.

$\theta_{400}=2.2667, \theta_{040}=3.65, \theta_{004}=2.8664, \theta_{220}=2.3377, \theta_{202}=2.2208, \theta_{400}=3.14$

The percent relative efficiency (PRE) of various estimators of S_y^2 with respect to conventional estimator s_y^2 has been computed and displayed in table 5.1.
Table 5.1: PRE of s_y^2, t_1, t_2 and min. MSE (t) with respect to s_y^2

Estimator	$PRE\left(., s_y^2\right)$
s_y^2	100
t_1	214.35
t_2	42.90
t	215.47

In Table 5.2 PRE of various estimators of s_y^2 in two-phase sampling with respect to S_y^2 are displayed.
Table 5.2: PRE of s_y^2, t_{1d}, t_{2d} and min. MSE (t_d) with respect to s_y^2

Estimator	$PRE\left(., s_y^2\right)$
s_y^2	100
t_{1d}	1470.76
t_{2d}	513.86
t_d	1472.77

6. CONCLUSION

From table 5.1 and 5.2 we infer that the proposed estimators t performs better than conventional estimator s_y^2 and other mentioned estimators.

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The limits of other sequences

Gheorghe Costovici[17]

ABSTRACT. We prove a Proposition in the theory of the sequences of real numbers (a generalization of the Proposition in [1]), and we also give some examples.

MAIN RESULTS

Proposition. Let p and q be integers such that $p > q \geq 1$. Suppose that the function $f : [1, \infty) \to R$ is continuous, decreasing and that $\lim\limits_{x\to\infty} f(x) = 0$.

Let $F : [1, \infty) \to R$ be a primitive of f. Suppose that $\lim\limits_{n\to\infty} (F(pqn) - F(qn)) = a \in \overline{R}$. Then $\lim\limits_{n\to\infty} \sum\limits_{i=\overline{1,(p-1)qn}} f(qn+i) = a$.

Proof. We consider the sequence

$$x_n = f(1) + f(2) + \cdots + f(qn) - F(qn).$$

By Lagrange's Theorem $\exists c_k \in (k, k+1)$ such that $F(k+1) - F(k) = F'(c_k) = f(c_k)$ for all $k = 1, 2, \ldots$. Then

$$\begin{gathered}(F(2) - F(1)) + (F(3) - F(2)) + \cdots + (F(qn+1) - F(qn)) = \\ = F(qn+1) - F(1) \leq f(1) + f(2) + \cdots + f(qn) = x_n + F(qn).\end{gathered}$$

From here we obtain $x_n \geq F(qn+1) - F(qn) - F(1) = f(c_{qn}) - F(1)$. Now, $\lim\limits_{x\to\infty} f(x) = 0$ and f decreasing $\Rightarrow f(c_{qn}) \geq 0 \Rightarrow x_n \geq -F(1)$. Hence the sequence x_n is lower-bounded.

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We also have

$$\begin{aligned} x_{n+1}-x_n &= f\left(qn+1\right)+f\left(qn+2\right)+\cdots+f\left(qn+q\right)-F\left(qn+q\right)+F\left(qn\right)= \\ &= f\left(qn+1\right)+f\left(qn+2\right)+\cdots+f\left(qn+q\right)-F\left(qn+q\right)+F\left(qn\right)- \\ &- F\left(qn+1\right)+F\left(qn+1\right)-F\left(qn+2\right)+F\left(qn+2\right)-\cdots- \\ &- F\left(qn+q-1\right)+F\left(qn+q-1\right)= \\ &= \left(f\left(qn+1\right)-F\left(qn+1\right)+F\left(qn\right)\right)+ \\ &+ \left(f\left(qn+2\right)-F\left(qn+2\right)+F\left(qn+1\right)\right)+\cdots+ \\ &+ \left(f\left(qn+q\right)-F\left(qn+q\right)+F\left(qn+q-1\right)\right)\le 0 \end{aligned}$$

and thus $x_{n+1}\le x_n$ $\forall n=1,2,\dots$. So the sequence x_n is convergent.
Now

$$x_{pn}-x_n=f\left(qn+1\right)+f\left(qn+2\right)+f\left(pqn\right)-\left(F\left(pqn\right)-F\left(qn\right)\right).$$

Then

$$\lim_{n\to\infty}\sum_{\underline{i=1,(p-1)qn}}f\left(qn+i\right)=\lim_{n\to\infty}\left[\left(x_{pn}-x_n\right)+\left(F\left(pqn\right)-F\left(qn\right)\right)\right]=a.$$

Remark. *For $q=1$ and $p=2$ we obtain the Proposition from [1].*

Exemples of functions which fulfil the hypotheses of Proposition.

1. $f\left(x\right)=\dfrac{1}{x+b}$, $b\ge 0$. $F\left(x\right)=\ln\left(x+b\right)$.

$$F\left(pqn\right)-F\left(qn\right)=\ln\frac{pqn+b}{qn+b}\underset{n\to\infty}{\to}\ln p.$$

2. $f\left(x\right)=\dfrac{1}{\left(x^2+b\right)^{1/2}}$, $b\ge 0$. $F\left(x\right)=\ln\left(x+\sqrt{x^2+b}\right)$.

$$\begin{aligned} F\left(pqn\right)-F\left(qn\right) &= \ln\frac{pqn+\sqrt{\left(pqn\right)^2+b}}{qn+\sqrt{\left(qn\right)^2+b}}= \\ &= \ln\frac{pqn\left(1+\sqrt{1+b/\left(pqn\right)^2}\right)}{qn\left(1+\sqrt{1+b/\left(qn\right)^2}\right)}\underset{n\to\infty}{\to}\ln p. \end{aligned}$$

3. $f(x)=\dfrac{1}{(x+b)^{\alpha}}$, $b\geq 0$, $0<\alpha<1$. $F(x)=\dfrac{(x+b)^{-\alpha+1}}{-\alpha+1}$.

$$\begin{aligned}F(pqn)-F(qn)&=\frac{1}{1-\alpha}\left((pqn+b)^{-\alpha+1}-(qn+b)^{-\alpha+1}\right)=\\&=\frac{1}{1-\alpha}(qn)^{-\alpha+1}\left[\left(p+\frac{b}{qn}\right)^{-\alpha+1}-\left(1+\frac{b}{qn}\right)^{-\alpha+1}\right]\underset{n\to\infty}{\longrightarrow}\infty.\end{aligned}$$

4. $f(x)=\dfrac{x^{\alpha-1}}{x^{\alpha}+b}$, $b\geq 0$, $\alpha\leq 1$.

$$\begin{aligned}f(x)&=\frac{1}{x+bx^{1-\alpha}}\Rightarrow\lim_{x\to\infty}f(x)=0.\\F(x)&=\frac{1}{\alpha}\ln\left(x^{\alpha}+b\right).\end{aligned}$$

$$F(pqn)-F(qn)=\frac{1}{\alpha}\ln\frac{(pqn)^{\alpha}+b}{(qn)^{\alpha}+b}=\frac{1}{\alpha}\ln\frac{p^{\alpha}+\frac{b}{(qn)^{\alpha}}}{1+\frac{b}{(qn)^{\alpha}}}\underset{n\to\infty}{\longrightarrow}\ln p.$$

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About Schur's inequality

Mihály Bencze[18]

ABSTRACT. In this paper we present the Schur inequality, and we give some applications

MAIN RESULTS

Theorem 1. (Schur) If $\alpha \in R$ and $x, y, z \geq 0$, then

$$(x-y)(x-z)x^{\alpha}+(y-z)(y-x)y^{\alpha}+(z-x)(z-y)z^{\alpha}\geq 0$$

Proof. If $\alpha \geq 0$, then we have

$$(x-y)(x-z)x^{\alpha}+(y-z)(y-x)y^{\alpha}+(z-x)(z-y)z^{\alpha}=$$

$$=(x-y)((x-z)x^{\alpha}-(y-z)y^{\alpha})+(x-z)(y-z)z^{\alpha}\geq$$

$$\geq (x-y)(y-z)(x^{\alpha}-y^{\alpha})+(x-z)(y-z)x^{\alpha}\geq 0$$

If $\alpha < 0$, then we have

$$(x-y)(x-z)x^{\alpha}+(y-z)(y-x)y^{\alpha}+(z-x)(z-y)z^{\alpha}=$$

$$=(x-y)(x-z)x^{\alpha}+(y-z)(-(x-y)y^{\alpha}+(x-z)x^{\alpha})\geq$$

$$\geq (x-y)(x-z)x^{\alpha}+(y-z)(x-z)(-y^{\alpha}+z^{\alpha})\geq 0$$

Theorem 2. If $f(x)=\sum\limits_{k=0}^{\infty}a_kx^k$, where $a_k\geq 0\ (k\in N)$, then

$$(x-y)(x-z)f(x)+(y-z)(y-x)f(y)+(z-x)(z-y)f(z)\geq 0$$

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for all $x, y, z \geq 0$.
Proof. Using the Theorem 1 we can write the following:

$$(x-y)(x-z)\left(a_k x^k\right)+(y-z)(y-x)\left(a_k y^k\right)+(z-x)(z-y)\left(a_k z^k\right) \geq 0,$$

and after summation we get:

$$(x-y)(x-z)\left(\sum_{k=0}^{\infty} a_k x^k\right)+(y-z)(y-x)\left(\sum_{k=0}^{\infty} a_k x^k\right)+$$
$$+(z-x)(z-y)\left(\sum_{k=0}^{\infty} a_k x^k\right) \geq 0,$$

or

$$(x-y)(x-z) f(x)+(y-z)(y-x) f(y)+(z-x)(z-y) f(z) \geq 0$$

Corollary 2.1. If $x, y, z > 0$, then

1). $(x-y)(x-z) a^x+(y-z)(y-x) a^y+(z-x)(z-y) a^z \geq 0$ for all $a \geq 1$

2). $(x-y)(x-z)\, tgx+(y-z)(y-x)\, tgy+(z-x)(z-y)\, tgz \geq 0$
3). $(x-y)(x-z)\left(\frac{1}{x}-ctgx\right)+(y-z)(y-x)\left(\frac{1}{y}-ctgy\right)+$
$(z-x)(z-y)\left(\frac{1}{z}-ctgz\right) \geq 0$
4). $(x-y)(x-z) \arcsin x+(y-z)(y-x) \arcsin y+$
$+(z-x)(z-y) \arcsin z \geq 0$ $(x, y, z \in[0,1])$
5). $(x-y)(x-z)\, shx+(y-z)(y-x)\, shy+(z-x)(z-y)\, shz \geq 0$
6). $(x-y)(x-z)\, chx+(y-z)(y-x)\, chy+(z-x)(z-y)\, chz \geq 0$
7). $\frac{(x-y)(x-z)}{1-x^k}+\frac{(y-z)(y-x)}{1-y^k}+\frac{(z-x)(z-y)}{1-z^k} \geq 0$ $\left(x, y, z \in(0,1),\, k \in N^*\right)$
8). $(x-y)(x-z) \ln (1-x)+(y-z)(y-x) \ln (1-y)+$
$(z-x)(z-y) \ln (1-z) \leq 0$ $(x, y, z \in(0,1))$
9). $(x-y)(x-z) \ln \frac{1+x}{1-x}+(y-z)(y-x) \ln \frac{1+y}{1-y}+(z-x)(z-y) \ln \frac{1+z}{1-z} \geq 0$
$(x, y, z \in(0,1))$
10). $(x-y)(x-z) \ln \frac{x+1}{x-1}+(y-z)(y-x) \ln \frac{y+1}{y-1}+(z-x)(z-y) \ln \frac{z+1}{z-1} \geq 0$
$(x, y, z>1)$

Proof. We take:

1). $f(x)=a^x=\sum\limits_{k=0}^{\infty} \frac{x^k(\ln a)^k}{k!}$

2). $f\left(x\right) = tgx = x + \frac{x^3}{3} + \frac{2x^5}{15} + ...$
3). $f\left(x\right) = \frac{1}{x} - ctgx = \frac{x}{3} + ...\frac{x^3}{45} + \frac{2x^5}{945} + ...$
4). $f\left(x\right) = \arcsin x = x + \frac{x^3}{2\cdot 3} + ... + \frac{(2n-1)!!x^{2n+1}}{(2n)!!(2n+1)} + ...$
5). $f\left(x\right) = shx = \sum\limits_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$
6). $f\left(x\right) = chx = \sum\limits_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$
7). $f\left(x\right) = \frac{1}{1-x^k} = 1 + x^k + x^{2k} + ...$
8). $f\left(x\right) = -\ln\left(1-x\right) = x + \frac{x^2}{2} + \frac{x^3}{3} + ...$
9). $f\left(x\right) = \ln\frac{1+x}{1-x} = 2\sum\limits_{k=0}^{\infty} \frac{x^{2k+1}}{2k+1}$
10). $f\left(x\right) = \ln\frac{x+1}{x-1} = 2\sum\limits_{k=0}^{\infty} \frac{1}{(2k+1)x^{2k+1}}$

Corollary 2.2. If $\alpha \in R$ and $x, y, z \geq 0$, then

$$\left(x-y\right)\left(x-z\right)ch\left(\alpha\ln x\right) + \left(y-z\right)\left(y-x\right)ch\left(\alpha\ln y\right) +$$

$$+\left(z-x\right)\left(z-y\right)ch\left(\alpha\ln z\right) \geq 0$$

Proof. From Theoprem 1 we have

$$\sum\left(x-y\right)\left(x-z\right)x^{\alpha} \geq 0$$

and

$$\sum\left(x-y\right)\left(x-z\right)x^{-\alpha} \geq 0$$

after addition we get:

$$\sum\left(x-y\right)\left(x-z\right)\left(x^{\alpha}+x^{-\alpha}\right) \geq 0 \text{ or}$$

$$\sum\left(x-y\right)\left(x-z\right)ch\left(\alpha\ln x\right) \geq 0$$

Remark. *If* $\alpha, x, y, z \in R$ *then*

$$\left(e^x-e^y\right)\left(e^x-e^z\right)ch\left(\alpha x\right) + \left(e^y-e^z\right)\left(e^y-e^x\right)ch\left(\alpha y\right) +$$

$$+\left(e^z-e^x\right)\left(e^z-e^y\right)ch\left(\alpha z\right) \geq 0$$

Proof. In Corollary 2.2. we take $x \to e^x$ etc.

Open Question 1. Determine all $x_k > 0$ $(k = 1, 2, ..., n)$ and all $\alpha \in R$ such that

$$(x_1 - x_2)(x_1 - x_3) \ldots (x_1 - x_n) x_1^\alpha + (x_2 - x_3) \ldots (x_2 - x_n)(x_2 - x_1) x_2^\alpha + \ldots$$

$$+ (x_n - x_1)(x_n - x_2) \ldots (x_n - x_{n-1}) x_n^\alpha \geq 0$$

Open Question 2. Determine all $f : R \to (0, +\infty)$ such that for all $x, y, z > 0$ holds

$$(x - y)(x - z) f(x) + (y - z)(y - x) f(y) + (z - x)(z - y) f(z) \geq 0$$

The function $f_\alpha(x) = |x|^\alpha$ offer infinitely many solutions.

Open Question 3. Determine all $x_k > 0$ $(k = 1, 2, ..., n)$ and $f, R \to (0, +\infty)$ such that

$$(x_1 - x_2)(x_1 - x_3) \ldots (x_1 - x_n) f(x_1) + (x_2 - x_3)(x_2 - x_4) \ldots (x_2 - x_1) f(x_2) + \ldots$$

$$+ (x_n - x_1)(x_n - x_2) \ldots (x_n - x_1) f(x_n) \geq 0$$

Open Question 4. Determine all $x, y, z > 0$ $(x \neq y \neq z)$ and $\alpha \in R$ such that

$$\frac{x^\alpha}{(x - y)(x - z)} + \frac{y^\alpha}{(y - z)(y - x)} + \frac{z^\alpha}{(z - x)(z - y)} \geq 0$$

(Same question in general case).

Open Question 5. Determine all $x, y, z > 0$ such that

$$(x - y)(x - z) \zeta(x) + (y - z)(y - x) \zeta(y) + (z - x)(z - y) \xi(z) \geq 0$$

where ζ denote the Riemann zeta function.

Open Question 6. Determine all $x, y, z > 0$ such that

$$(x - y)(x - z) \Gamma(x) + (y - z)(y - x) \Gamma(y) + (z - x)(z - y) \Gamma(z) \geq 0,$$

when Γ denote the Euler‘s gamma function.

Corollary 2.3. If $x, y, z \in (0,1)$, then

1). $\frac{(x-y)(x-z)x}{1-x} + \frac{(y-z)(y-x)y}{1-y} + \frac{(z-x)(z-y)z}{1-z} \geq 0$
2). $\frac{(x-y)(x-z)x}{(1-x)^2} + \frac{(y-z)(y-x)y}{(1-y)^2} + \frac{(z-x)(z-y)z}{(1-z)^2} \geq 0$
3). $\frac{(x-y)(x-z)x(x+1)}{(1-x)^3} + \frac{(y-z)(y-x)y(y+1)}{(1-y)^3} + \frac{(z-x)(z-y)z(z+1)}{(1-z)^3} \geq 0$

Proof. In Theorem 2 we take:
1). $f(x) = \frac{x}{1-x} = \sum\limits_{n=1}^{\infty} x^n$
2). $f(x) = \frac{x}{(1-x)^2} = \sum\limits_{n=1}^{\infty} nx^n$
3). $f(x) = \frac{x(x+1)}{(1-x)^3} = \sum\limits_{n=1}^{\infty} n^2x^n$

Corollary 2.4. We have the following inequalities:

1). $\frac{(x-y)(x-z)}{(1-x)^t} + \frac{(y-z)(y-x)}{(1-y)^t} + \frac{(z-x)(z-y)}{(1-z)^t} \geq 0$, where $x, y, z \in (0,1)$ and $t \geq 0$
2). $\frac{(x-y)(x-z)}{(1-x)e^x} + \frac{(y-z)(y-x)}{(1-y)e^y} + \frac{(z-x)(z-y)}{(1-z)e^z} \geq 0$, where $x, y, z \in (0,1)$
3). $\frac{(x-y)(x-z)(1-x)}{1-2x} + \frac{(y-z)(y-x)(1-y)}{1-2y} + \frac{(z-x)(z-y)(1-z)}{1-2z} \geq 0$, where
$x, y, z \in \left(0, \frac{1}{2}\right) \cup (1, +\infty)$
4). $\frac{(x-y)(x-z)}{2-e^x} + \frac{(y-z)(y-x)}{2-e^y} + \frac{(z-x)(z-y)}{2-e^z} \geq 0$, where $x, y, z \in (0, \ln 2)$
5). $(x-y)(x-z)\arcsin^2 x + (y-z)(y-x)\arcsin^2 y +$
$+ (z-x)(z-y)\arcsin^2 z \geq 0$, where $x, y, z \in (0,1)$
6). $(x-y)(x-z)\ln\left(\frac{1-\sqrt{1-4x}}{2x}\right) + (y-z)(y-x)\ln\left(\frac{1-\sqrt{1-4y}}{2y}\right) +$
$+ (z-x)(z-y)\ln\left(\frac{1-\sqrt{1-4z}}{2z}\right) \geq 0$, where $x, y, z \in \left(0, \frac{1}{4}\right)$
7). $\frac{(x-y)(x-z)\left(1-\sqrt{1-4x}\right)}{x} + \frac{(y-z)(y-x)\left(1-\sqrt{1-4y}\right)}{y} + \frac{(z-x)(z-y)\left(1-\sqrt{1-4z}\right)}{z} \geq 0$,
where $x, y, z \in \left(0, \frac{1}{4}\right)$
8). $(x-y)(x-z)x(1-4x)^{-\frac{3}{2}} + (y-z)(y-x)y(1-4y)^{-\frac{3}{2}} +$
$+ (z-x)(z-y)z(1-4z)^{-\frac{3}{2}} \geq 0$,where $x, y, z \in \left(0, \frac{1}{4}\right)$
9). $\frac{(x-y)(x-z)x(2x+1)}{(1-4x)^{\frac{3}{2}}} + \frac{(y-z)(y-x)y(2y+1)}{(1-4y)^{\frac{3}{2}}} + \frac{(z-x)(z-y)z(2z+1)}{(1-4z)^{\frac{3}{2}}} \geq 0$, where
$x, y, z \in \left(0, \frac{1}{4}\right)$
10). $\frac{(x-y)(x-z)\arcsin x}{x} + \frac{(y-z)(y-x)\arcsin y}{y} + \frac{(z-x)(z-y)\arcsin z}{z} \geq 0$, where
$x, y, z \in (0,1)$

11). $\frac{(x-y)(x-z)x\arcsin x}{\sqrt{1-x^2}}+\frac{(y-z)(y-x)y\arcsin y}{\sqrt{1-y^2}}+\frac{(z-x)(z-y)z\arcsin z}{\sqrt{1-z^2}}\ge 0$, where $x,y,z\in(0,1)$

12). $(x-y)(x-z)\int\limits_0^x\left(\frac{\arcsin t}{t}\right)^2dt+(y-z)(y-x)\int\limits_0^y\left(\frac{\arcsin t}{t}\right)^2dt+$
$+(z-x)(z-y)\int\limits_0^z\left(\frac{\arcsin t}{t}\right)^2dt\ge 0$, where $x,y,z\in(0,1)$

13). $(x-y)(x-z)\left(\frac{x^2}{1-x^2}+\frac{x\arcsin x}{\sqrt{(1-x^2)^3}}\right)+$
$+(y-z)(y-x)\left(\frac{y^2}{1-y^2}+\frac{y\arcsin y}{\sqrt{(1-y^2)^3}}\right)+(z-x)(z-y)\left(\frac{z^2}{1-z^2}+\frac{z\arcsin z}{\sqrt{(1-z^2)^3}}\right)\ge 0,$
for all $x,y,z\in(0,1)$

14). $\frac{(x-y)(x-z)}{\sqrt{1-4x}}\left(\frac{1-\sqrt{1-4x}}{2x}\right)^n+\frac{(y-z)(y-x)}{\sqrt{1-4y}}\left(\frac{1-\sqrt{1-4y}}{2y}\right)^n+$
$+\frac{(z-x)(z-y)}{\sqrt{1-4z}}\left(\frac{1-\sqrt{1-4z}}{2z}\right)^n\ge 0$ for all $x,y,z\in\left(0,\frac{1}{4}\right)$ and $n\in N$.

15). $(x-y)(x-z)\left(\frac{1-\sqrt{1-4x}}{2x}\right)^n+(y-z)(y-x)\left(\frac{1-\sqrt{1-4y}}{2y}\right)^n+$
$+(z-x)(z-y)\left(\frac{1-\sqrt{1-4z}}{2z}\right)^n\ge 0$ for all $x,y,z\in\left(0,\frac{1}{4}\right)$ and $n\in N$.

16). $(x-y)(x-z)x\int\limits_0^1 e^{-xt\ln t}dt+(y-z)(y-x)y\int\limits_0^1 e^{-yt\ln t}dt+$
$+(z-x)(z-y)z\int\limits_0^1 e^{-zt\ln t}dt\ge 0$ for all $x,y,z>0$

17). $(x-y)(x-z)\int\limits_0^x\frac{\ln(1-t)}{t}dt+(y-z)(y-x)\int\limits_0^y\frac{\ln(1-t)}{t}dt+$
$+(z-x)(z-y)\int\limits_0^z\frac{\ln(1-t)}{t}dt\le 0$ for all $x,y,z\in(0,1)$

18). $\frac{(x-y)(x-z)}{2(1-x)+\sqrt{1-4x}+\sqrt{1-4x^2}}+\frac{(y-z)(y-x)}{2(1-y)+\sqrt{1-4y}+\sqrt{1-4y^2}}+\frac{(z-x)(z-y)}{2(1-z)+\sqrt{1-4z}+\sqrt{1-4z^2}}\ge 0$
for all $x,y,z\in\left(0,\frac{1}{4}\right)$

19). $(x-y)(x-z)\int\limits_0^{\frac{\pi}{2}}\left(1-x^2\sin^2 t\right)^{-\frac{1}{2}}dt+(y-z)(y-x)\cdot$
$\cdot\int\limits_0^{\frac{\pi}{2}}\left(1-y^2\sin^2 t\right)^{-\frac{1}{2}}dt+(z-x)(z-y)\int\limits_0^{\frac{\pi}{2}}\left(1-z^2\sin^2 t\right)^{-\frac{1}{2}}dt\ge 0$, for all $x,y,z\in(0,1)$.

Proof. In Theorem 2 we take:

1). $f(x)=\frac{1}{(1-x)^t}=1+\sum\limits_{k=1}^{\infty}\frac{t(t+1)\ldots(t+k-1)}{k!}x^k$

2). $f(x)=\frac{1}{(1-x)e^x}=\sum\limits_{k=1}^{\infty}\frac{d(k)x^k}{k!}$, where $d(k)$ is the number of deranjamentes of $1, 2, ..., k$ (i.e. permutations σ, such that for all $i, \sigma(i) \neq i$)

3). $f(x) = \frac{1-x}{1-2x} = 1 + \sum\limits_{k=1}^{\infty} f(k) x^k$, where $f(k)$ denote the number of ways to write the positive integer k as an ordered sum of positive integers.

4). $f(x) = \frac{1}{2-e^x} = 1 + \sum\limits_{k=1}^{\infty} \frac{g(k)x^k}{k!}$, where $g(k)$ denote the number of ways to write an $k-$ elements set as an ordered union of pairwise disjoint nonempty subsets.

5). $f(x) = (\arcsin x)^2 = \frac{1}{2} \sum\limits_{k=1}^{\infty} \frac{(2x)^{2k}}{k^2 \binom{2k}{k}}$

6). $f(x) = \ln\left(\frac{1-\sqrt{1-4x}}{2x}\right) = \frac{1}{2} \sum\limits_{k=1}^{\infty} \frac{x^k \binom{2k}{k}}{k}$

7). $f(x) = \frac{1-\sqrt{1-4x}}{2x} = 2 \sum\limits_{k=1}^{\infty} \frac{x^k \binom{2k}{k}}{k+1}$

8). $f(x) = x(1-4x)^{-\frac{3}{2}} = \frac{1}{2} \sum\limits_{k=0}^{\infty} k \binom{2k}{k} x^k$

9). $f(x) = \frac{x(2x+1)}{(1-4x)^{\frac{3}{2}}} = \frac{1}{2} \sum\limits_{k=1}^{\infty} k^2 \binom{2k}{k} x^k$

10). $f(x) = \frac{\arcsin x}{x} = 2 \sum\limits_{k=1}^{\infty} \frac{\binom{2k}{k}}{2k+1} \left(\frac{x}{2}\right)^{2k}$

11). $f(x) = \frac{x \arcsin x}{\sqrt{1-x^2}} = \frac{1}{2} \sum\limits_{k=1}^{\infty} \frac{(2x)^{2k}}{k \binom{2k}{k}}$

12). $f(x) = \int\limits_0^x \left(\frac{\arcsin t}{t}\right)^2 dt = \frac{1}{4} \sum\limits_{k=1}^{\infty} \frac{(2x)^{2k}}{k^3 \binom{2k}{k}}$

13). $f(x) = \frac{x^2}{1-x^2} + \frac{x \arcsin x}{\sqrt{(1-x^2)^3}} = \sum\limits_{k=1}^{\infty} \frac{(2x)^{2k}}{\binom{2k}{k}}$

14). $f(x) = \frac{1}{\sqrt{1-4x}} \left(\frac{1-\sqrt{1-4x}}{2x}\right)^n = \sum\limits_{k=0}^{\infty} \frac{2k+n}{k} x^k$

15). $f(x) = \left(\frac{1-\sqrt{1-4x}}{2x}\right)^n = \sum\limits_{k=0}^{\infty} \frac{n}{k+n} \binom{2k+n-1}{k} x^k$

16). $f(x) = x \int\limits_0^1 e^{-xt \ln t} dt = \sum\limits_{k=1}^{\infty} \frac{x^k}{k^k}$

17). $f(x) = \int\limits_0^x \frac{\ln(1-t)}{t} dt = -\sum\limits_{k=1}^{\infty} \frac{x^k}{k^2}$

18). $f(x) = \frac{4}{2(1-x)+\sqrt{1-4x}+\sqrt{1-4x^2}} = 1 + \sum\limits_{k=1}^{\infty} f(k) x^k$

19). $f(x) = \int\limits_0^{\frac{\pi}{2}} \left(1 - x^2 \sin^2 t\right)^{-\frac{1}{2}} = \frac{\pi}{2} \sum\limits_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^2}{(k!)^2} x^{2k}$

Corollary 2.5. We have the following inequalities:
1). $\frac{(x-y)(x-z)}{\sqrt{1-2tx+x^2}}+\frac{(y-z)(y-x)}{\sqrt{1-2ty+y^2}}+\frac{(z-x)(z-y)}{\sqrt{1-2tz+z^2}}\geq 0$ for all $x,y,z\in(0,1)$ and $0<t<\min\left\{\frac{1+x^2}{x},\frac{1+y^2}{y},\frac{1+z^2}{z}\right\}$
2). $\frac{(x-y)(x-z)x\left(e^{tx}-1\right)}{e^x-1}+\frac{(y-z)(y-x)y\left(e^{ty}-1\right)}{e^y-1}+\frac{(z-x)(z-y)z\left(e^{tz}-1\right)}{e^z-1}\geq 0$ for all $x,y,z>0$ and $t\geq 1$
3). $\frac{(x-y)(x-z)\left(4-x^2\right)}{4-4tx+x^2}+\frac{(y-z)(y-x)\left(4-y^2\right)}{4-4ty+y^2}+\frac{(z-x)(z-y)\left(4-z^2\right)}{4-4tz+z^2}\geq 0$ for all $x,y,z\in(0,2)$ and $t\geq 1$
4). $\frac{(x-y)(x-z)x}{1-xt-x^2}+\frac{(y-z)(y-x)y}{1-yt-y^2}+\frac{(z-x)(z-y)z}{1-zt-z^2}\geq 0$ for all $x,y,z\in\left(0,\frac{1-t^2}{t}\right)$ and $t\in(0,1)$
5). $\frac{(x-y)(x-z)(2-xt)}{1-xt-x^2}+\frac{(y-z)(y-x)(2-yt)}{1-yt-y^2}+\frac{(z-x)(z-y)(2-zt)}{1-zt-z^2}\geq 0$ for all $x,y,z\in\left(0,\frac{1-t^2}{t}\right)$ and $t\in(0,1)$

Proof. In Theorem 2 we take
1). $f(x)=\frac{1}{\sqrt{1-2tx+x^2}}=\sum\limits_{k=1}^{\infty}P_k(t)\,x^k$, where $P_k(t)$ is the k-th Legendre‘s polynomial: $P_0(t)=1$, $P_1(t)=t$, $P_2(t)=\frac{1}{2}\left(3t^2-1\right)$, in general $(k+1)\,P_{k+1}(t)-(2k+1)\,tP_k(t)+kP_{k-1}(t)=0$
2). $f(x)=\frac{x\left(e^{tx}-1\right)}{e^x-1}=\sum\limits_{k=1}^{\infty}\frac{1}{k!}\varphi_k(t)\,x^k$, where $\varphi_k(t)$ is the Bernoulli‘s function $\varphi_1(t)=t$, $\varphi_2(t)=t(t-1)$, $\varphi_3(t)=t(t-1)\left(t-\frac{1}{2}\right)$, $\varphi_4(t)=t^2(t-1)^2$, in general

$$\varphi_k(t+1)-\varphi_k(t)=kt^{k-1}$$

3). $f(x)=\frac{4-x^2}{4-4tx+x^2}=\sum\limits_{k=0}^{\infty}T_k(t)\,x^k$, where $T_k(t)$ is the k-th Chebishev‘s polynomial

$$T_k(t)=\frac{1}{2^k}\left(\left(t+\sqrt{t^2-1}\right)^k+\left(t-\sqrt{t^2-1}\right)^k\right)$$

4). $f(x)=\frac{x}{1-xt-x^2}=\sum\limits_{k=1}^{\infty}F_k(t)\,x^k$, where $F_k(t)$ is the $k-$th Fibonacci‘s polynomial:

$$F_k(t)=\sum_{p=0}^{\left[\frac{k-1}{2}\right]}\binom{k-1-p}{p}t^{k-1-2p}$$

5). $f(x) = \frac{2-xt}{1-xt-x^2} = \sum\limits_{k=1}^{\infty} L_k(t)\, x^k$, where $L_k(t)$ is the $k-th$ Lucas‘s polynomial:

$$L_k(t) = \sum_{p=0}^{\left[\frac{k}{2}\right]} \frac{k}{k-p} \binom{k-p}{p} t^{k-2p}$$

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Several proofs and generalizations of a fractional inequality with constraints

Fuhua Wei and Shanhe Wu[19]

ABSTRACT. Ten different proofs are given for a fractional inequality with constraints. Finally, two generalized forms are established by introducing exponent parameters and additive terms.

1. INTRODUCTION

The 2nd problem given at the 36th IMO held at Toronto (Canada) in 1995 was:

Problem 1. Let a, b, c be positive real numbers with $abc = 1$. Prove that

$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)}\geq\frac{3}{2} \tag{1}$$

In this paper, we show several different proofs and generalized forms of the inequality (1).

2. SEVERAL PROOFS FOR THE INEQUALITY (1)

Proof 1. It follows from the condition $abc = 1$ that

$$\frac{1}{a^3(b+c)}=\frac{b^2c^2}{a(b+c)},\ \ \frac{1}{b^3(c+a)}=\frac{c^2a^2}{b(c+a)},\ \ \frac{1}{c^3(a+b)}=\frac{a^2b^2}{c(a+b)}.$$

Now, the inequality (1) is equivalent to

$$\frac{b^2c^2}{a(b+c)}+\frac{c^2a^2}{b(c+a)}+\frac{a^2b^2}{c(a+b)}\geq\frac{3}{2}.$$

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By Cauchy-Schwarz inequality (see [1]), we have

$$\left(\lambda_1^2+\lambda_2^2+\lambda_3^3\right)\left(\frac{b^2c^2}{\lambda_1^2}+\frac{c^2a^2}{\lambda_2^2}+\frac{a^2b^2}{\lambda_3^2}\right)\geq(bc+ca+ab)^2.$$

Using a substitution

$$\lambda_1^2=a(b+c),\ \ \lambda_2^2=b(c+a),\ \ \lambda_3^2=c(a+b)$$

in the above inequality, and applying the arithmetic-geometric means inequality, we obtain

$$\frac{b^2c^2}{a(b+c)}+\frac{c^2a^2}{b(c+a)}+\frac{a^2b^2}{c(a+b)}\geq\frac{1}{2}(bc+ca+ab)\geq\frac{3}{2}\sqrt[3]{(abc)^2}=\frac{3}{2}.$$

Proof 2. We note that the inequality (1) is equivalent to

$$\frac{b^2c^2}{ab+ac}+\frac{c^2a^2}{bc+ba}+\frac{a^2b^2}{ca+cb}\geq\frac{3}{2}.$$

Let

$$K=\frac{b^2c^2}{ab+ac}+\frac{c^2a^2}{bc+ba}+\frac{a^2b^2}{ca+cb}.$$

Using Cauchy-Schwarz inequality and arithmetic-geometric means inequality gives

$$\begin{aligned}&[(ab+ac)+(bc+ba)+(ca+cb)]\,K\geq\\ \geq\ &\left(\sqrt{ab+ac}\cdot\frac{bc}{\sqrt{ab+ac}}+\sqrt{bc+ba}\cdot\frac{ca}{\sqrt{bc+ba}}+\sqrt{ca+cb}\cdot\frac{ab}{\sqrt{ca+cb}}\right)^2=\\ =\ &(bc+ca+ab)^2\geq3(bc+ca+ab)\sqrt[3]{(bc)(ca)(ab)}=3(bc+ca+ab).\end{aligned}$$

Hence $K\geq\frac{3}{2}$. The desired conclusion follows.

Proof 3. Note that for $a>0$,

$$a+\frac{1}{a}\geq2\Longleftrightarrow a\geq2-\frac{1}{a}.$$

We thus have

$$\frac{1}{a^3(b+c)}=\frac{1}{2a}\left[\frac{2}{a^2(b+c)}\right]\geq\frac{1}{2a}\left(2-\frac{a^2(b+c)}{2}\right)=\frac{1}{a}-\frac{ab+ac}{4}.$$

Similarly

$$\frac{1}{b^3(c+a)} \geq \frac{1}{b} - \frac{bc+ba}{4},$$
$$\frac{1}{c^3(a+b)} \geq \frac{1}{c} - \frac{ca+cb}{4}.$$

Adding the above inequalities yields

$$\frac{1}{a^3\,(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)} \geq \frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\frac{1}{2}(ab+bc+ca) = \frac{1}{2}(ab+bc+ca).$$

Finally, the arithmetic-geometric means inequality leads us to the required inequality.

Proof 4. The inequality (1) is equivalent to

$$\frac{b^2c^2}{a(b+c)} + \frac{c^2a^2}{b(c+a)} + \frac{a^2b^2}{c(a+b)} \geq \frac{3}{2}.$$

On the other hand, we have for $\lambda > 0$,

$$\begin{aligned}
\frac{b^2c^2}{a(b+c)} + \lambda a(b+c) &\geq 2\sqrt{\lambda}bc,\\
\frac{c^2a^2}{b(c+a)} + \lambda b(c+a) &\geq 2\sqrt{\lambda}ca,\\
\frac{a^2b^2}{c(a+b)} + \lambda c(a+b) &\geq 2\sqrt{\lambda}ab.
\end{aligned}$$

Adding the above inequalities yields

$$\begin{aligned}
\frac{b^2c^2}{a(b+c)} + \frac{c^2a^2}{b(c+a)} + \frac{a^2b^2}{c(a+b)} &\geq (2\sqrt{\lambda}-2\lambda)(ab+bc+ca)\\
&\geq 6(\sqrt{\lambda}-\lambda)\sqrt[3]{(abc)^2} = 6(\sqrt{\lambda}-\lambda).
\end{aligned}$$

Choosing $\lambda = \frac{1}{4}$ gives

$$\frac{b^2c^2}{a(b+c)} + \frac{c^2a^2}{b(c+a)} + \frac{a^2b^2}{c(a+b)} \geq \frac{3}{2},$$

which is the required inequality.

Proof 5. We make the substitution $bc = x,\ ca = y,\ ab = z,\ x + y + z = s.$ Then

$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(a+c)}+\frac{1}{c^3(a+b)}=\frac{x^2}{y+z}+\frac{y^2}{z+x}+\frac{z^2}{x+y}=\frac{x^2}{s-x}+\frac{y^2}{s-y}+\frac{z^2}{s-z}.$$

We consider the probability distribution sequence of random variable ξ below:

$$p\left(\xi=\frac{x}{s-x}\right)=\frac{s-x}{2s},\quad p\left(\xi=\frac{y}{s-y}\right)=\frac{s-y}{2s},\quad p\left(\xi=\frac{z}{s-z}\right)=\frac{s-z}{2s}.$$

It follows that

$$E\xi=\frac{x}{s-x}\cdot\frac{s-x}{2s}+\frac{y}{s-y}\cdot\frac{s-y}{2s}+\frac{z}{s-z}\cdot\frac{s-z}{2s}=\frac{x+y+z}{2s}=\frac{1}{2},$$

$$E\xi^2=\left(\frac{x}{s-x}\right)^2\frac{s-x}{2s}+(-z)=\frac{1}{2s}\left(\frac{x^2}{s-x}+\frac{y^2}{s-y}+\frac{z^2}{s-z}\right).$$

According to $D(\xi) = E\xi^2 - (E\xi)^2 > 0$, we have

$$\frac{1}{2s}\left(\frac{x^2}{s-x}+\frac{y^2}{s-y}+\frac{z^2}{s-z}\right)\geq\frac{1}{4},$$

so

$$\frac{x^2}{s-x}+\frac{y^2}{s-y}+\frac{z^2}{s-z}\geq\frac{1}{2}s=\frac{1}{2}(x+y+z)\geq\frac{3}{2}\sqrt[3]{xyz}=\frac{3}{2}.$$

Hence

$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(a+c)}+\frac{1}{c^3(a+b)}\geq\frac{3}{2}.$$

Proof 6. Let $bc = x,\ ca = y,\ ab = z$. The inequality (1) is equivalent to

$$\frac{x^2}{y+z}+\frac{y^2}{z+x}+\frac{z^2}{x+y}\geq\frac{3}{2}.$$

By symmetry, we may assume that $x \geq y \geq z$, then

$$\frac{x}{y+z} \geq \frac{y}{z+x} \geq \frac{z}{x+y}.$$

Using the rearrangement inequality (see [2]) gives

$$\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} \geq z \cdot \frac{x}{y+z} + x \cdot \frac{y}{x+z} + y \cdot \frac{z}{x+y},$$

$$\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} \geq y \cdot \frac{x}{y+z} + z \cdot \frac{y}{z+x} + x \cdot \frac{z}{x+y},$$

Adding the above inequalities yields

$$2\left(\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y}\right) \geq x+y+z \geq 3\sqrt[3]{xyz} = 3,$$

The required inequality follows.

Proof 7. Apply the same substitution as in Proof 6. The inequality (1) is equivalent to

$$\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} \geq \frac{3}{2}.$$

Since (x^2, y^2, z^2) and $(\frac{1}{y+z}, \frac{1}{z+x}, \frac{1}{x+y})$ are similarly sorted sequences, it follows from the rearrangement inequality that

$$\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} \geq \frac{1}{2}\left(\frac{y^2+z^2}{y+z} + \frac{z^2+x^2}{z+x} + \frac{x^2+y^2}{x+y}\right).$$

By the power mean inequality, we have

$$\frac{y^2+z^2}{y+z} + \frac{z^2+x^2}{z+x} + \frac{x^2+y^2}{x+y} \geq \frac{y+z}{2} + \frac{z+x}{2} + \frac{x+y}{2} \geq$$

$$\geq 6\left(\frac{y}{2} \cdot \frac{z}{2} \cdot \frac{z}{2} \cdot \frac{x}{2} \cdot \frac{x}{2} \cdot \frac{y}{2}\right)^{\frac{1}{6}} = 3,$$

this yields

$$\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} \geq \frac{3}{2}.$$

Proof 8. Let $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = t$ $(t > 0)$, then

$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(a+c)}+\frac{1}{c^3(a+b)}=\frac{a^{-2}}{b^{-1}+c^{-1}}+\frac{b^{-2}}{c^{-1}+a^{-1}}+\frac{c^{-2}}{a^{-1}+b^{-1}}=$$

$$=\frac{a^{-2}}{t-a^{-1}}+\frac{b^{-2}}{t-b^{-1}}+\frac{c^{-2}}{t-c^{-1}}.$$

Consider the following function:

$$g(x)=\frac{x^2}{t-x} \quad (0<x<t).$$

Since

$$g''(x)=\frac{2t^2}{(t-x)^3}>0 \quad (0<x<t),$$

we conclude that the function g is convex on $(0,\ t)$.
Using Jensen's inequality gives

$$g(a^{-1})+g(b^{-1})+g(c^{-1})\geq 3g(\frac{a^{-1}+b^{-1}+c^{-1}}{3})=\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\geq$$

$$\geq\frac{3}{2}\sqrt[3]{\frac{1}{a}\cdot\frac{1}{b}\cdot\frac{1}{c}}=\frac{3}{2}.$$

Proof 9. Consider the following function:

$$f\left(x\right)=\left(\frac{bc}{\sqrt{ab+ac}}x-\sqrt{ab+ac}\right)^2+\left(\frac{ca}{\sqrt{bc+ab}}x-\sqrt{bc+ab}\right)^2+$$

$$+\left(\frac{ab}{\sqrt{ac+bc}}x-\sqrt{ac+bc}\right)^2=\left(\frac{b^2c^2}{ab+ac}+\frac{c^2a^2}{bc+ab}+\frac{a^2b^2}{ac+bc}\right)x^2-$$

$$-2\left(ab+bc+ca\right)x+2\left(ab+bc+ca\right).$$

Since $f\left(x\right)\geq 0$ for $x\in R$, we have the discriminant $\Delta\leq 0$, that is

$$4\left(ab+bc+ac\right)^2-8\left(\frac{b^2c^2}{ab+ac}+\frac{c^2a^2}{bc+ab}+\frac{a^2b^2}{ac+bc}\right)\left(ab+bc+ac\right)\leq 0.$$

Thus

$$\frac{b^2c^2}{ab+ac}+\frac{c^2a^2}{bc+ab}+\frac{a^2b^2}{ac+bc}\geq\frac{1}{2}\left(ab+bc+ac\right)\geq\frac{3}{2}\sqrt[3]{a^2b^2c^2}=\frac{3}{2},$$

which leads to

$$\frac{1}{a^3\left(b+c\right)}+\frac{1}{b^3\left(c+a\right)}+\frac{1}{c^3\left(a+b\right)}\geq\frac{3}{2}.$$

Proof 10. Construct the following vectors:

$$\begin{aligned}\overrightarrow{OA} &= \left(\sqrt{a\left(b+c\right)},\ \sqrt{b\left(c+a\right)},\ \sqrt{c\left(a+b\right)}\right),\\ \overrightarrow{OB} &= \left(\frac{bc}{\sqrt{a\left(b+c\right)}},\ \frac{ca}{\sqrt{b\left(c+a\right)}},\ \frac{ab}{\sqrt{c\left(a+b\right)}}\right).\end{aligned}$$

We denote by $\theta\,(0\leq\theta\leq\pi)$ the angle of vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$.
Since

$$\left|\overrightarrow{OA}\right|=\sqrt{2\left(ab+bc+ca\right)},$$

$$\left|\overrightarrow{OB}\right|=\sqrt{\frac{b^2c^2}{a\left(b+c\right)}+\frac{c^2a^2}{b\left(c+a\right)}+\frac{a^2b^2}{c\left(a+b\right)}},$$

we have

$$\overrightarrow{OA}\cdot\overrightarrow{OB}=\left|\overrightarrow{OA}\right|\left|\overrightarrow{OB}\right|\cos\theta=\sqrt{2\left(ab+bc+ca\right)}\cdot$$

$$\cdot\sqrt{\frac{b^2c^2}{a\left(b+c\right)}+\frac{c^2a^2}{b\left(c+a\right)}+\frac{a^2b^2}{c\left(a+b\right)}}\cos\theta\leq$$

$$\leq\sqrt{2\left(ab+bc+ca\right)}\sqrt{\frac{b^2c^2}{a\left(b+c\right)}+\frac{c^2a^2}{b\left(c+a\right)}+\frac{a^2b^2}{c\left(a+b\right)}}.$$

On the other hand, we have

$$\overrightarrow{OA}\cdot\overrightarrow{OB}=ab+bc+ca.$$

Thus

$$\frac{b^2c^2}{a\left(b+c\right)}+\frac{c^2a^2}{b\left(c+a\right)}+\frac{a^2b^2}{c\left(a+b\right)}\geq\frac{1}{2}\left(ab+bc+ca\right)\geq\frac{3}{2}\sqrt[3]{a^2b^2c^2}=\frac{3}{2},$$

which leads us to the inequality (1).

GENERALIZATIONS OF THE INEQUALITY (1)

Theorem 1. Let a, b, c be positive real numbers such that $abc = 1$, and let $\lambda \geq 2$. Then

$$\frac{1}{a^\lambda(b+c)} + \frac{1}{b^\lambda(c+a)} + \frac{1}{c^\lambda(a+b)} \geq \frac{3}{2}. \tag{2}$$

Proof. Let $a = \frac{1}{x},\ b = \frac{1}{y},\ c = \frac{1}{z}$. Then

$$\frac{1}{a^\lambda(b+c)} + \frac{1}{b^\lambda(c+a)} + \frac{1}{c^\lambda(a+b)} = \frac{x^{\lambda-1}}{y+z} + \frac{y^{\lambda-1}}{z+x} + \frac{z^{\lambda-1}}{x+y},$$

By symmetry, we may assume that $x \geq y \geq z$, then

$$x^{\lambda-2} \geq y^{\lambda-2} \geq z^{\lambda-2}, \frac{1}{y+z} \geq \frac{1}{z+x} \geq \frac{1}{x+y}$$

and

$$\frac{x^{\lambda-2}}{y+z} \geq \frac{y^{\lambda-2}}{z+x} \geq \frac{z^{\lambda-2}}{x+y}.$$

Using the rearrangement inequality gives

$$\frac{x^{\lambda-1}}{y+z} + \frac{y^{\lambda-1}}{z+x} + \frac{z^{\lambda-1}}{x+y} \geq z \cdot \frac{x^{\lambda-2}}{y+z} + x \cdot \frac{y^{\lambda-2}}{z+x} + y \cdot \frac{z^{\lambda-2}}{x+y},$$

$$\frac{x^{\lambda-1}}{y+z} + \frac{y^{\lambda-1}}{z+x} + \frac{z^{\lambda-1}}{x+y} \geq y \cdot \frac{x^{\lambda-2}}{y+z} + z \cdot \frac{y^{\lambda-2}}{z+x} + x \cdot \frac{z^{\lambda-2}}{x+y}.$$

Adding the above inequalities yields

$$\begin{aligned}\frac{x^{\lambda-1}}{y+z} + \frac{y^{\lambda-1}}{z+x} + \frac{z^{\lambda-1}}{x+y} &\geq \frac{1}{2}(x^{\lambda-2} + y^{\lambda-2} + z^{\lambda-2}) \\ &\geq \frac{3}{2}\sqrt[3]{x^{\lambda-2}y^{\lambda-2}z^{\lambda-2}} \\ &= \frac{3}{2}.\end{aligned}$$

The inequality (2) is proved.

Theorem 2. Let $x_1, x_2, \ldots, x_n$ be positive real numbers such that $x_1x_2\cdots x_n = 1$, and let $n \geq 3,\ \lambda \geq 3$. Then

$$\sum_{1\le k<l\le n} \frac{1}{(\prod_{1\le i\le n,\ i\ne k,l} x_i)^{\lambda-1}(\sum_{1\le i<j\le n,\ i\ne k,l} x_i x_j)} \ge \frac{n(n-1)}{(n+1)(n-2)}. \tag{3}$$

Proof. Applying the generalized Radon's inequality (see [3-6]):

$$\sum_{i=1}^{n} \frac{a_i^p}{b_i} \ge n^{2-p}(\sum_{i=1}^{n} a_i)^p/(\sum_{i=1}^{n} b_i)$$

$(a_i > 0,\ b_i > 0,\ i = 1, 2, \ldots, n,\ p \ge 2$ or $p \le 0)$, we deduce that

$$\sum_{1\le k<l\le n} \frac{1}{(\prod_{1\le i\le n,\ i\ne k,l} x_i)^{\lambda-1}(\sum_{1\le i<j\le n,\ i\ne k,l} x_i x_j)} =$$

$$= \sum_{1\le k<l\le n} \frac{1}{[(\prod_{1\le i\le n} x_i)/x_k x_l)]^{\lambda-1}[(\sum_{1\le i<j\le n} x_i x_j) - x_k x_l)]}$$

$$= \sum_{1\le k<l\le n} \frac{(x_k x_l)^{\lambda-1}}{(\sum_{1\le i<j\le n} x_i x_j) - x_k x_l} \ge \left[\frac{n(n-1)}{2}\right]^{3-\lambda} \cdot$$

$$\cdot \frac{(\sum_{1\le k<l\le n} x_k x_l)^{\lambda-1}}{\sum_{1\le k<l\le n} [(\sum_{1\le i<j\le n} x_i x_j) - x_k x_l)]}$$

$$= \left[\frac{n(n-1)}{2}\right]^{3-\lambda} \left[\frac{n(n-1)}{2} - 1\right]^{-1} (\sum_{1\le k<l\le n} x_k x_l)^{\lambda-2} \ge$$

$$\ge \frac{2}{n^2-n-2}\left[\frac{n(n-1)}{2}\right]^{3-\lambda} \left[\frac{n(n-1)}{2}(\prod_{1\le k\le n} x_k)^{\frac{2}{n}}\right]^{\lambda-2} = \frac{n(n-1)}{(n+1)(n-2)}.$$

This completes the proof of Theorem 2.

Remark. *In a special case when $n = 3,\ \lambda = 3,\ x_1 = a,\ x_2 = b,\ x_3 = c$, the inequality (3) would reduce to the inequality (1).*

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An extension of inequality $\sum \sin \frac{A}{2} \le \frac{3}{2}$

Mihály Bencze and Shanhe Wu[20]

ABSTRACT. In this paper extend the inequality $\sum \sin \frac{A}{2} \le \frac{3}{2}$ valid in any triangle, and we give some applications.

INTRODUCTION

For the classical inequality $\sum \sin \frac{A}{2} \le \frac{3}{2}$ we know many proofs. In following we extend this inequality, which give some new refinements.

Theorem 1. If $x, y, z > 0$, then

$$\sum \sqrt{\frac{xy}{(x+z)(y+z)}} \le \sqrt{\frac{2\left(\sum x\right)\left(\sum xy\right)}{\prod (x+y)}} \le \frac{3}{2}$$

Proof. Using Cauchy-Schwarz inequality, we obtain

$$\left(\sum \sqrt{\frac{xy}{(x+z)(y+z)}}\right)^2 = \left(\sum \sqrt{xy} \cdot \frac{1}{\sqrt{(x+z)(y+z)}}\right)^2 \le$$

$$\le \left(\sum xy\right) \sum \frac{1}{(x+z)(y+z)} = \frac{2\left(\sum x\right)\left(\sum xy\right)}{\prod (x+y)}$$

but

$$\frac{2\left(\sum x\right)\left(\sum xy\right)}{\prod (x+y)} \le \frac{9}{4} \Leftrightarrow 8\left(\sum x\right)\left(\sum xy\right) \le 9 \prod (x+y) \Leftrightarrow$$

$$\Leftrightarrow \left(\sum x\right)\left(\sum xy\right) \ge 9xyz$$

which is true.

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Corollary 1.1. In all triangle ABc holds:
1). $\sum\sqrt{\frac{a+b}{c}}\le\sqrt{\frac{s^2+r^2+4Rr}{Rr}}\le\frac{3}{2}\sqrt{\frac{s^2+r^2+2Rr}{2Rr}}$
2). $\sum\sin\frac{A}{2}\le\sqrt{\frac{4R+r}{2R}}\le\frac{3}{2}$
3). $\sum\sqrt{\frac{h_a+h_b}{h_c}}\le\sqrt{\frac{s^2+r^2+4Rr}{Rr}}\le\frac{3}{2}\sqrt{\frac{s^2+r^2+2Rr}{2Rr}}$
4). $\sum\sqrt{\frac{r_a+r_b}{r_c}}\le\sqrt{\frac{2(4R+r)}{r}}\le 3\sqrt{\frac{R}{r}}$
5). If ABC is acute triangle, then

$$\sum\sqrt{\frac{\cos A+\cos B}{\cos C}}\le 2\sqrt{\frac{2\left(R+r\right)\left(s^2+r^2-4R^2\right)}{R\left(s^2-\left(2R+r\right)^2\right)}}\le\frac{3}{2}\sqrt{\frac{r\left(s^2+r^2+2Rr\right)}{R\left(s^2-\left(2R+r\right)^2\right)}}$$

6). $\sum\sqrt{\frac{ctg\frac{A}{2}+ctg\frac{B}{2}}{ctg\frac{C}{2}}}\le\sqrt{\frac{2(4R+r)}{r}}\le 3\sqrt{\frac{R}{r}}$
7). $\sum\sqrt{\frac{tg\frac{A}{2}+tg\frac{B}{2}}{tg\frac{C}{2}}}\le\sqrt{2}\le 3\sqrt{\frac{R}{4R+r}}$
8). $\sum\sqrt{\frac{\sin^2\frac{A}{2}+\sin^2\frac{B}{2}}{\sin^2\frac{C}{2}}}\le\sqrt{\frac{(2R-r)(s^2+r^2-8Rr)}{Rr^2}}\le\frac{3}{2}\sqrt{\frac{(2R-r)(s^2+r^2-8Rr)-2Rr^2}{2Rr^2}}$
9). $\sum\sqrt{\frac{\cos^2\frac{A}{2}+\cos^2\frac{B}{2}}{\cos^2\frac{C}{2}}}\le\sqrt{\frac{(4R+r)\left(s^2+(4R+r)^2\right)}{Rs^2}}\le\frac{3}{2}\sqrt{\frac{(4R+r)^3+s^2+(2R+r)}{2Rs^2}}$

Proof. In Theorem 1 we take
$(x,y,z)\in\{(a,b,c);(s-a,s-b,s-c);(h_a,h_b,h_c);(r_a,r_b,r_c)$,
$\left(\cos\frac{A}{2},\cos\frac{B}{2},\cos\frac{C}{2}\right),\left(ctg\frac{A}{2},ctg\frac{B}{2},ctg\frac{C}{2}\right),\left(tg\frac{A}{2},tg\frac{B}{2},tg\frac{C}{2}\right)$,
$\left(\sin^2\frac{A}{2},\sin^2\frac{B}{2},\sin^2\frac{C}{2}\right),\left(\cos^2\frac{A}{2},\cos^2\frac{B}{2},\cos^2\frac{C}{2}\right)\}$

Theorem 2. If $x,y,z>0$, then

$$\sum\sqrt{\frac{xy}{\left(x+z\right)\left(y+z\right)}}\le\sqrt{\frac{3xyz}{\prod\left(x+y\right)}\sum\frac{x+y}{2}}$$

Proof. Using the inequality

$$\sqrt{u}+\sqrt{v}+\sqrt{w}\le\sqrt{3\left(u+v+w\right)}$$

we have

$$\sum\sqrt{\frac{xy}{\left(x+z\right)\left(y+z\right)}}=\sqrt{\frac{xyz}{\prod\left(x+y\right)}}\sum\sqrt{\frac{x+y}{z}}\le\sqrt{\frac{3xyz}{\prod\left(x+y\right)}\sum\frac{x+y}{z}}$$

Corollary 1.1. In all triangle ABc holds:

1). $\sum \sqrt{\frac{ab}{(a+c)(b+c)}} \le \sqrt{\frac{3(s^2+r^2-2Rr)}{s^2+r^2+2Rr}}$

2). $\sum \sin \frac{A}{2} \le \sqrt{\frac{3(2R-r)}{2R}}$

3). $\sum \sqrt{\frac{h_a h_b}{(h_a+h_c)(h_b+h_c)}} \le \sqrt{\frac{3(s^2+r^2-2Rr)}{s^2+r^2+2Rr}}$

4). $\sum \sqrt{\frac{r_a r_b}{(r_a+r_c)(r_b+r_c)}} \le \sqrt{\frac{3(2R-r)}{2R}}$

5). In all acute triangle ABC holds

$\sum \sqrt{\frac{\cos A \cos B}{\cos A + \cos B}} \le \sqrt{\frac{3\left((2R+r)^2+s^2r-2R(s^2+r^2+2Rr)\right)}{r(s^2+r^2+2Rr)}}$

6). $\sum \sqrt{\frac{ctg\frac{A}{2}ctg\frac{B}{2}}{\left(ctg\frac{A}{2}+ctg\frac{C}{2}\right)\left(ctg\frac{B}{2}+ctg\frac{C}{2}\right)}} \le \sqrt{\frac{3(2R-r)}{2R}}$

7). $\sum \sqrt{\frac{tg\frac{A}{2}tg\frac{B}{2}}{\left(tg\frac{A}{2}+tg\frac{C}{2}\right)\left(tg\frac{B}{2}+tg\frac{C}{2}\right)}} \le \sqrt{\frac{3(2R-r)}{2R}}$

8). $\sum \sqrt{\frac{\sin\frac{A}{2}\sin\frac{B}{2}}{\left(\sin^2\frac{A}{2}+\sin^2\frac{C}{2}\right)\left(\sin^2\frac{B}{2}+\sin^2\frac{C}{2}\right)}} \le \sqrt{3}$

9). $\sum \sqrt{\frac{\cos\frac{A}{2}\cos\frac{B}{2}}{\left(\cos^2\frac{A}{2}+\cos^2\frac{C}{2}\right)\left(\cos^2\frac{B}{2}+\cos^2\frac{C}{2}\right)}} \le \sqrt{3}$

Proof. See Corollary 1.1.

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The limits of some real sequences

Gheorghe Costovici[21]

ABSTRACT. In Proposition P_2, via Proposition P_1, we shall establish the limits of some sequences.

MAIN RESULTS

Proposition P_1. If a_i, $i = \overline{1,p}$ satisfy the (triungular) system

$$C_{2p}^{k} + a_1 C_{2p-2}^{k-1} + a_2 C_{2p-4}^{k-2} + \cdots + a_{k-1} C_{2p-2k+2}^{1} + a_k = 0, \quad k = \overline{1,p},$$

then we have

$$\left(t+\frac{1}{t}\right)^{2p} + a_1\left(t+\frac{1}{t}\right)^{2p-2} + a_2\left(t+\frac{1}{t}\right)^{2p-4} + \cdots$$
$$+a_{p-2}\left(t+\frac{1}{t}\right)^{4} + a_{p-1}\left(t+\frac{1}{t}\right)^{2} + a_p = t^{2p} + \frac{1}{t^{2p}}. \tag{1}$$

If b_i, $i = \overline{1,q}$ satisfy the (triangular) system

$$C_{2q+1}^{k} + b_1 C_{2q-1}^{k-1} + b_2 C_{2q-3}^{k-2} + \cdots + b_{k-1} C_{2q-2k+3}^{1} + b_k = 0, \quad k = \overline{1,q},$$

then we have

$$\left(t+\frac{1}{t}\right)^{2q+1} + b_1\left(t+\frac{1}{t}\right)^{2q-1} + b_2\left(t+\frac{1}{t}\right)^{2q-3} + \cdots$$
$$+b_{q-1}\left(t+\frac{1}{t}\right)^{3} + b_q\left(t+\frac{1}{t}\right) = t^{2q+1} + \frac{1}{t^{2q+1}}. \tag{2}$$

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Proof of (1).

$$A=\left(t+\frac{1}{t}\right)^{2p}+a_1\left(t+\frac{1}{t}\right)^{2p-2}+a_2\left(t+\frac{1}{t}\right)^{2p-4}+\cdots+a_i\left(t+\frac{1}{t}\right)^{2p-2i}+$$

$$\cdots+a_{p-2}\left(t+\frac{1}{t}\right)^{4}+a_{p-1}\left(t+\frac{1}{t}\right)^{2}+a_p=$$

$$=\Big[\left(t^{2p}+\frac{1}{t^{2p}}\right)+C_{2p}^{1}\left(t^{2p-2}+\frac{1}{t^{2p-2}}\right)+C_{2p}^{2}\left(t^{2p-4}+\frac{1}{t^{2p-4}}\right)+\cdots+$$

$$+C_{2p}^{i}\left(t^{2p-2i}+\frac{1}{t^{2p-2i}}\right)+\cdots+C_{2p}^{p-1}\left(t^{2}+\frac{1}{t^{2}}\right)+C_{2p}^{p}\Big]+$$

$$+a_1\Big[\left(t^{2p-2}+\frac{1}{t^{2p-2}}\right)+C_{2p-2}^{1}\left(t^{2p-4}+\frac{1}{t^{2p-4}}\right)+\cdots+$$

$$+C_{2p-2}^{i-1}\left(t^{2p-2i}+\frac{1}{t^{2p-2i}}\right)+\cdots+C_{2p-2}^{p-2}\left(t^{2}+\frac{1}{t^{2}}\right)+C_{2p-2}^{p-1}\Big]+$$

$$+a_2\Big[\left(t^{2p-4}+\frac{1}{t^{2p-4}}\right)+C_{2p-4}^{1}\left(t^{2p-6}+\frac{1}{t^{2p-6}}\right)+\cdots+$$

$$+C_{2p-4}^{i-2}\left(t^{2p-2i}+\frac{1}{t^{2p-2i}}\right)+\cdots+C_{2p-4}^{p-3}\left(t^{2}+\frac{1}{t^{2}}\right)+C_{2p-4}^{p-2}\Big]+$$

$$+\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots+$$

$$+a_{i-1}\Big[\left(t^{2p-2i+2}+\frac{1}{t^{2p-2i+2}}\right)+C_{2p-2i+2}^{1}\left(t^{2p-2i}+\frac{1}{t^{2p-2i}}\right)+\cdots+$$

$$+C_{2p-2i+2}^{p-i}\left(t^{2}+\frac{1}{t^{2}}\right)+C_{2p-2i+2}^{p-i+1}\Big]+$$

$$+a_{i}\Big[\left(t^{2p-2i}+\frac{1}{t^{2p-2i}}\right)+C_{2p-2i}^{1}\left(t^{2p-2i-2}+\frac{1}{t^{2p-2i-2}}\right)+\cdots+$$

$$+C_{2p-2i}^{p-i-1}\left(t^{2}+\frac{1}{t^{2}}\right)+C_{2p-2i}^{p-i}\Big]+$$

$$+\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots+$$

$$+a_{p-2}\left[\left(t^{4}+\frac{1}{t^{4}}\right)+C_{4}^{1}\left(t^{2}+\frac{1}{t^{2}}\right)+C_{4}^{2}\right]+$$

$$+a_{p-1}\left[\left(t^{2}+\frac{1}{t^{2}}\right)+C_{2}^{1}\right]+$$

$$+a_p\,[1]\,.$$

We sum up this table-like expression along diagonals and we have

$$A=\left(t^{2p}+\frac{1}{t^{2p}}\right)+\sum_{i=\overline{1,p-1}}\left(C_{2p}^{i}+a_1C_{2p-2}^{i-1}+a_2C_{2p-4}^{i-2}+\cdots+a_{i-1}C_{2p-2i+2}^{1}+a_i\right)\cdot$$
$$\cdot\left(t^{2p-2i}+\frac{1}{t^{2p-2i}}\right)+\left(C_{2p}^{p}+a_1C_{2p-2}^{p-1}+a_2C_{2p-4}^{p-2}+\cdots+a_{p-2}C_4^2+a_{p-1}C_2^1+a_p\right).$$

Taking into account the hypotheses we obtain $A=t^{2p}+\dfrac{1}{t^{2p}}$.
Proof of (2).

$$B=\left(t+\frac{1}{t}\right)^{2q+1}+b_1\left(t+\frac{1}{t}\right)^{2q-1}+b_2\left(t+\frac{1}{t}\right)^{2q-3}+\cdots+b_i\left(t+\frac{1}{t}\right)^{2q-2i+1}+$$
$$\cdots+b_{q-1}\left(t+\frac{1}{t}\right)^{3}+b_q\left(t+\frac{1}{t}\right)=$$
$$=\Bigg[\left(t^{2q+1}+\frac{1}{t^{2q+1}}\right)+C_{2q+1}^{1}\left(t^{2q-1}+\frac{1}{t^{2q-1}}\right)+C_{2q+1}^{2}\left(t^{2q-3}+\frac{1}{t^{2q-3}}\right)+\cdots+$$
$$+C_{2q+1}^{i}\left(t^{2q+1-2i}+\frac{1}{t^{2q+1-2i}}\right)+\cdots+C_{2q+1}^{q-1}\left(t^{3}+\frac{1}{t^{3}}\right)+C_{2q+1}^{q}\left(t+\frac{1}{t}\right)\Bigg]+$$

$$+b_1\Bigg[\left(t^{2q-1}+\frac{1}{t^{2q-1}}\right)+C_{2q-1}^{1}\left(t^{2q-3}+\frac{1}{t^{2q-3}}\right)+\cdots+$$

$$+C_{2q-1}^{i-1}\left(t^{2q+1-2i}+\frac{1}{t^{2q+1-2i}}\right)+\cdots+C_{2q-1}^{q-2}\left(t^{3}+\frac{1}{t^{3}}\right)+C_{2q-1}^{q-1}\left(t+\frac{1}{t}\right)\Bigg]+$$

$$+b_2\Bigg[\left(t^{2q-3}+\frac{1}{t^{2q-3}}\right)+C_{2q-3}^{1}\left(t^{2q-5}+\frac{1}{t^{2q-5}}\right)+\cdots+$$

$$+C_{2q-3}^{i-1}\left(t^{2q+1-2i}+\frac{1}{t^{2q+1-2i}}\right)+\cdots+C_{2q-3}^{q-3}\left(t^{3}+\frac{1}{t^{3}}\right)+C_{2q-3}^{q-2}\left(t+\frac{1}{t}\right)\Bigg]+$$

$$+\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots+$$

$$+b_{i-1}\Bigg[\left(t^{2q-2i+3}+\frac{1}{t^{2q-2i+3}}\right)+C_{2q-2i+3}^{1}\left(t^{2q-2i+1}+\frac{1}{t^{2q-2i+1}}\right)+\cdots+$$

$$+C_{2q-2i+3}^{q-i}\left(t^3+\frac{1}{t^3}\right)+C_{2q-2i+3}^{q-i+1}\left(t+\frac{1}{t}\right)\Big]+$$

$$+b_i\Big[\left(t^{2q-2i+1}+\frac{1}{t^{2q-2i+1}}\right)+C_{2q-2i+1}^{1}\left(t^{2q-2i-1}+\frac{1}{t^{2q-2i-1}}\right)+\cdots+$$

$$+C_{2q-2i+1}^{q-i-1}\left(t^3+\frac{1}{t^3}\right)+C_{2q-2i+1}^{q-i}\left(t+\frac{1}{t}\right)\Big]+$$

$$+\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots+$$

$$+b_{q-1}\left[\left(t^3+\frac{1}{t^3}\right)+C_3^1\left(t+\frac{1}{t}\right)\right]+b_{p-1}\left[\left(t+\frac{1}{t}\right)\right].$$

We sum up this table along diagonals and we get

$$B=\left(t^{2q+1}+\frac{1}{t^{2q+1}}\right)+\sum_{i=\overline{1,q}}\left(C_{2q+1}^{i}+b_1C_{2q-1}^{i-1}+b_2C_{2q-3}^{i-2}+\cdots+b_{i-1}C_{2q-2i+3}^{1}+b_i\right)\cdot$$
$$\cdot\left(t^{2q+1-2i}+\frac{1}{t^{2q+1-2i}}\right).$$

Taking into account the hypotheses we obtain $B=t^{2q+1}+\dfrac{1}{t^{2q+1}}$.

Proposition P_2. If a_i, $i=\overline{1,p}$ satisfy the system

$$C_{2p}^{k}+a_1C_{2p-2}^{k-1}+a_2C_{2p-4}^{k-2}+\cdots+a_{k-1}C_{2p-2k+2}^{1}+a_k=0,\quad k=\overline{1,p}$$

and the sequence x_n is given by the conditions $x_0=a>2$ and $x_n=x_{n-1}^{2p}+a_1x_{n-1}^{2p-2}+a_2x_{n-1}^{2p-4}+\cdots+a_{p-1}x_{n-1}^{2}+a_p$ for $n\ge 1$, then $\lim\limits_{n\to\infty}x_n^{1/(\alpha(2p)^n)}=\left(\dfrac{a+\sqrt{a^2-4}}{2}\right)^{1/\alpha}$ where α is a real number $\neq 0$.

If b_i, $i=\overline{1,q}$ satisfy the system

$$C_{2q+1}^{k}+b_1C_{2q-1}^{k-1}+b_2C_{2q-3}^{k-2}+\cdots+b_{k-1}C_{2q-2k+3}^{1}+b_k=0,\quad k=\overline{1,q}$$

and the sequence y_n is given by the conditions $y_0=a>2$ and $y_n=y_{n-1}^{2q+1}+b_1y_{n-1}^{2q-1}+b_2y_{n-1}^{2q-3}+\cdots+b_{q-1}y_{n-1}^{3}+b_qy_{n-1}$ for $n\ge 1$, then $\lim\limits_{n\to\infty}y_n^{1/(\alpha(2q+1)^n)}=\left(\dfrac{a+\sqrt{a^2-4}}{2}\right)^{1/\alpha}$ where α is a real number $\neq 0$.

Proof for x_n. Denote $z+\frac{1}{z}=a$ and we have $z^2-az+1=0$ with the roots $z_{1,2}=\frac{a\pm\sqrt{a^2-4}}{2}$. We choose $z=\frac{a+\sqrt{a^2-4}}{2}>\frac{a}{2}>1$ and express x_n in terms of this z. We show, by the method of mathematical induction, that we have $x_n=z^{(2p)^n}+\frac{1}{z^{(2p)^n}}$ for all $n\geq 0$. For $n=0$ we have $x_0=a=z+\frac{1}{z}=z^{(2p)^0}+\frac{1}{z^{(2p)^0}}$, holding true. For $n=1$ we have

$$x_1=x_0^{2p}+a_1x_0^{2p-2}+a_2x_0^{2p-4}+\cdots+a_{p-1}x_0^2+a_p=$$

$$=\left(z+\frac{1}{z}\right)^{2p}+a_1\left(z+\frac{1}{z}\right)^{2p-2}+a_2\left(z+\frac{1}{z}\right)^{2p-4}+\ldots$$

$$+a_{p-1}\left(z+\frac{1}{z}\right)^2+a_p\overset{(1)}{=}\left(z^{2p}+\frac{1}{z^{2p}}\right)=z^{(2p)^1}+\frac{1}{z^{(2p)^1}},$$

also true. Suppose that $x_n=z^{(2p)^n}+\frac{1}{z^{(2p)^n}}$ for $n\geq 1$ arbitrarily fixed, and examine

$$x_{n+1}=x_n^{2p}+a_1x_n^{2p-2}+a_2x_n^{2p-4}+\cdots+a_{p-1}x_n^2+a_p=$$
$$=\left(z^{(2p)^n}+\frac{1}{z^{(2p)^n}}\right)^{2p}+a_1\left(z^{(2p)^n}+\frac{1}{z^{(2p)^n}}\right)^{2p-2}+a_2\left(z^{(2p)^n}+\frac{1}{z^{(2p)^n}}\right)^{2p-4}+$$
$$+\cdots+a_{p-1}\left(z^{(2p)^n}+\frac{1}{z^{(2p)^n}}\right)^2+a_p\overset{(1)}{=}\left(z^{(2p)^n}\right)^{2p}+\frac{1}{\left(z^{(2p)^n}\right)^{2p}}=z^{(2p)^{n+1}}+\frac{1}{z^{(2p)^{n+1}}}.$$

Then $x_n=z^{(2p)^n}+\frac{1}{z^{(2p)^n}}$ for all $n\geq 0$. Now

$$x_n^{1/(\alpha(2p)^n)}=\left(z^{(2p)^n}+\frac{1}{z^{(2p)^n}}\right)^{1/(\alpha(2p)^n)}=\left(z^{(2p)^n}\right)^{1/(\alpha(2p)^n)}\left(1+\frac{1}{z^{2(2p)^n}}\right)^{1/(\alpha(2p)^n)}.$$

From here we have $\lim\limits_{n\to\infty}x_n^{1/(\alpha(2p)^n)}=z^{1/\alpha}=\left(\frac{a+\sqrt{a^2-4}}{2}\right)^{1/\alpha}$.

Proof for y_n. Denote $z+\frac{1}{z}=a$ and we have $z^2-az+1=0$ with the roots $z_{1,2}=\frac{a\pm\sqrt{a^2-4}}{2}$. Choose $z=\frac{a+\sqrt{a^2-4}}{2}>\frac{a}{2}>1$ and express y_n in terms of this z. We show, by the method of mathematical induction, that we have $y_n = z^{(2q+1)^n} + \dfrac{1}{z^{(2q+1)^n}}$ for all $n \geq 0$. For $n = 0$ we have $y_0 = a = z + \dfrac{1}{z} = z^{(2q+1)^0} + \dfrac{1}{z^{(2q+1)^0}}$, true. For $n = 1$ we have

$$y_1 = y_0^{2q+1} + b_1 y_0^{2q-1} + b_2 y_0^{2q-3} + \cdots + b_{q-1} y_0^3 + b_q y_0 =$$

$$= \left(z + \frac{1}{z}\right)^{2q+1} + b_1 \left(z + \frac{1}{z}\right)^{2q-1} + b_2 \left(z + \frac{1}{z}\right)^{2q-3} + \ldots$$

$$+ b_{q-1} \left(z + \frac{1}{z}\right)^3 + b_q \left(z + \frac{1}{z}\right) \overset{(2)}{=} \left(z^{2q+1} + \frac{1}{z^{2q+1}}\right) = z^{(2q+1)^1} + \frac{1}{z^{(2q+1)^1}},$$

also true. Suppose that $y_n = z^{(2q+1)^n} + \dfrac{1}{z^{(2q+1)^n}}$ for $n \geq 1$ arbitrarily fixed, and examine

$$\begin{aligned} y_{n+1} &= y_n^{2q+1} + b_1 y_n^{2q-1} + b_2 y_n^{2q-3} + \cdots + b_{q-1} y_n^3 + b_q y_n = \\ &= \left(z^{(2q+1)^n} + \frac{1}{z^{(2q+1)^n}}\right)^{2q+1} + b_1 \left(z^{(2q+1)^n} + \frac{1}{z^{(2q+1)^n}}\right)^{2q-1} + \\ &+ b_2 \left(z^{(2q+1)^n} + \frac{1}{z^{(2q+1)^n}}\right)^{2q-3} + \cdots + b_{q-1} \left(z^{(2q+1)^n} + \frac{1}{z^{(2q+1)^n}}\right)^3 + \\ &+ b_q \left(z^{(2q+1)^n} + \frac{1}{z^{(2q+1)^n}}\right)^1 \overset{(2)}{=} \\ &= \left(z^{(2q+1)^n}\right)^{2q+1} + \frac{1}{\left(z^{(2q+1)^n}\right)^{2q+1}} = z^{(2q+1)^{n+1}} + \frac{1}{z^{(2q+1)^{n+1}}}. \end{aligned}$$

So $y_n = z^{(2q+1)^n} + \dfrac{1}{z^{(2q+1)^n}}$ for all $n \geq 0$. Now

$$\begin{aligned} y_n^{1/(\alpha(2q+1)^n)} &= \left(z^{(2q+1)^n} + \frac{1}{z^{(2q+1)^n}}\right)^{1/(\alpha(2q+1)^n)} = \\ &= \left(z^{(2q+1)^n}\right)^{1/(\alpha(2q+1)^n)} \left(1 + \frac{1}{z^{2(2q+1)^n}}\right)^{1/(\alpha(2q+1)^n)}. \end{aligned}$$

From here we have $\lim\limits_{n\to\infty} y_n^{1/(\alpha(2q+1)^n)} = z^{1/\alpha} = \left(\dfrac{a + \sqrt{a^2 - 4}}{2}\right)^{1/\alpha}$.

Some peculiar cases.
For $p = \overline{1,4}$, $q = \overline{1,4}$ in P_2, we respectively obtain the sequences,

$$x_n = x_{n-1}^2 - 2, \quad x_n = x_{n-1}^4 - 4x_{n-1}^2 - 2,$$
$$x_n = x_{n-1}^6 - 6x_{n-1}^4 + 9x_{n-1}^2 - 2,$$
$$x_n = x_{n-1}^8 - 8x_{n-1}^6 + 20x_{n-1}^4 - 16x_{n-1}^2 + 2,$$
$$y_n = y_{n-1}^3 - 3y_{n-1}, \quad y_n = y_{n-1}^5 - 5y_{n-1}^3 + 5y_{n-1},$$
$$y_n = y_{n-1}^7 - 7y_{n-1}^5 + 14y_{n-1}^3 - 7y_{n-1},$$
$$y_n = y_{n-1}^9 - 9y_{n-1}^7 + 27y_{n-1}^5 - 30y_{n-1}^3 + 9y_{n-1}$$

for $n \geq 1$ and $x_0 = y_0 = a > 2$.
Then, the corresponding sequences $x_n^{1/\alpha 2^n}$, $x_n^{1/\alpha 4^n}$, $x_n^{1/\alpha 6^n}$, $x_n^{1/\alpha 8^n}$, $y_n^{1/\alpha 3^n}$, $y_n^{1/\alpha 5^n}$, $y_n^{1/\alpha 7^n}$, $y_n^{1/\alpha 9^n}$, have, all of then, the same limit.

Remark. *For $p = 1$ and $\alpha = 1$ in P_2, the sequence thus obtained has been studied by Oswaldo Larreal in [1].*

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About some elementary inequalities

Mihály Bencze[22]

ABSTRACT. In this paper we present some elementary inequalities, and its applications.

MAIN RESULTS

Theorem 1. If $x > 0$ then

$$\frac{x^3+1}{x^2+1} \geq \sqrt{x^2-x+1} \geq \sqrt[4]{\frac{x^4+1}{2}} \geq \sqrt{\frac{x^2+1}{2}} \geq \frac{x+1}{2} \geq \sqrt{x} \geq$$

$$\geq \frac{2x}{x+1} \geq \frac{x\left(x^2+1\right)}{x^3+1} \geq \frac{x\left(x+1\right)^2}{2\left(x^3+1\right)}$$

Proof. We have the followings:

$$\frac{x^3+1}{x^2+1} \geq \sqrt{x^2-x+1} \Leftrightarrow x\left(x-1\right)^2 \geq 0 \tag{1}$$

$$\sqrt{x^2-x+1} \geq \sqrt[4]{\frac{x^4+1}{2}} \Leftrightarrow \left(x-1\right)^4 \geq 0 \tag{2}$$

$$\sqrt[4]{\frac{x^4+1}{2}} = \sqrt[4]{\frac{\left(x^2\right)^2+1}{2}} \geq \sqrt[4]{\left(\frac{x^2+1}{2}\right)^2} = \sqrt{\frac{x^2+1}{2}} \tag{3}$$

$$\sqrt[4]{\frac{x^2+1}{2}} \geq \sqrt{\left(\frac{x^2+1}{2}\right)^2} = \frac{x+1}{2} \tag{4}$$

$$\frac{x+1}{2} \geq \sqrt{x} \Leftrightarrow \left(\sqrt{x}-1\right)^2 \geq 0 \tag{5}$$

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$$\sqrt{x} \geq \frac{2x}{x+1} \Leftrightarrow \left(\sqrt{x}-1\right)^2 \geq 0 \tag{6}$$

$$\frac{2x}{x+1} \geq \frac{x\left(x^2+1\right)}{x^3+1} \Leftrightarrow (x-1)^2 \geq 0 \tag{7}$$

$$\frac{x\left(x^2+1\right)}{x^3+1} \geq \frac{x\left(x+1\right)^2}{2\left(x^3+1\right)} \Leftrightarrow (x-1)^2 \geq 0 \tag{8}$$

Equality holds if and only if $x = 1$.

Corollary 1.1. If $a_k > 0$ $(k = 1, 2, ..., n)$ and

$$\sum_{cyclic} a_1 a_2 = n,$$

then

$$\sum_{cyclic} \frac{\left(a_1^3+1\right)\left(a_2^3+1\right)}{\left(a_1^2+1\right)\left(a_2^2+1\right)} \geq n$$

Proof. Using the Theorem 1 we get

$$\sum_{cyclic} \frac{\left(a_1^3+1\right)\left(a_2^3+1\right)}{\left(a_1^2+1\right)\left(a_2^2+1\right)} \geq \sum_{cyclic} \sqrt[4]{\frac{\left(a_1^4+1\right)\left(a_2^4+1\right)}{4}} \geq \sum_{cyclic} \sqrt{\frac{a_1^2a_2^2+1}{2}} \geq$$

$$\geq \sum_{cyclic} \frac{a_1a_2+1}{2} = n$$

Equality holds if and only if $a_1 = a_2 = ... = a_n = 1$.

Corollary 1.2. If $a_k > 0$ $(k = 1, 2, ..., n)$ and

$$\sum_{k=1}^{n} a_k = n, \text{ then } \sum_{k=1}^{n} \frac{a_k - 1}{a_k^3 + 1} \leq 0$$

Proof. We have

$$\sum_{k=1}^{n} \left(1 + \frac{a_k - 1}{a_k^3 + 1}\right) = \sum_{k=1}^{n} \frac{a_k\left(a_k^2+1\right)}{a_k^3+1} \leq \sum_{k=1}^{n} \frac{a_k+1}{2} = n,$$

therefore

$$n+\sum_{k=1}^{n}\frac{a_k-1}{a_k^3+1}\leq n$$

and finally

$$\sum_{k=1}^{n}\frac{a_k-1}{a_k^3+1}\leq 0$$

Equality holds if and only if $a_1=a_2=...=a_n=1$.

Corollary 1.3. If $a_k>0$ $(k=1,2,...,n)$ and $\sum\limits_{cyclic}a_1^2a_2^2=n$, then

$$\sum_{cyclic}\left(\frac{\left(a_1^3+1\right)\left(a_2^3+1\right)}{\left(a_1^2+1\right)\left(a_2^2+1\right)}\right)^2\geq n$$

Poof. We have

$$\sum_{cyclic}\left(\frac{\left(a_1^3+1\right)\left(a_2^3+1\right)}{\left(a_1^2+1\right)\left(a_2^2+1\right)}\right)^2\geq\sum_{cyclic}\left(\sqrt[4]{\frac{\left(a_1^4+1\right)\left(a_2^4+1\right)}{4}}\right)^2\geq$$

$$\geq\sum_{cyclic}\frac{a_1^2a_2^2+1}{2}=n$$

Corollary 1.4. If $a_k>0$ $(k=1,2,...,n)$ and $\sum\limits_{cyclic}a_1a_2a_3=n$, then

$$\sum_{cyclic}\frac{\left(a_1^6+1\right)\left(a_2^6+1\right)\left(a_3^6+1\right)}{\left(a_1^4+1\right)\left(a_2^4+1\right)\left(a_3^4+1\right)}\geq n$$

Poof. We have

$$\sum_{cyclic}\frac{\left(a_1^6+1\right)\left(a_2^6+1\right)\left(a_3^6+1\right)}{\left(a_1^4+1\right)\left(a_2^4+1\right)\left(a_3^4+1\right)}\geq$$

$$\geq\sum\sqrt{\left(a_1^4-a_1^2+1\right)\left(a_2^4-a_2^2+1\right)\left(a_3^4-a_3^2+1\right)}\geq\sum a_1a_2a_3=n$$

because

$$a_1^4-a_1^2+1\geq a_1^2 \text{ etc.}$$

Equality holds if and only if $a_1=a_2=...=a_n=1$.

Corollary 1.5. If $a, b > 0$, then

$$\frac{a^3+b^3}{a^2+b^2} \geq \sqrt{a^2-ab+b^2} \geq \sqrt[4]{\frac{a^4+b^4}{2}} \geq \sqrt{\frac{a^2+b^2}{2}} \geq \frac{a+b}{2} \geq \sqrt{ab} \geq$$

$$\geq \frac{2}{\frac{1}{a}+\frac{1}{b}} \geq \frac{ab\left(a^2+b^2\right)}{a^3+b^3} \geq \frac{ab\,(a+b)^2}{2\left(a^3+b^3\right)}$$

Proof. In Theorem 1 we take $x = \frac{a}{b}$.

Equality holds if and only if $a = b$.

Corollary 1.6. If $x, y > 0$, then

$$\frac{\left(x^3+1\right)^2\left(y^3+1\right)^2}{\left(x^2+1\right)^2\left(y^2+1\right)^2} \geq \left(x^2-x+1\right)\left(y^2-y+1\right) \geq \sqrt{\frac{\left(x^4+1\right)\left(y^4+1\right)}{4}} \geq$$

$$\geq \frac{x^2y^2+1}{2}$$

Proof. From Theorem 1 we get

$$\left(\frac{\left(x^3+1\right)\left(y^3+1\right)}{\left(x^2+1\right)\left(y^2+1\right)}\right)^2 \geq \left(\sqrt{\left(x^2-x+1\right)\left(y^2-y+1\right)}\right)^2 \geq$$

$$\geq \left(\sqrt[4]{\frac{\left(x^4+1\right)\left(y^4+1\right)}{4}}\right)^2 \geq \left(\sqrt[4]{\frac{\left(1+x^2y^2\right)^2}{4}}\right)^2 = \frac{x^2y^2+1}{2}$$

Remark. *In [1] problem 1.62 is proved that for all $x, y \in R$ holds*

$$\left(x^2-x+1\right)\left(y^2-y+1\right) \geq \frac{x^2y^2+1}{2}$$

Corollary 1.6 offer a refinement and a new proof for that, namelly

$$\left(x^2-x+1\right)\left(y^2-y+1\right) \geq \sqrt[4]{\frac{\left(x^4+1\right)\left(y^4+1\right)}{4}} \geq \frac{x^2y^2+1}{2}$$

for all $x, y \in R$.

Corollary 1.7. If $a_k > 0$ $(k = 1, 2, ..., n)$, then

$$\prod_{k=1}^{n}\left(\frac{a_k^3+1}{a_k^2+1}\right)^2 \geq \prod_{k=1}^{n}\left(a_k^2 - a_k + 1\right) \geq \sqrt{\prod_{k=1}^{n}\frac{a_k^4+1}{2}} \geq$$

$$\geq \left(\frac{1+\left(\prod\limits_{k=1}^{n} a_k\right)^{\frac{4}{n}}}{2}\right)^{\frac{n}{2}}$$

which is a generalization and a refinement of problem 1.62 ([1]).
Proof. Using the Theorem 1, we get

$$\prod_{k=1}^{n}\left(\frac{a_k^3+1}{a_k^2+1}\right)^2 \geq \prod_{k=1}^{n}\left(a_k^2 - a_k + 1\right) \geq \left(\sqrt[4]{\prod_{k=1}^{n}\frac{a_k^4+1}{2}}\right)^2 \geq$$

$$\geq \left(\frac{1+\left(\prod\limits_{k=1}^{n} a_k\right)^{\frac{4}{n}}}{2}\right)^{\frac{n}{2}}$$

Corollary 1.8. If $t \in \left(0, \frac{\pi}{2}\right)$, then

$$(\sin t + \cos t)\,(1 - \sin t \cos t) \geq \sqrt{1 - \sin t \cos t} \geq \sqrt[4]{\frac{1}{2} - \sin^2 t \cos^2 t} \geq$$

$$\geq \frac{1}{\sqrt{2}} \geq \frac{\sin t + \cos t}{2} \geq \sqrt{\sin t \cos t} \geq \frac{2 \sin t \cos t}{\sin t + \cos t} \geq$$

$$\geq \frac{\sin t \cos t}{(\sin t + \cos t)\,(1 - \sin t \cos t)} \geq \frac{\sin t \cos t\,(\sin t + \cos t)}{2\,(1 - \sin t \cos t)}$$

Proof. In Theorem 1 we take $x = tgt$.

Corollary 1.9. If $a_k > 0$ $(k = 1, 2, ..., n)$, then

$$2\sum_{cyclic}\frac{\left(a_1^3+1\right)\left(a_2^3+1\right)\left(a_3^3+1\right)}{\left(a_1^2+1\right)\left(a_2^2+1\right)\left(a_3^2+1\right)} \geq \sum_{cyclic}\frac{a_3\left(a_1a_2+1\right)\left(a_3^2+1\right)}{a_3^3+1}$$

A generalization of problem 3349 Crux Mathematicorum.

Proof. We have

$$\frac{\left(a_1^3+1\right)\left(a_2^3+1\right)}{\left(a_1^2+1\right)\left(a_2^2+1\right)} \geq \sqrt[4]{\frac{\left(a_1^4+1\right)\left(a_2^4+1\right)}{4}} \geq \sqrt{\frac{a_1^2a_2^2+1}{2}} \geq \frac{a_1a_2+1}{2}$$

and $\frac{a_3^3+1}{a_3^2+1} \geq \sqrt{a_3}$ or $\frac{a_3^3+1}{a_3^2+1} \geq \frac{a_3\left(a_3^2+1\right)}{a_3^3+1}$, after multiplication we get

$$\sum_{cyclic} \frac{\left(a_1^3+1\right)\left(a_2^3+1\right)\left(a_3^3+1\right)}{\left(a_1^2+1\right)\left(a_2^2+1\right)\left(a_3^2+1\right)} \geq \sum_{cyclic} \left(\frac{a_1a_2+1}{2}\right)\left(\frac{a_3\left(a_3^2+1\right)}{a_3^3+1}\right)$$

Equality holds if and only if $a_1 = a_2 = ... = a_n = 1$.

If $n = 3$, then we obtain problem 3349 Crux Mathematicorum.

Corollary 1.10. If $a_k > 0$ $(k = 1, 2, ..., n)$, then

$$2\sum_{cyclic} \frac{\left(a_1^3+1\right)\left(a_2^3+1\right)\left(a_3^3+1\right)}{\left(a_1^2+1\right)\left(a_2^2+1\right)\left(a_3^2+1\right)} \geq \sum_{cyclic} \frac{a_1a_2\left(a_3+1\right)\left(a_1^2a_2^2+1\right)}{a_1^3a_2^3+1}$$

A generalization of problem 3349 Crux Mathematicorum.

Proof. We have

$$\frac{\left(a_1^3+1\right)\left(a_2^3+1\right)}{\left(a_1^2+1\right)\left(a_2^2+1\right)} \geq \frac{a_1a_2+1}{2} \geq \frac{a_1a_2\left(a_1^2a_2^2+1\right)}{a_1^3a_2^3+1}$$

and $\frac{a_3^3+1}{a_3^2+1} \geq \frac{a_3+1}{2}$ after multiplication we get:

$$\sum_{cyclic} \frac{\left(a_1^3+1\right)\left(a_2^3+1\right)\left(a_3^3+1\right)}{\left(a_1^2+1\right)\left(a_2^2+1\right)\left(a_3^2+1\right)} \geq \sum_{cyclic} \left(\frac{a_1a_2\left(a_1^2a_2^2+1\right)}{a_1^3a_2^3+1}\right)\left(\frac{a_3+1}{2}\right) \text{ etc.}$$

Equality holds if and only if $a_1 = a_2 = ... = a_n = 1$.

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An extension of Ky Fan‘s inequality

József Sándor[23]

ABSTRACT. We offer a generalization of the inequality $\frac{A_n}{A'_n} \geq \frac{G_n}{G'_n}$, due to Ky Fan.

Let A_n, G_n denote the arithmetic, respectivelly geometric means and of x_i $\left(i = \overline{1,n}\right)$, where $x_i \in \left(0, \frac{1}{2}\right]$. Put $A'_n = A'_n(x_i) = A_n(1 - x_i)$; $G'_n = G'_n(x_i) = G_n(1 - x_i)$. The famous inequality of Ky Fan (see e.g. [1]) states that

$$\frac{A_n}{A'_n} \geq \frac{G_n}{G'_n} \tag{1}$$

Define $A_k = A_k(a_i) = \left(\frac{a_1^k + ... + a_n^k}{n}\right)^{1/k}$ for $k \neq 0$ and $a_i > 0$ $\left(i = \overline{1,n}\right)$, while $A_0 = A_o(a_i) = \lim\limits_{k \to 0} A_k = \sqrt[k]{a_1 ... a_n} = G_n(a) = G_n$.

Then the following extension of (1) holds true:

Theorem. For all $x_i \in \left(0, \frac{1}{2}\right]$ and all $k \in R$ one has

$$1 + \left[\frac{\left(\frac{1-x_1}{x_1}\right)^k + ... + \left(\frac{1-x_n}{x_n}\right)^k}{n}\right]^{\frac{1}{k}} \geq \left(\frac{\frac{1}{x_1^{2k-1}} + ... + \frac{1}{x_n^{2k-1}}}{n}\right)^{\frac{1}{2k-1}} \tag{2}$$

Proof. Remark that for $k = O$, inequality (2) beceomes

$$\sqrt[n]{\frac{1-x_1}{x_1} ... \frac{1-x_n}{x_n}} \geq \frac{x_1 + ... + x_n}{n} - 1,$$

which is relation (1), i.e. the Ky Fan inequality.
Put now $\frac{1-x_i}{x_i} = a_i$ in relation (2). Since $x_i \in \left(0, \frac{1}{2}\right]$, we get $a_i \geq 1$.
After some elementary transformations, relation (2) becomes

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$$1+\left(\frac{a_1^k+...+a_n^k}{n}\right)^{\frac{1}{k}} \geq \left[\frac{(a_1+1)^{2k-1}+...+(a_n+1)^{2k-1}}{n}\right]^{\frac{1}{2k-1}} \tag{3}$$

where $k \neq 0$.
Let $a_k = y_k^{\frac{1}{k}}$ $(y_k > 0)$ and consider the application $f(y) = \left(1 + y^{\frac{1}{k}}\right)^{2k-1}$; where $y \geq 1$. It is immediate that this function is convex for $k < 0$ or $0 < k \leq \frac{1}{2}$, or $k \geq 1$; and concave for $k \in \left[\frac{1}{2}, 1\right]$.
Using the Jensen inequality for convex (concave) functions, relation (3) follows at one for $k \neq \frac{1}{2}$.
For $k = \frac{1}{2}$, however we have to prove the inequality

$$1+\left(\frac{\sqrt{a_1}+...+\sqrt{a_n}}{n}\right)^2 \geq \sqrt[n]{(1+a_1)...(1+a_n)} \tag{4}$$

Letting $a_k = y_k^2$ and $f(y) = \log\left(1+y^2\right)$ which is concave, by Jensen‘s inequality follows at one again (4).

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A refinement of some geometrical inequalities

Mihály Bencze[24]

ABSTRACT. In this paper we present a new inequality, which give some refinements for classical inequalities.

MAIN RESULTS

Theorem 1. If $x, y, z > 0$, then

$$\frac{x+y}{2}\left(\frac{16xyz}{(x+y)(y+z)(z+x)}-1\right)+\frac{\left(\sqrt{x}+\sqrt{y}\right)^4-16xy}{2\left(\sqrt{x}+\sqrt{y}\right)^2}\leq\sqrt{xy}$$

Proof. After elementary calculus we get

$$z\left(\sqrt{x}+\sqrt{y}\right)^2\leq(x+z)(y+z)$$

and finally $\left(z-\sqrt{xy}\right)^2\geq 0$.

Corollary 1.1. In all triangle ABC holds

1). $\frac{a+b}{2}\left(\frac{32R}{s^2+r^2+2Rr}-1\right)+\frac{\left(\sqrt{a}+\sqrt{b}\right)^4-16ab}{2\left(\sqrt{a}+\sqrt{b}\right)^2}\leq\sqrt{ab}$ and his permutations

2). $\frac{c}{2}\left(\frac{4r}{R}-1\right)+\frac{\left(\sqrt{s-a}+\sqrt{s-b}\right)^4-16(s-a)(s-b)}{2\left(\sqrt{s-a}+\sqrt{s-b}\right)^2}\leq\sqrt{(s-a)(s-b)}$ and his permutations. This is a refinement of inequality

$$\frac{c}{2}\left(\frac{4r}{R}-1\right)\leq\sqrt{(s-a)(s-b)},$$

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3). $\frac{h_a+h_b}{2}\left(\frac{32Rr}{s^2+r^2+2Rr}-1\right)+\frac{\left(\sqrt{h_a}+\sqrt{h_b}\right)^4-16h_ah_b}{2\left(\sqrt{h_a}+\sqrt{h_b}\right)^2}\leq\sqrt{h_ah_b}$ and his permutations.

4). $\frac{r_a+r_b}{2}\left(\frac{4r}{R}-1\right)+\frac{\left(\sqrt{r_a}+\sqrt{r_b}\right)^4-16r_ar_b}{2\left(\sqrt{r_a}+\sqrt{r_b}\right)^2}\leq\sqrt{r_ar_b}$ and his permutations.

5). $\frac{\sin^2\frac{A}{2}+\sin^2\frac{B}{2}}{\sin^2\frac{C}{2}}\left(\frac{32Rr^2}{(2R-r)(s^2+r^2-8Rr)-2Rr^2}-1\right)+$
$+\frac{\left(\sin\frac{A}{2}+\sin\frac{B}{2}\right)^4-16\sin^2\frac{A}{2}\sin^2\frac{B}{2}}{2\left(\sin\frac{A}{2}+\sin\frac{B}{2}\right)^2}\leq\sin\frac{A}{2}\sin\frac{B}{2}$ and his permutations.

6). $\frac{\cos^2\frac{A}{2}+\cos^2\frac{B}{2}}{\cos^2\frac{C}{2}}\left(\frac{32s^2R}{(4R+r)^3+s^2(2R-r)}-1\right)+\frac{\left(\cos\frac{A}{2}+\cos\frac{B}{2}\right)^4-16\cos^2\frac{A}{2}\cos^2\frac{B}{2}}{2\left(\cos\frac{A}{2}+\cos\frac{B}{2}\right)^2}\leq$
$\leq\cos\frac{A}{2}\cos\frac{B}{2}$ and his permutations.

Proof. In Theorem 1 we take $(x,y,z)\in$ $\left\{(a,b,c),(s-a,s-b,s-c),(h_a,h_b,h_c),(r_a,r_b,r_c),\left(\sin^2\frac{A}{2},\sin^2\frac{B}{2},\sin^2\frac{C}{2}\right),\left(\cos^2\frac{A}{2},\cos^2\frac{B}{2},\cos^2\frac{C}{2}\right)\right\}$.

Corollary 1.2. In all triangle ABC holds

1). $\frac{a+b}{2}\left(\frac{32Rr}{s^2+r^2+2Rr}-1\right)\leq\sqrt{ab}$
2). $\frac{c}{2}\left(\frac{4r}{R}-1\right)\leq\sqrt{(s-a)(s-b)}$
3). $\frac{h_a+h_b}{2}\left(\frac{32Rr}{s^2+r^2+2Rr}-1\right)\leq\sqrt{h_ah_b}$
4). $\frac{r_a+r_b}{2}\left(\frac{4r}{R}-1\right)\leq\sqrt{r_ar_b}$
5). $\frac{\sin^2\frac{A}{2}+\sin^2\frac{B}{2}}{2}\left(\frac{32Rr^2}{(2R-r)(s^2+r^2-8Rr)-2Rr^2}-1\right)\leq\sin\frac{A}{2}\sin\frac{B}{2}$
6). $\frac{\cos^2\frac{A}{2}+\cos^2\frac{B}{2}}{2}\left(\frac{32Rs^2}{(4R+r)+s^2(2R-r)}-1\right)\leq\cos\frac{A}{2}\cos\frac{B}{2}$

Corollary 1.3. In all triangle ABC holds
1). $2s\left(\frac{32Rr}{s^2+r^2+2Rr}-1\right)\leq\sum\sqrt{ab}$
2). $s\left(\frac{4r}{R}-1\right)\leq\sum\sqrt{(s-a)(s-b)}$
3). $\frac{s^2+r^2+4Rr}{2R}\left(\frac{32Rr}{s^2+r^2+2Rr}-1\right)\leq\sum\sqrt{h_ah_b}$
4). $(4R+r)\left(\frac{4r}{R}-1\right)\leq\sum\sqrt{r_ar_b}$
5). $\frac{2R-r}{2R}\left(\frac{32Rr^2}{(2R-r)(s^2+r^2-8Rr)-2Rr^2}-1\right)\leq\sum\sin\frac{A}{2}\sin\frac{B}{2}$
6). $\frac{4R+r}{2R}\left(\frac{32Rs^2}{(4R+r)^3+s^2(2R-r)}-1\right)\leq\sum\cos\frac{A}{2}\cos\frac{B}{2}$

Theorem 2. In all triangle ABC $(a \neq b \neq c)$ we have

$$\frac{\sqrt{3}}{18R^2r} \leq \sum \frac{1}{a\left(a-b\right)\left(a-c\right)} \leq \frac{\sqrt{3}}{36Rr^2}$$

(A refinement of Euler's inequality $R \geq 2r$)
Proof. We have

$$\sum \frac{1}{a\left(a-b\right)\left(a-c\right)} = \frac{1}{abc} = \frac{1}{4sRr}$$

but $3\sqrt{3}r \leq s \leq \frac{3\sqrt{3}R}{2}$ which finish the assertion.

Corollary 2.1. In all triangle ABC $(a \neq b \neq c)$ we have

$$\frac{2\sqrt{3}}{9Rr^2} \leq \sum \frac{1}{\left(s-a\right)\left(b-a\right)\left(c-a\right)} \leq \frac{\sqrt{3}}{9r^3}$$

(A refinement of Euler's inequality).
Proof. We have

$$\sum \frac{1}{\left(s-a\right)\left(b-a\right)\left(c-a\right)} = \frac{1}{\left(s-a\right)\left(s-b\right)\left(s-c\right)} = \frac{1}{sr^2}$$

but $3\sqrt{3} \leq s \leq \frac{3\sqrt{3}}{2}R$ etc.

Corollary 2.2. In all triangle ABC $(a \neq b \neq c)$ we have

$$\frac{4}{27R^2r} \leq \sum \frac{1}{r_a\left(r_a-r_b\right)\left(r_a-r_c\right)} \leq \frac{1}{27r^3}$$

(A refinement of Euler's inequality).
Proof. We have

$$\sum \frac{1}{r_a\left(r_a-r_b\right)\left(r_a-r_c\right)} = \frac{1}{r_ar_br_c} = \frac{1}{s^2r} \text{ etc.}$$

Corollary 2.3. In all triangle ABC $(a \neq b \neq c)$ holds

$$\frac{2}{27Rr^2} \leq \sum \frac{1}{h_a\left(h_a-h_b\right)\left(h_a-h_c\right)} \leq \frac{R}{54r^4}$$

(A refinement of Euler's inequality).
Proof. We have

$$\sum \frac{1}{h_a\left(h_a-h_b\right)\left(h_a-h_c\right)}=\frac{1}{h_a h_b h_c}=\frac{1}{2s^2r^2} \text{ etc.}$$

Corollary 2.4. In all triangle ABC $(a \neq b \neq c)$ we have

$$3\sqrt{3} \leq \sum \frac{ctg\frac{A}{2}}{\left(tg\frac{A}{2}-tg\frac{B}{2}\right)\left(tg\frac{A}{2}-tg\frac{C}{2}\right)} \leq \frac{3\sqrt{3}R}{2r}$$

(A refinement of Euler‘s inequality).
Proof. We have

$$\sum \frac{ctg\frac{A}{2}}{\left(tg\frac{A}{2}-tg\frac{B}{2}\right)\left(tg\frac{A}{2}-tg\frac{C}{2}\right)}=\frac{1}{\prod tg\frac{A}{2}}=\frac{s}{r} \text{ etc.}$$

Corollary 2.5. In all triangle ABC $(a \neq b \neq c)$ we have

$$\frac{2\sqrt{3}r}{9R} \leq \sum \frac{tg\frac{A}{2}}{\left(ctg\frac{A}{2}-ctg\frac{B}{2}\right)\left(ctg\frac{A}{2}-ctg\frac{C}{2}\right)} \leq \frac{\sqrt{3}}{9}$$

(A refinement of Euler‘s inequality).
Proof. We have

$$\sum \frac{tg\frac{A}{2}}{\left(ctg\frac{A}{2}-ctg\frac{B}{2}\right)\left(ctg\frac{A}{2}-ctg\frac{C}{2}\right)}=\frac{1}{\prod ctg\frac{A}{2}}=\frac{r}{s} \text{ etc.}$$

Corollary 2.6. In all triangle ABC $(a \neq b \neq c)$ we have

$$\sum \frac{1}{\sin^2\frac{A}{2}\left(\sin^2\frac{A}{2}-\sin^2\frac{B}{2}\right)\left(\sin^2\frac{A}{2}-\sin^2\frac{C}{2}\right)} > 64$$

Proof. We have

$$\sum \frac{1}{\sin^2\frac{A}{2}\left(\sin^2\frac{A}{2}-\sin^2\frac{B}{2}\right)\left(\sin^2\frac{A}{2}-\sin^2\frac{C}{2}\right)}=\frac{16R^2}{r^2} \geq 64.$$

Corollary 2.7. In all triangle ABC $(a \neq b \neq c)$ we have

$$\frac{64}{27} \leq \sum \frac{1}{\cos^2\frac{A}{2}\left(\cos^2\frac{A}{2}-\cos^2\frac{B}{2}\right)\left(\cos^2\frac{A}{2}-\cos^2\frac{C}{2}\right)} \leq \frac{16R^2}{27r^2}$$

(A refinement of Euler‘s inequality).
Proof. We have

$$\sum \frac{1}{\cos^2\frac{A}{2}\left(\cos^2\frac{A}{2}-\cos^2\frac{B}{2}\right)\left(\cos^2\frac{A}{2}-\cos^2\frac{C}{2}\right)}=\frac{16R^2}{s^2} \text{ etc.}$$

Theorem 3. In all triangle ABC $(a \neq b \neq c)$ we have

$$6\sqrt{3}r \leq \sum \frac{a^3}{(a-b)(a-c)} \leq 3\sqrt{3}R$$

(A refinement of Euler's inequality).
Proof. We have

$$\sum \frac{a^3}{(a-b)(a-c)} = a+b+c = 2s$$

but $3\sqrt{3}r \leq s \leq \frac{3\sqrt{3}}{2}R$ etc.

Corollary 3.1. In all triangle ABC $(a \neq b \neq c)$ we have

$$3\sqrt{3}r \leq \sum \frac{(s-a)^3}{(b-a)(c-a)} \leq \frac{3\sqrt{3}}{2}R$$

(A refinement of Euler's inequality).
Proof. We have

$$\sum \frac{(s-a)^3}{(b-a)(c-a)} = \sum (s-a) = s \text{ etc.}$$

Corollary 3.2. In all triangle ABC $(a \neq b \neq c)$ we have

$$\frac{2r(7r+R)}{R} \leq \sum \frac{h_a^3}{(h_a-h_b)(h_a-h_c)} \leq \frac{27R^2+16Rr+4r^2}{8R}$$

(A refinement of Euler's inequality).
Proof. We have

$$\sum \frac{h_a^3}{(h_a-h_b)(h_a-h_c)} = \sum h_a = \frac{s^2+r^2+4Rr}{2R} \text{ etc.}$$

Corollary 3.3. In all triangle ABC $(a \neq b \neq c)$ we have

$$9r \leq \sum \frac{r_a^3}{(r_a-r_b)(r_a-r_c)} \leq \frac{9R}{2}$$

(A refinement of Euler's inequality).
Proof. We have

$$\sum \frac{r_a^3}{(r_a-r_b)(r_a-r_c)} = 4R+r \text{ etc.}$$

Corollary 3.4. In all triangle ABC $(a \neq b \neq c)$ we have

$$3\left(4R+r\right)r\sqrt{3} \leq \sum \frac{tg^3\frac{A}{2}}{\left(tg\frac{A}{2}-tg\frac{B}{2}\right)\left(tg\frac{A}{2}-tg\frac{C}{2}\right)} \leq \frac{3\left(4R+r\right)R\sqrt{3}}{2}$$

(A refinement of Euler‘s inequality).
Proof. We have

$$\sum \frac{tg^3\frac{A}{2}}{\left(tg\frac{A}{2}-tg\frac{B}{2}\right)\left(tg\frac{A}{2}-tg\frac{C}{2}\right)} = \frac{4R+r}{s} \text{ etc.}$$

Corollary 3.5. In all triangle ABC $(a \neq b \neq c)$ we have

$$3\sqrt{3} \leq \sum \frac{ctg^3\frac{A}{2}}{\left(ctg\frac{A}{2}-ctg\frac{B}{2}\right)\left(ctg\frac{A}{2}-ctg\frac{C}{2}\right)} \leq \frac{3\sqrt{3}R}{2r}$$

(A refinement of Euler‘s inequality).
Proof. We have

$$\sum \frac{ctg^3\frac{A}{2}}{\left(ctg\frac{A}{2}-ctg\frac{B}{2}\right)\left(ctg\frac{A}{2}-ctg\frac{C}{2}\right)} = \frac{s}{r} \text{ etc.}$$

Corollary 3.6. In all triangle ABC $(a \neq b \neq c)$ we have

$$\sum \frac{\sin^6\frac{A}{2}}{\left(\sin^2\frac{A}{2}-\sin^2\frac{B}{2}\right)\left(\sin^2\frac{A}{2}-\sin^2\frac{C}{2}\right)} > \frac{3}{4}$$

Proof. We have

$$\sum \frac{\sin^6\frac{A}{2}}{\left(\sin^2\frac{A}{2}-\sin^2\frac{B}{2}\right)\left(\sin^2\frac{A}{2}-\sin^2\frac{C}{2}\right)} = 1-\frac{r}{2R} \geq \frac{3}{4}$$

Corollary 3.7. In all triangle ABC $(a \neq b \neq c)$ we have

$$\sum \frac{\cos^6\frac{A}{2}}{\left(\cos^2\frac{A}{2}-\cos^2\frac{B}{2}\right)\left(\cos^2\frac{A}{2}-\cos^2\frac{C}{2}\right)} < \frac{9}{4}$$

Proof. We have

$$\sum \frac{\cos^6\frac{A}{2}}{\left(\cos^2\frac{A}{2}-\cos^2\frac{B}{2}\right)\left(\cos^2\frac{A}{2}-\cos^2\frac{C}{2}\right)} = \sum \cos^2\frac{A}{2} = 2+\frac{r}{2R} \leq \frac{9}{4}$$

Theorem 4. In all triangle ABC holds

$$4r\left(20r-R\right)\leq\sum\frac{a^4}{\left(a-b\right)\left(a-c\right)}\leq\frac{81R^2}{4}-r^2-4Rr$$

(A refinement of Euler‘s inequality).
Proof. We have

$$\sum\frac{a^4}{\left(a-b\right)\left(a-c\right)}=\sum a^2+\sum ab=3s^2-r^2-4Rr\text{ etc.}$$

Corollary 4.1. In all triangle ABC $(a\neq b\neq c)$ we have

$$2r\left(13r-2R\right)\leq\sum\frac{\left(s-a\right)^4}{\left(b-a\right)\left(c-a\right)}\leq\frac{27R^2}{4}-r^2-4Rr$$

(A refinement of Euler‘s inequality).
Proof. We have

$$\sum\frac{\left(s-a\right)^4}{\left(b-a\right)\left(c-a\right)}=\sum\left(s-a\right)^2+\sum\left(s-a\right)\left(s-b\right)=s^2-r^2-4Rr$$

Corollary 4.2. In all triangle ABC $(a\neq b\neq c)$ we have

$$\frac{37R^2}{4}+8Rr+r^2\leq\sum\frac{r_a^4}{\left(r_a-r_b\right)\left(r_a-r_c\right)}\leq16R^2+8Rr-26r^2$$

(A refinement of Euler‘s inequality).
Proof. We have

$$\sum\frac{r_a^4}{\left(r_a-r_b\right)\left(r_a-r_c\right)}=\left(4R+r\right)^2-s^2\text{ etc.}$$

Corollary 4.3. In all triangle ABC $(a\neq b\neq c)$ we have

$$\left(\frac{2\left(4R+r\right)}{3\sqrt{3}R}\right)^2-1\leq\sum\frac{tg^4\frac{A}{2}}{\left(tg\frac{A}{2}-tg\frac{B}{2}\right)\left(tg\frac{A}{2}-tg\frac{C}{2}\right)}\leq\left(\frac{4R+r}{3\sqrt{3}}\right)^2-1$$

(A refinement of Euler‘s inequality).
Proof. We have

$$\sum\frac{tg^4\frac{A}{2}}{\left(tg\frac{A}{2}-tg\frac{B}{2}\right)\left(tg\frac{A}{2}-tg\frac{C}{2}\right)}=\left(\frac{4R+r}{s}\right)^2-1\text{ etc.}$$

Corollary 4.4. In all triangle ABC $(a \neq b \neq c)$ we have

$$26-\frac{4R}{r}\leq\sum\frac{ctg^4\frac{A}{2}}{\left(ctg\frac{A}{2}-ctg\frac{B}{2}\right)\left(ctg\frac{A}{2}-ctg\frac{C}{2}\right)}\leq\left(\frac{3\sqrt{3}R}{2r}\right)^2-\frac{4R}{r}-1$$

(A refinement of Euler‘s inequality).
Proof. We have

$$\sum\frac{ctg^4\frac{A}{2}}{\left(ctg\frac{A}{2}-ctg\frac{B}{2}\right)\left(ctg\frac{A}{2}-ctg\frac{C}{2}\right)}=\left(\frac{s}{r}\right)^2-\frac{4R}{r}-1$$

Corollary 4.5. In all triangle ABC $(a \neq b \neq c)$ we have

$$\frac{37R^2+12r^2-32Rr}{64R^2}\leq\sum\frac{\sin^8\frac{A}{2}}{\left(\sin^2\frac{A}{2}-\sin^2\frac{B}{2}\right)\left(\sin^2\frac{A}{2}-\sin^2\frac{C}{2}\right)}\leq$$
$$\leq\frac{2R^2-Rr-3r^2}{2R^2}$$

(A refinement of Euler‘s inequality).
Proof. We have

$$\sum\frac{\sin^8\frac{A}{2}}{\left(\sin^2\frac{A}{2}-\sin^2\frac{B}{2}\right)\left(\sin^2\frac{A}{2}-\sin^2\frac{C}{2}\right)}=\frac{16R^2+3r^2-8Rr-s^2}{16R^2}$$

Corollary 4.6. In all triangle ABC $(a \neq b \neq c)$ we have

$$3\left(1+\frac{r}{4R}\right)^2-\frac{27}{64}\leq\sum\frac{\cos^8\frac{A}{2}}{\left(\cos^2\frac{A}{2}-\cos^2\frac{B}{2}\right)\left(\cos^2\frac{A}{2}-\cos^2\frac{C}{2}\right)}=$$
$$=3\left(1+\frac{r}{4R}\right)^2-\left(\frac{3\sqrt{3}r}{4R}\right)^2$$

(A refinement of Euler‘s inequality).
Proof. We have

$$\sum\frac{\cos^8\frac{A}{2}}{\left(\cos^2\frac{A}{2}-\cos^2\frac{B}{2}\right)\left(\cos^2\frac{A}{2}-\cos^2\frac{C}{2}\right)}=3\left(1+\frac{r}{4R}\right)^2-\left(\frac{s}{2R}\right)^2 \text{ etc.}$$

Theorem 5. In all triangle ABC holds

$$\frac{R+7r}{4s^2R^2r} \leq \sum \frac{1}{a^2(a-b)(a-c)} \leq \frac{27R^2+16Rr+4r^2}{16s^2R^2r^2}$$

(A refinement of Euler's inequality).
Proof. We have

$$\sum \frac{1}{a^2(a-b)(a-c)} = \frac{\sum ab}{a^2b^2c^2} = \frac{s^2+r^2+4Rr}{16s^2R^2r^2} \text{ etc.}$$

Corollary 5.1. In all triangle ABC $(a \neq b \neq c)$ we have

$$\frac{4(4R+r)}{27R^2r^3} \leq \sum \frac{1}{(s-a)^2(b-a)c(-a)} \leq \frac{4R+r}{27r^5}$$

(A refinement of Euler's inequality).
Proof. We have

$$\sum \frac{1}{(s-a)^2(b-a)c(-a)} = \frac{4R+r}{s^2r^3} \text{ etc.}$$

Corollary 5.2. In all triangle ABC $(a \neq b \neq c)$ we have

$$\frac{2}{27Rr^2} \leq \sum \frac{1}{h_a^2(h_a-h_b)(h_a-h_c)} \leq \frac{R}{54r^4}$$

(A refinement of Euler's inequality).
Proof. We have

$$\sum \frac{1}{h_a^2(h_a-h_b)(h_a-h_c)} = \frac{R}{2s^2r^2} \text{ etc.}$$

Corollary 5.3. In all triangle ABC $(a \neq b \neq c)$ we have

$$\frac{4}{27R^2r^2} \leq \sum \frac{1}{r_a^2(r_a-r_b)(r_a-r_c)} \leq \frac{1}{27r^4}$$

(A refinement of Euler's inequality).
Proof. We have

$$\sum \frac{1}{r_a^2(r_a-r_b)(r_a-r_c)} = \frac{1}{s^2r^2} \text{ etc.}$$

Corollary 5.4. In all triangle ABC $(a \neq b \neq c)$ we have

$$27 \leq \sum \frac{ctg^2 \frac{A}{2}}{\left(tg\frac{A}{2} - tg\frac{B}{2}\right)\left(tg\frac{A}{2} - tg\frac{C}{2}\right)} \leq \frac{27R^2}{4r^2}$$

(A refinement of Euler‘s inequality).
Proof. We have

$$\sum \frac{ctg^2 \frac{A}{2}}{\left(tg\frac{A}{2} - tg\frac{B}{2}\right)\left(tg\frac{A}{2} - tg\frac{C}{2}\right)} = \left(\frac{s}{r}\right)^2 \text{ etc.}$$

Corollary 5.5. In all triangle ABC $(a \neq b \neq c)$ we have

$$\frac{4\,(4R+r)\,r}{27R^2} \leq \sum \frac{tg^2 \frac{A}{2}}{\left(ctg\frac{A}{2} - ctg\frac{B}{2}\right)\left(ctg\frac{A}{2} - ctg\frac{C}{2}\right)} \leq \frac{4R+r}{27r}$$

(A refinement of Euler‘s inequality).
Proof. We have

$$\sum \frac{tg^2 \frac{A}{2}}{\left(ctg\frac{A}{2} - ctg\frac{B}{2}\right)\left(ctg\frac{A}{2} - ctg\frac{C}{2}\right)} = \frac{(4R+r)\,r}{s^2} \text{ etc.}$$

Corollary 5.6. In all triangle ABC $(a \neq b \neq c)$ we have

$$\frac{64\,(7r-2R)\,R^2}{r^3} \leq \sum \frac{1}{\sin^4\frac{A}{2}\left(\sin^2\frac{A}{2} - \sin^2\frac{B}{2}\right)\left(\sin^2\frac{A}{2} - \sin^2\frac{C}{2}\right)} \leq$$
$$\leq \frac{4\left(27R^2 + 4r^2 - 32Rr\right)R^2}{r^4}$$

(A refinement of Euler‘s inequality).
Proof. We have

$$\sum \frac{1}{\sin^4\frac{A}{2}\left(\sin^2\frac{A}{2} - \sin^2\frac{B}{2}\right)\left(\sin^2\frac{A}{2} - \sin^2\frac{C}{2}\right)} = \frac{16\left(s^2 + r^2 - 8Rr\right)R^2}{r^4} \text{ etc.}$$

Corollary 5.7. In all triangle ABC $(a \neq b \neq c)$ we have

$$\frac{64R^2\left(24R^2 + 2Rr + 7r^2\right)}{s^4} \leq$$

$$\leq \sum \frac{1}{\cos^4\frac{A}{2}\left(\cos^2\frac{A}{2} - \cos^2\frac{B}{2}\right)\left(\cos^2\frac{A}{2} - \cos^2\frac{C}{2}\right)} \leq \frac{4R^2\left(91R^2 + 32Rr + 4r^2\right)}{s^4}$$

(A refinement of Euler‘s inequality).
Proof. We have

$$\sum \frac{1}{\cos^4 \frac{A}{2}\left(\cos^2 \frac{A}{2} - \cos^2 \frac{B}{2}\right)\left(\cos^2 \frac{A}{2} - \cos^2 \frac{C}{2}\right)} = \frac{16\left(s^2 + (4R+r)^2\right)R^2}{s^4} \text{ etc.}$$

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$e^{i\pi}=-1$

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On certain conjectures in prime number theory

József Sándor[25]

ABSTRACT. Some remarks between connections of certain famous conjectures on prime differences are pointed out.

Let p_n denote the n^{th} prime number. Put $d_n = p_{n+1} - p_n$. As

$$\frac{d_1 + d_2 + ... + d_n}{n} = \frac{p_{n+1} - 2}{n} \sim \log n \sim \log p_n \text{ as } n \to \infty$$

(by the prime number theorem $p_n \sim n \log n$); we can say that the average order of d_n is $\log p_n$.

MAIN RESULTS

The study of gaps between primes has a long history.

In 1920 H. Crámer ([1]) proved that the famous Riemann hypothesis implies that

$$d_n = O\left(p_n^{1/2} \cdot \log p_n\right) \tag{1}$$

We note that the Riemann hypothesis implies a weaker hypothesis, namely, the so-called Lindelöf hypothesis, to the effect that for any $\varepsilon > 0$ one has

$$\zeta\left(\frac{1}{2} + it\right) = O\left(t^{\varepsilon}\right) \text{ as } t \to \infty$$

A.E. Ingham showed (see e.g. [11]) that by assuming only the Lindelöf hypothesis, one has for any $\varepsilon > 0$

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$$d_n < p_n^{1/2+\varepsilon}, \text{ for } n \geq n_0 \tag{2}$$

(i.e. if n is sufficiently large).
Clearly, relation (1) is stronger than (2) as $c \log p_n < p_n^{\varepsilon}$ for any fixed constants $c, \varepsilon > 0$; if n is sufficiently large.
In 1937, based on probability arguments, Crámer conjectured ([5]) even a stronger relation than (1), namely that

$$d_n = O\left((\log p_n)^2\right) \tag{3}$$

The strong inequality (3) perhaps is not true; and indeed, recently A. Granville ([4], [6]) conjectured that for infinitely many n one has

$$d_n > k\,(\log p_n)^2, \tag{4}$$

where $k = 2 \cdot e^{-\gamma} \approx 1,12292...$ (e and γ being the two Euler constants). See also H. Maier [7].
Crámer conjectured in fact a more precise form of (3), namely that

$$\limsup_{n\to\infty} \frac{d_n}{(\log p_n)^2} = c \tag{5}$$

where c is finite constant (perhaps; $c = 1$).
As noted by K. Soundararajan [9], (5) is for beyond what "reasonable" conjectures such as the Riemann hypothesis would imply.
An old conjecture says that there is always a prime between two consecutive squares (see e.g. [8], [11]). Even this lies slightly beyond the reach of the Riemann hypothesis, and all it would imply is that

$$\limsup_{n\to\infty} \frac{d_n}{\sqrt{p_n}} \leq 4, \tag{6}$$

which is weaker than (5) with a finite value of c.
A. Schinzel conjectured (see [8]) that between x and $x + (\log x)^2$ there is a prime, if $x \geq x_0$ ($x_0 \approx 7,1374035$).
Thus $p_n < p_{n+1} < p_n + (\log p_n)^2$ if $p_n \geq 8$ $(i.e.\ n \geq 4)$, so this would imply

$$\limsup_{n\to\infty} \frac{p_{n+1} - p_n}{(\log p_n)^2} \leq 1, \tag{7}$$

i.e. c of (5) is ≤ 1. This would imply also the conjecture (3) of Crámer.
Schinzel‘s conjecture gives also

$$0 < \frac{p_{n+1} - p_n}{\sqrt{p_n}} < \frac{(\log p_n)^2}{\sqrt{p_n}} \to 0 \text{ as } n \to \infty,$$

so

$$\frac{d_n}{\sqrt{p_n}} \to 0 \text{ as } n \to \infty \tag{8}$$

As

$$\sqrt{p_{n+1}} - \sqrt{p_n} = \frac{p_{n+1} - p_n}{\sqrt{p_n} + \sqrt{p_{n+1}}} = \frac{p_{n+1} - p_n}{\sqrt{p_n}\left(1 + \sqrt{\frac{p_{n+1}}{p_n}}\right)}$$

and since $\frac{p_{n+1}}{p_n} \to 1$, we get that (8) is equivalent with

$$\sqrt{p_{n+1}} - \sqrt{p_n} \to 0 \text{ as } n \to \infty \tag{9}$$

In our paper [10] it is proved that (and even stronger results)

$$\liminf_{n\to\infty} \sqrt[4]{p_n}\left(\sqrt{p_{n+1}} - \sqrt{p_n}\right) = 0 \tag{10}$$

Motivated by our paper, D. Andrica ([8]) conjectured that

$$\sqrt{p_{n+1}} - \sqrt{p_n} < 1 \text{ for all } n \geq 1 \tag{11}$$

As for $x \geq 121$ one has $(\log x)^2 < 2\sqrt{x} + 1$, we get that

$$(\log p_n)^2 < 2\sqrt{p_n} + 1$$

if $p_n \geq 121$, so

$$p_{n+1} - p_n < (\log p_n)^2 < 2\sqrt{p_n} + 1, \text{ i.e.}$$

(11) holds true, which means that Schinzel‘s conjecture implies the Andrica conjecture. Another conjecture, due to Piltz is that for any $\varepsilon > 0$

$$d_n = O\left(p_n^{\varepsilon}\right) \tag{12}$$

As $(\log p_n)^2 / p_n^{\varepsilon} \to 0$ as $n \to \infty$, clearly (3) is stronger than (12).
On the other hand, (12) is stronger than (11), for sufficiently large n, as letting e.g. $\varepsilon = \frac{1}{3}$ in (12) we get

$$p_{n+1} - p_n < Mp_n^{\frac{1}{3}} < 2p_n^{\frac{1}{2}} + 1 \text{ if } n \geq n_0$$

But (12) implies even relation (2), as from $p_{n+1} - p_n < M \cdot p_n^{\frac{1}{3}}$ it follows immediately that $\frac{p_{n+1}-p_n}{\sqrt{p_n}} \to 0$ as $n \to \infty$, i.e. (8), which is equivalent to (9).
Inequality (11) seems to be true; at least for $n = 4$ we get
$\sqrt{p_5} - \sqrt{p_4} = \sqrt{11} - \sqrt{7} \approx 0,670873$ which is the largest value among the first 10^5 primes.
In 2001 Baker, Harman and Pintz ([3]) obtained an unconditional result on the maximum value of d_n. They proved that, without any hypothess one has

$$d_n < p_n^{0,525} \text{ for } n \geq n_0 \tag{13}$$

and perhaps there are hopes to prove unconditionally inequality (2) for not only $\varepsilon = 0,025$ but any $\varepsilon > 0$.
Recently, a major advance was made by Goldston, Pintz and Yildirim (see e.g. [9]); who proved without any assumption that

$$\liminf_{n\to\infty} \frac{d_n}{\log p_n} = 0$$

Clearly, this implies also that the lim inf of expression (8) is true; but remains open the proof of the similar fact for the lim sup.
As noted by K. Soundarajan [9], theorem (14) shows that given $\varepsilon > 0$, for infinitely many n, the interval $[n, n + \varepsilon \cdot \log n]$ contains at least two primes.

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New inequality in tetrahedron

Mihály Bencze[26]

ABSTRACT. In this paper we will prove a new inequality for the tetrahedron.

MAIN RESULTS

Theorem 1. (A refinement of Finsler-Hadwiger‘s inequality). In all triangle ABC holds

$$6\left(ab+bc+ca\right)-5\left(a^2+b^2+c^2\right)\leq 4\sqrt{3}\sigma\left[ABC\right]\leq$$

$$\leq 2\left(ab+bc+ca\right)-\left(a^2+b^2+c^2\right)$$

Proof. Using the Schur‘s theorem for all $x, y, z > 0$ and $t \in R$ we get

$$x^t\left(x-y\right)\left(x-z\right)+y^t\left(y-z\right)\left(y-x\right)+z^t\left(z-x\right)\left(z-y\right)\geq 0$$

If $t = 2$ and $x = a, y = b, z = c$, then we obtain

$$\sum a^4+abc\sum a\geq\sum ab\left(a^2+b^2\right)$$

or

$$4\left(\sum ab\right)^2+\left(\sum a^2\right)^2-4\left(\sum ab\right)\left(\sum a^2\right)\geq$$

$$\geq 3\left(\sum a\right)\prod\left(b+c-a\right)$$

which is equivalent with

$$4\sqrt{3}\sigma\left[ABC\right]\leq 2\sum ab-\sum a^2$$

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Using again the Schur‘s theorem we get

$$\left(2\sum xy-\sum x^2\right)\left(\sum x\right)\leq 9xyz$$

but $27xyz\leq\left(\sum x\right)^3$, therefore

$$2\sum xy-\sum x^2\leq\sqrt{3xyz\left(\sum x\right)}$$

If $x+y=a$, $y+z=b$, $z+x=c$, then we obtain
$6\sum ab-5\sum a^2\leq 4\sqrt{3}\sigma\left[ABC\right]$.

Corollary 1.1. In all triangle ABC holds

$$4r^2+16Rr-s^2\leq\sqrt{3}sr\leq r\left(4R+r\right)$$

Proof. In Theorem 1 we use that

$$\sum ab=s^2+r^2+4Rr,\ \ \sum a^2=2\left(s^2-r^2-4Rr\right)$$

and $\sigma\left[ABC\right]=sr$.

Theorem 2. Let $ABCD$ be a tetrahedron with h_A and m_A the lengths of the altitude and the median from vertex A to the opposite face BCD, respectively, and V denote the volume of the tetrahedron. If

$$M=(BC-CD)^2+(BC-BD)^2+(BC-AC)^2+(BC-AB)^2+$$

$$+(CD-BD)^2+(CD-AD)^2+(CD-AC)^2+(BD-AD)^2+(BD-AB)^2+$$

$$+(AD-AC)^2+(AD-AB)^2+(AB-AC)^2,$$

then

$$\left(\sum h_A\right)\left(\sum m_A^2\right)\geq\frac{128V}{\sqrt{3}}+\frac{32r}{9}M,$$

where r denote the radius of insphere of tetrahedron.
Proof. We have $\sum m_A^2=\frac{4}{9}\sum AB^2$. Denote S_A the area of face BCD etc., and using the Theorem 1, we obtain

$$\begin{cases} BC^2+CD^2+DB^2 \geq 4S_A\sqrt{3}+(BC-CD)^2+(BC-DB)^2+(CD-DB)^2 \\ AC^2+CD^2+DA^2 \geq 4S_B\sqrt{3}+(AC-CD)^2+(AC-DA)^2+(CD-DA)^2 \\ AB^2+BD^2+DA^2 \geq 4S_C\sqrt{3}+(AB-BD)^2+(AB-DA)^2+(BD-DA)^2 \\ AB^2+BC^2+AC^2 \geq 4S_D\sqrt{3}+(AB-BC)^2+(AB-AC)^2+(BC-AC)^2 \end{cases}$$

After addition we get

$$2\sum AB^2 \geq 4\sqrt{3}\sum S_A + M \text{ or } \frac{9}{2}\sum m_A^2 \geq 4\sqrt{3}\sum S_A + M \text{ or}$$

$$\sum m_A^2 \geq \frac{8\sqrt{3}}{9}\sum S_A + \frac{2}{9}M.$$

Multiplying by $\sum h_A$ we obtain:

$$\left(\sum h_A\right)\left(\sum m_A^2\right) \geq \frac{8\sqrt{3}}{9}\left(\sum S_A\right)\left(\sum h_A\right) + \frac{2}{9}M\left(\sum h_A\right).$$

If we suppose that $S_A \leq S_B \leq S_C \leq S_D$, then $h_A \geq h_B \geq h_C \geq h_D$ and from Chebishev's inequality holds

$$\left(\sum S_A\right)\left(\sum h_A\right) \geq 4\sum S_A h_A = 48V$$

Because $\sum \frac{1}{h_A} = \frac{1}{r}$, therefore $\sum h_A \geq \frac{16}{\sum \frac{1}{h_A}} = 16r$, thus

$$\left(\sum h_A\right)\left(\sum m_A^2\right) \geq \frac{8\sqrt{3}}{9}\left(\sum S_A\right)\left(\sum h_A\right) + \frac{2}{9}M\left(\sum h_A\right) \geq$$

$$\geq \frac{8\sqrt{3}}{9}\cdot 48V + \frac{2}{9}M\cdot 16r = \frac{128V}{\sqrt{3}} + \frac{32r}{9}M$$

Equality holds if and only if $AB = BC = CD = DA = BD = AC$.

Open Question. Using the notation of Theorem 2, determine the best constants $c_1, c_2 > 0$ such that

$$\frac{128V}{\sqrt{3}} + c_1 MR \geq \left(\sum h_A\right)\left(\sum m_A^2\right) \geq \frac{128V}{\sqrt{3}} + c_2 Mr,$$

where R denote the radius of circumsphere of tetrahedron.

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Inequalities involving $\sigma_2(n)$

József Sándor[27]

ABSTRACT. We offer certain inequalities involving $\sigma_2(n)$ for odd or even values of n. Some consequences are pointed out, too.

MAIN RESULTS

1. In a former (see [2]) we have proved the following double inequality for all $n \geq 2$:

$$\frac{(\sigma(n))^2}{d(n)} < \sigma_2(n) < \frac{\pi^2}{6} \cdot n^2 \tag{1}$$

In fact, the right side of (1) may be improved in some series.

Let $J_2(n)$ denote the Jordan totient of order 2 (see e.g. [1]), i.e.

$$J_2(n) = n^2 \cdot \prod_{p|n} \left(1 - \frac{1}{p^2}\right), \tag{2}$$

where p runs throught the prime divisors of n.

If $n = p_1^{a_1} ... p_r^{a_r}$ is the prime factorization of n, then

$$\sigma_2(n) = \prod_{i=1}^{r} \frac{p_i^{2a_i+2} - 1}{p_i^2 - 1} = \prod_{i=1}^{r} \frac{p_i^{2a_i+2}\left(1 - \frac{1}{p_i^{2a_i+2}}\right)}{\left(p_i^2 - 1\right)} < \prod_{i=1}^{r} \frac{p_i^{2a_i+2}}{p_i^2 - 1} =$$

$$= n^2 \prod_{p|n} \frac{p^2}{p^2 - 1} = n^2 \prod_{p|n} \frac{1}{1 - \frac{1}{p^2}}, \text{ i.e.}$$

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$$\frac{\sigma_2(n)}{n^2} < \prod_{p|n} \frac{1}{1-\frac{1}{p^2}} = \frac{n^2}{J_2(n)}, \text{ so:}$$

$$\frac{\sigma_2(n)}{n^2} < \frac{n^2}{J_2(n)} \text{ for } n \geq 2 \tag{3}$$

Now, as $\frac{n^2}{J_2(n)} = \prod\limits_{p|n} \frac{1}{1-\frac{1}{p^2}} < \prod\limits_{p \text{ prime}} \frac{1}{1-\frac{1}{p^2}}$ (since for $p \nmid n$, $1-\frac{1}{p^2} < 1$) and by $\prod\limits_{p \text{ prime}} \frac{1}{1-\frac{1}{p^2}} = \zeta(2) = \frac{\pi^2}{6}$ by Euler‘s formula, so we get

$$\frac{\sigma_2(n)}{n^2} < \frac{n^2}{J_2(n)} < \frac{\pi^2}{6} \text{ for } n \geq 2 \tag{4}$$

2. The method shown in [2] however offers an improvement of the constant $\frac{\pi^2}{6}$, for odd values of n.

Namely, as in [2], write

$$\sigma_2(n) = \sum_{d|n} d^2 = \sum_{d|n} \left(\frac{n}{d}\right)^2 = n^2 \cdot \sum_{d|n} \frac{1}{d^2}$$

Now, if $n = 2m-1$ is an odd number, then all divisors d of n are odd, too; so

$$\sum_{d|2m-1} \frac{1}{d^2} \leq \frac{1}{1^2} + \frac{1}{3^2} + ... + \frac{1}{(2m-1)^2}$$

Since $\frac{1}{1^2} + ... + \frac{1}{(2m-1)^2} < \sum\limits_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}$, we get:

$$\sigma_2(2m-1) < (2m-1)^2 \frac{\pi^2}{8}, \; m \geq 1 \tag{5}$$

i.e. an improvement of right-side of (1) for odd values of n.

3. A lower bound for $\sigma_2(n)$, distinct from that of the left-side of (1) is offered for even values of n:

If n is even, then

$$\sigma_2(n) \geq \frac{5}{4} n^2 \tag{6}$$

Indeed, let $n = 2^k N$ with $k \geq 1$, N odd. Since $\sigma_2\left(2^k\right) = \frac{2^{2k+2}-1}{3}$ and σ_2 is multiplicative, with $\sigma_2(N) \geq N^2$, we have $\sigma_2(n) \geq \frac{2^{2k+2}-1}{3} \cdot N^2 = \frac{4 \cdot 2^{2k}-1}{3 \cdot 2^{2k}} \cdot n^2$. It is easy to verify that

$$\frac{4 \cdot 2^{2k}-1}{3 \cdot 2^{2k}} \geq \frac{5}{4}, \tag{7}$$

or equivalently $2^{2k} \geq 2^2$, valid for all $k \geq 1$, (7) follows and this finishes the proof of (6).

Letting $n = 2m$ in (6), we get

$$\sigma_2(2n) \geq 5m^2 \tag{8}$$

with equality only for $m = 1$.

Corollary 1.

$$\sigma_2(2m) > \sigma_2(2m-1), \text{ for any } m \geq 1 \tag{9}$$

This follows by (5) and (8), and remarking that

$$5m^2 > \frac{\pi^2}{8}(2m-1)^2$$

This follows e.g. by $\pi^2 < 10$.

As $\pi^2(2n+1)^2 < 40n^2$; i.e. $\pi(2n+1) < 2n\sqrt{10}$, where $\sqrt{10} \approx 3,16$ holds true for any $n \geq 8$ (i.e. $2n\left(\sqrt{10}-\pi\right) > \pi = 3,14...$), we got the following:

Corollary 2. For any $m \geq 8$ one has

$$\sigma_2(2m) > \sigma_2(2m+1) \tag{10}$$

Remark. $\sigma_2(6) = \sigma_2(7) = 50$; *as* $1^2 + 2^2 + 3^2 + 6^2 = 50$ *and* $1^2 + 7^2 = 10$. *Since* $\sigma_2(2m) \neq \sigma_2(2k)$ *for* $m \neq k$ *and* $\sigma_2(2m-1) \neq \sigma_2(2k-1)$ *for* $m \neq k$*; clearly, by (9) and (10); the only solution to the equation*

$$\sigma_2(n) = \sigma_2(n+1) \tag{11}$$

is $n = 6$.

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A property of polynomials

Mihály Bencze and Lili Smarandache[28]

ABSTRACT. In this paper we prove a property of polynomials which have all the roots real numbers.

MAIN RESULTS

Theorem Denote $x_1, x_2, ..., x_n$ all the real roots of the equation

$$x^n + a_1 x^{n-1} + ... + a_n = 0,$$

where $n \geq 2$. If $x_1 \leq x_2 \leq ... \leq x_n$, then

$$\frac{2(n-1)a_1^2 - 4na_2}{n(n-1)} \leq (x_n - x_1)^2 \leq \frac{2(n-1)a_1^2 - 4na_2}{n}$$

Proof. Starting from Lagrange identity we get:

$$n\sum_{k=1}^{n} x_k^2 - \left(\sum_{k=1}^{n} x_k\right)^2 = \sum_{1\leq i<j\leq n} (x_i - x_j)^2 \leq \frac{n(n-1)}{2}(x_n - x_1)^2$$

but

$$n\sum_{k=1}^{n} x_k^2 - \left(\sum_{k=1}^{n} x_k\right)^2 = n\left(\left(\sum_{k=1}^{n} x_k\right)^2 - 2\sum_{1\leq i<j\leq n} x_i x_j\right) - \left(\sum_{k=1}^{n} x_k\right)^2 =$$

$$= (n-1)a_1^2 - 2na_2$$

therefore

$$(x_n - x_1)^2 \geq \frac{2(n-1)a_1^2 - 4na_2}{n(n-1)}$$

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For the right side inequality we start from

$$\sum_{1\le i<j\le n}(x_i-x_j)^2 \ge \sum_{k=1}^{n-1}\left((x_1-x_k)^2+(x_k-x_n)^2\right) =$$

$$= \frac{1}{2}\left(2\sum_{k=1}^{n-1}(x_1-x_k)^2+(x_k-x_n)^2+2\,(x_1-x_n)^2\right) \ge$$

$$\ge \frac{1}{2}\left(\sum_{k=1}^{n-1}(x_1-x_n)^2\right)+(x_1-x_n)^2 = \frac{n}{2}\,(x_1-x_n)^2$$

therefore

$$(x_1-x_n)^2 \le \frac{2\,(n-1)\,a_1^2-4na_2}{n}$$

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Two inequalities for real numbers

Nikolaos Karagiannis [29]

We will formulate and prove the following two inequalities:

Theorem

I. If $x_1, \ldots, x_{n+1} > 0$ such that

$$\prod_{i=1}^{n+1}(1+x_i) = \sum_{i=1}^{n+1} \prod_{j=1, j\neq i}^{n+1} (1+x_j)$$

then

$$(n+1)^{n+1} x_1 \cdots x_{n+1} \leq n^{n+1}(1+x_1)\cdots(1+x_{n+1}) \qquad \text{for every } n \in \mathbb{N}$$

II. Prove that the following inequality holds

$$\sum_{n=1}^{\infty}\sum_{\kappa=1}^{n+1} \frac{\kappa^2}{(2n+4)!} < \frac{1}{20}$$

*P*roof. I.

$$\prod_{i=1}^{n+1}(1+x_i) = \sum_{i=1}^{n+1} \prod_{\substack{j=1\\ j\neq i}}^{n+1}(1+x_j) \Longleftrightarrow (1+x_1)\cdots(1+x_{n+1}) =$$

$$= (1+x_2)\cdots(1+x_{n+1}) + \ldots + (1+x_1)\cdots(1+x_n) \Longleftrightarrow$$

$$\Longleftrightarrow 1 = \frac{1}{1+x_1} + \ldots + \frac{1}{1+x_{n+1}} \Longleftrightarrow 1 = \frac{1+x_1-x_1}{1+x_1} + \ldots$$

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$$+\frac{1+x_{n+1}-x_{n+1}}{1+x_{n+1}} \Longleftrightarrow 1=1-\frac{x_1}{1+x_1}+\ldots+1-\frac{x_{n+1}}{1+x_{n+1}} \Longleftrightarrow 1=$$

$$=(n+1)-\left(\frac{x_1}{1+x_1}+\ldots+\frac{x_{n+1}}{1+x_{n+1}}\right) \Longleftrightarrow n=\frac{x_1}{1+x_1}+\ldots+\frac{x_{n+1}}{1+x_{n+1}}$$

However for the positive real numbers:

$$\frac{1+x_1}{x_1}, \frac{1+x_2}{x_2}, \ldots, \frac{1+x_{n+1}}{x_{n+1}}$$

from Cauchy's inequality, one has

$$\sqrt[n+1]{\frac{(1+x_1)\cdots(1+x_{n+1})}{x_1\cdots x_{n+1}}} \geq \frac{n+1}{\frac{x_1}{1+x_1}+\ldots+\frac{x_{n+1}}{1+x_{n+1}}} \Longleftrightarrow$$

$$\Longleftrightarrow \sqrt[n+1]{\frac{(1+x_1)\cdots(1+x_{n+1})}{x_1\cdots x_{n+1}}} \geq \frac{n+1}{n} \Longleftrightarrow \frac{(1+x_1)\cdots(1+x_{n+1})}{x_1\cdots x_{n+1}} \geq$$

$$\geq \frac{(n+1)^{n+1}}{n^{n+1}}$$

Therefore

$$(n+1)^{n+1}x_1\cdots x_{n+1} \leq n^{n+1}(1+x_1)\cdots(1+x_{n+1}) \qquad \text{for every } n\in\mathbb{N}$$

*P*roof.
II. We know that

$$e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots+\frac{x^n}{n!}+\ldots \text{for every} x\in\mathbb{R} \tag{1}$$

Thus

$$e^{-x}=\sum_{n=0}^{\infty}\frac{(-x)^n}{n!} \quad \text{for every } x\in\mathbb{R} \tag{2}$$

From (1) and (2) we obtain:

$$\frac{e^x+e^{-x}}{2}=\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x^n}{n!}+\frac{(-1)^n x^n}{n!}\right)=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\ldots=\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$$

Therefore

$$\frac{e^x + e^{-x}}{2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \quad \text{for every } x \in \mathbb{R} \tag{3}$$

Differentiating both parts of (3) with respect to x we get

$$\frac{e^x - e^{-x}}{2} = \sum_{n=1}^{\infty} \frac{2nx^{2n-1}}{(2n)!} \Longleftrightarrow$$

$$\Longleftrightarrow \frac{e^x - e^{-x}}{2} = \sum_{n=0}^{\infty} \frac{2(n+1)x^{2n+1}}{(2n+2)!} \quad \text{for every } x \in \mathbb{R} \tag{4}$$

Differentiating both parts of (4) with respect to x we get:

$$\frac{e^x + e^{-x}}{2} = \sum_{n=1}^{\infty} \frac{2(n+1)(2n+1)x^{2n}}{(2n+2)!} \quad \text{for every } x \in \mathbb{R} \tag{5}$$

Differentiating both parts of (5) with respect to x we get:

$$\frac{e^x - e^{-x}}{2} = \sum_{n=2}^{\infty} \frac{4n(n+1)(2n+1)}{(2n+2)!} x^{2n-1} = 24 \sum_{n=2}^{\infty} \frac{1^2 + 2^2 + \ldots + n^2}{(2n+2)!} x^{2n-1}$$

for every $x \in \mathbb{R}$. Thus

$$\frac{e^x - e^{-x}}{48} = \sum_{n=1}^{\infty} \frac{1^2 + 2^2 + \ldots + (n+1)^2}{(2n+4)!} x^{2n+1} \quad \text{for every } x \in \mathbb{R} \tag{6}$$

If we set $x = 1$, (6) implies :

$$\frac{e - e^{-1}}{48} = \sum_{n=1}^{\infty} \frac{1^2 + 2^2 + \ldots + (n+1)^2}{(2n+4)!} = \frac{1^2 + 2^2}{(2 \cdot 1 + 4)!} + \frac{1^2 + 2^2 + 3^2}{(2 \cdot 2 + 4)!} + \ldots$$

$$= \frac{1}{(2 \cdot 1 + 4)!} \sum_{\kappa=1}^{1+1} \kappa^2 + \frac{1}{(2 \cdot 2 + 4)!} \sum_{\kappa=1}^{2+1} \kappa^2 + \ldots = \sum_{n=1}^{\infty} \frac{1}{(2n+4)!} \sum_{\kappa=1}^{n+1} \kappa^2 =$$

$$= \sum_{n=1}^{\infty} \sum_{\kappa=1}^{n+1} \frac{\kappa^2}{(2n+4)!}$$

Therefore

$$\sum_{n=1}^{\infty}\sum_{\kappa=1}^{n+1}\frac{\kappa^2}{(2n+4)!}=\frac{e-e^{-1}}{48}<0.0489<\frac{1}{20}.$$

Hence

$$\sum_{n=1}^{\infty}\sum_{\kappa=1}^{n+1}\frac{\kappa^2}{(2n+4)!}<\frac{1}{20}.$$

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An inequality for the number of divisors of n

József Sándor and Lehel Kovács[30]

ABSTRACT. M.I. Isravilov and I. Allikov proved in 1980 that $d(n) < n^{2/3}$ for $n > 12$. The authors proved in [2] that $d(n) < \sqrt{n}$ for $n \geq 1262$. We shall prove here that $d(n) \leq 4n^{1/3}$ for all $n \geq 1$.

MAIN RESULTS

Let $d(n)$ denote the number of distinct positive divisors of n.

Then $d(1) = 1$ and if $n = \prod_{i=1}^{r} p_i^{a_i}$ is the prime factorization of n (p_i distinct primes; $a_i \geq 1$), then it is well-known that

$$d(n) = (a_1 + 1) \ldots (a_r + 1) \tag{1}$$

The classical inequality (see e.g. [1], [2])

$$d(n) < 2\sqrt{n} \tag{2}$$

is valid for all $n \geq 1$. Though the inequality

$$d(n) < n^{2/3} \text{ for } n > 12 \tag{3}$$

by Isravilar and Allikov holds true, remark that for $n \geq 64$ (2) is stronger than (3), as $2n^{1/2} \leq n^{2/3}$ is true for $n \geq 2^6 = 64$.

A stronger inequality, namely

$$d(n) < \sqrt{n}, \ n \geq 1262 \tag{4}$$

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has been proved in our note [2].

For greater values of n, however we can deduce the following stronger relation:

$$d\,(n) \leq 4n^{1/3} \text{ for } n \geq 1 \tag{5}$$

Indeed, as $4n^{1/3} \leq n^{1/2}$ for $n \geq 4^6 = 4096$, we get that (5) is stronger than (4) for $n \geq 4096$.

Theorem. For all $n \geq 1$ one has

$$d\,(n) < 4\sqrt[3]{n} \tag{6}$$

Proof. As $1 < 4$, (6) is true for $n = 1$. Let $n \geq 2$, and use relation (1).

First we need the following.

Lemma. For all $p \geq 11$ one has

$$p^{a/3} > a + 1; \tag{7}$$

For all $p \geq 2$ one has

$$(a+1)\,p^{-a/3} \leq M\,(p)\,, \tag{8}$$

where

$$M\,(p) = \frac{3}{\log p} \cdot p^{(\log p - 3)/3\log p} \tag{9}$$

Proof. The inequality (7), written equivalently $p^a > (a+1)^3$ follows by $p^a \geq 11^a$ and from

$$11^a > (a+1)^3\,, a \geq 1 \tag{10}$$

inductively. Indeed, (10) is valid for $a = 1$, as $11 > 2^3 = 8$. Now, assuing (10) for a, one has $11^{a+1} = 11 \cdot 11^a > 11\,(a+1)^3 > (a+2)^3$, as

$$\frac{(a+2)^3}{(a+1)^3} = \left(1 + \frac{1}{a+1}\right)^3 \leq \left(1 + \frac{1}{2}\right)^3 = \frac{27}{8} < 11;$$

finishing the proof of (10).

For the proof of (8), consider the application

$$f(a) = (a+1)\, p^{-a/3}$$

As $f'(a) = p^{-a/3}\left[1 - \frac{(a+1)\log p}{3}\right]$, we get that the function f has a maximum at $a = a_0 = \frac{3}{\log p} - 1$, and the maximum value of $f(a_0)$ is

$$M(p) = \frac{3}{\log p} \cdot p^{(\log p - 3)/3\log p},$$

i.e. (8) is valid, with (9).

This finishes the proof of Lemma.

Now, for the proof of the theorem write n as

$$n = \prod_{p \le 7} p^a \cdot \prod_{p \ge 11} p^a$$

As for each term of the second product one has

$$d(p^a) = a + 1 < p^{a/3}$$

while for each term of the first product we have $d(p^a) \le p^{a/3} M(p)$, we get that for all $n \ge 2$ one has

$$d(n) \le M(2) \cdot M(3) \cdot M(5) \cdot M(7) \prod_{p \ge 2} p^{a/3} = K \cdot n^{1/3}$$

By using a computer, one can find that $M(2) = 2,006063; M(3) = 1,448847..., M(5) = 1,172580..., M(7) = 1,084934...$, so $K \approx 3,697... < 4$, finishing the proof of the theorem.

Remark. *As in $d(n) < K\sqrt[3]{n}$, one has $K \approx 3,...$, we get even a stronger relation than inequality (6). For example, $K < 3,7$.*

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On refinements of the arithmetic-geometric- harmonic mean inequality

József Sándor[31]

ABSTRACT. We offer a survey of results on upper and lower bounds for $A_n - G_n, A_n - H_n, G_n - H_n$.

Recently, in paper [1] the following result has been proved:

If $x_k > 0$ $\left(k = \overline{1,n}\right)$, then for

$$\triangle_n(x) = A_n(x) - G_n(x) \tag{1}$$

one has

$$\triangle_n(x) \le \frac{1}{n} \cdot S_n(x), \tag{2}$$

where $A_n(x) = \frac{x_1+...+x_n}{n}$, $G_n(x) = \sqrt[n]{x_1...x_n}$;
$S_n(x) = \sum\limits_{1\le i<j\le n} \left(\sqrt{x_i} - \sqrt{x_j}\right)^2$.

In 1958 H. Kober [3] proved that (2) holds true, and that one has also

$$\frac{1}{n(n-1)} \cdot S_n(x) \le \triangle_n(x) \tag{3}$$

More generally, he proved that, if $A_n(x,w)$ and $G_n(x,w)$ are the weighted arithmetic, respectivelly geometric means, with weight $w = (w_1, ..., w_n)$, where $w_i > 0$, $w_1 + ... + w_n = 1$; and denoting

$$\triangle_n(x,w) = A_n(x,w) - G_n(x,w)\,;\ S_n(x,w) = \sum_{1\le i<j\le n} w_i w_j \left(\sqrt{x_i} - \sqrt{x_j}\right)^2,$$

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then

$$\frac{1}{n-1}\min\left\{w_1,...,w_n\right\} \leq \frac{\triangle_n\left(x,w\right)}{S_n\left(x,w\right)} \leq \max\left\{w_1,...,w_n\right\} \tag{4}$$

Clearly, when $w_1 = ... = w_n = \frac{1}{n}$, from (4) we reobtain (2) and (3).

Another type of results were obtained in 1978 by D.I. Cartwright and M.J. Field ([4]):

If $0 < x_1 \leq x_2 \leq ... \leq x_n$, then

$$\frac{1}{2x_n}\sum_{i=1}^{n} w_i\left(x_i - A_n\right)^2 \leq A_n - G_n \leq \frac{1}{2x_1}\sum_{i=1}^{n} w_i\left(x_i - A_n\right)^2 \tag{51}$$

H. Alzer [5] improved the lower bound of (5) to

$$\frac{1}{2x_n}\sum_{i=1}^{n} w_i\left(x_i - G_n\right)^2 \leq A_n - G_n \tag{6}$$

Clearly, one has also

$$A_n - G_n \leq \frac{1}{2x_1}\sum_{i=1}^{n} w_i\left(x_i - G_n\right)^2 \tag{7}$$

but as $G_n \leq A_n$ (so $-A_n \leq -G_n$), clearly, the upper bound in (5) is stronger than (7).

In 2000 A. McD. Mercer [5] proved, by using another argument, that:

$$A_n - H_n \geq \frac{1}{2x_n}\sum_{i=1}^{n} w_i\left(x_i - A_n\right)^2 \tag{8}$$

$$A_n - H_n \geq \frac{1}{2x_n}\sum_{i=1}^{n} w_i\left(x_i - H_n\right)^2 \tag{9}$$

$$\frac{G_n}{2x_n^2}\sum_{i=1}^{n} w_i\left(x_i - G_n\right)^2 \leq G_n - H_n \leq \frac{G_n}{2x_1^2}\sum_{i=1}^{n} w_i\left(x_i - G_n\right)^2 \tag{10}$$

$$\frac{G_n}{2x_n^2}\sum_{i=1}^{n} w_i\left(x_i - H_n\right)^2 \leq G_n - H_n \leq \frac{G_n}{2x_1^2}\sum_{i=1}^{n} w_i\left(x_i - H_n\right)^2 \tag{11}$$

In another paper [7], he proved the following:

If $x_1 \leq x_2 \leq ... \leq x_n$, with at least one sign of inequality strict, then

$$P\left(x_n\right)\sum_{i=1}^{n} w_i\left(x_i - A_n\right)^2 < A_n - G_n < P\left(x_1\right)\sum_{i=1}^{n} w_i\left(x_i - A_n\right)^2 \qquad (12)$$

and

$$Q\left(x_n\right)\sum_{i=1}^{n} w_i\left(x_i - G_n\right)^2 < A_n - G_n < Q\left(x_1\right)\sum_{i=1}^{n} w_i\left(x_i - G_n\right)^2 \qquad (13)$$

where

$$P\left(x\right) = \frac{x - G_n}{2x\left(x - A_n\right)} \text{ and } Q\left(x\right) = \frac{x - G_n}{2x\left(x - G_n\right) - 2G_n\left(A_n - G_n\right)}$$

Regarding Kober's result (4), we note that Kober has shown that the upper bound is always reached. However, P. Diananda [2] has proved that this is not always the case with the lower bound. In fact, he proved the following result:

If not all x_i $\left(i = \overline{1,n}\right)$ are equal, then

$$\frac{1}{1 - \min\left\{w_1, ..., w_n\right\}} \leq \frac{\triangle_n\left(x, w\right)}{S_n\left(x, w\right)} \leq \frac{1}{\min\left\{w_1, ..., w_n\right\}} \qquad (14)$$

Here both bounds are always reached.

For other results, we quote the onograph by D.S. Mitrinovic [8].

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Identities and inequalities in a quadrilateral

Ovidiu T. Pop[32]

ABSTRACT. In this paper we will prove some identities and inequalities in cyclic and tangential quadrilaterals.

1. INTRODUCTION

Let $ABCD$ be a convex quadrilateral and we note $AB = a$, $BC = b$, $CD = c$, $DA = d$, $BD = e$, $AC = f$, $s = \dfrac{a+b+c+d}{2}$, $AC \cap BD = \{M\}$, the measure of the angles AMB is φ and Δ is the area of quadrilateral $ABCD$.

If $ABCD$ is a cyclic and tangential quadrilateral, let R, r be the radii of the circumscribed circle, respectively inscribed circle of quadrilateral $ABCD$.

It is well-known that the sides a, b, c, d are the solutions of the equation (see [4])

$$x^4 - 2sx^3 + (s^2 + 2r^2 + 2r\sqrt{4R^2 + r^2})x^2 - 2rs(\sqrt{4R^2 + r^2} + r)x + r^2s^2 = 0 \quad (1.1)$$

and the following inequalities are true (see [3])

$$2\sqrt{2r(\sqrt{4R^2 + r^2} - r)} \le s, \quad (1.2)$$

with equality if and only if $ABCD$ is a square (when $R = r\sqrt{2}$) and isosceles trapezoid (when $R \neq \sqrt{2}$),

$$s \le \sqrt{4R^2 + r^2} + r, \quad (1.3)$$

with equality if and only if $ABCD$ is a orthodiagonal quadrilateral and

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$$2\sqrt{2r(\sqrt{4R^2+r^2}-r)} \leq s \leq \sqrt{4R^2+r^2}+r, \tag{1.4}$$

when the equalities hold simultaneous if and only if $ABCD$ is a square (when $R = r\sqrt{2}$), and if $R \neq \sqrt{2}$ at least one inequality from (1.4) is strict.

On the other hand, the L. Fejes Tóth Inequality

$$R \geq \sqrt{2} \tag{1.5}$$

holds, with equality if and only if $ABCD$ is a square and the following identities

$$ef = 2r(\sqrt{4R^2+r^2}+r) \tag{1.6}$$

and

$$\Delta = sr \tag{1.7}$$

hold.

If quadrilateral $ABCD$ is cyclic, then

$$e^2 = \frac{(ac+bd)(ab+cd)}{ad+bc}, \tag{1.8}$$

$$ef = ac+bd, \tag{1.91}$$

$$\frac{e}{f} = \frac{ab+cd}{ad+bc} \tag{1.10}$$

and

$$16R^2\Delta^2 = (ab+cd)(ac+bd)(ad+bc), \tag{1.11}$$

called Girard's relation.

2. IDENTITIES AND INEQUALITIES WITH R_1, R_2, R_3, R_4

Let $ABCD$ be a convex quadrilateral and we note with R_1, R_2, R_3, R_4 the radii of the circumscribed circles to the triangles AMB, BMC, CMD and DMA.

Lemma 2.1. The following identity

$$R_1+R_2+R_3+R_4=\frac{sef}{2\Delta} \tag{2.1}$$

and

$$a=\frac{2sR_1}{R_1+R_2+R_3+R_4},\ b=\frac{2sR_2}{R_1+R_2+R_3+R_4},$$

$$c=\frac{2sR_3}{R_1+R_2+R_3+R_4},\ d=\frac{2sR_4}{R_1+R_2+R_3+R_4} \tag{2.2}$$

hold.

*P*roof. From the sine theorem we deduce that $R_1=\frac{a}{2\sin\varphi}$, $R_2=\frac{b}{2\sin\varphi}$, $R_3=\frac{c}{2\sin\varphi}$, $R_4=\frac{d}{2\sin\varphi}$, from where $R_1+R_2+R_3+R_4=\frac{a+b+c+d}{2\sin\varphi}=\frac{s}{\sin\varphi}$. But $\Delta=\frac{ef\sin\varphi}{2}$ and then (2.1) follows. On the other hand, we have that $\frac{R_1}{a}=\frac{R_2}{b}=\frac{R_3}{c}=\frac{R_4}{d}=\frac{1}{2\sin\varphi}$, from where $b=\frac{R_2}{R_1}a$, $c=\frac{R_3}{R_1}a$ and $d=\frac{R_4}{R_1}a$. From the relations above, by summing, we obtain that $2s=a+b+c+d=a+\frac{R_2}{R_1}a+\frac{R_3}{R_1}a+\frac{R_4}{R_1}a=\frac{a}{R_1}(R_1+R_2+R_3+R_4)$ and then relations from (2.2) also result.

Lemma 2.2. If quadrilateral $ABCD$ is a cyclic and tangential quadrilateral, then

$$a=\frac{2sR_1}{\sqrt{4R^2+r^2}+r}. \tag{2.3}$$

*P*roof. From (2.2), by taking (2.1), (1.6) and (1.7) into account, relation (2.3) is obtained.

Theorem 2.1. If $ABCD$ is a cyclic and tangential quadrilateral, then R_1, R_2, R_3, R_4 are the solutions of the equation

$$16s^2x^4-16s^2(\sqrt{4R^2+r^2}+r)x^3+4(s^2+2r^2+2r\sqrt{4R^2+r^2})(\sqrt{4R^2+r^2}+r)^2x^2$$

$$-4r(\sqrt{4R^2+r^2}+r)^4x+r^2(\sqrt{4R^2+r^2}+r)^4=0. \tag{2.4}$$

*P*roof. By taking (2.3) into account, we have a solution for equation (1.1) by replacing one from (2.3) in (1.1), after calculus, we obtain that R_1 verifies equation (2.4).

Theorem 2.2. If quadrilateral $ABCD$ is a cyclic and tangential quadrilateral, then the following identities

$$\sum R_1=\sqrt{4R^2+r^2}+r, \tag{2.5}$$

$$\sum R_1R_2=\frac{(s^2+2r^2+2r\sqrt{4R^2+r^2})(\sqrt{4R^2+r^2}+r)^2}{4s^2}, \tag{2.6}$$

$$\sum R_1R_2R_2=\frac{r(\sqrt{4R^2+r^2}+r)^4}{4s^2}, \tag{2.7}$$

$$R_1R_2R_2R_4=\frac{r^2(\sqrt{4R^2+r^2}+r)^4}{16s^2}, \tag{2.8}$$

$$\sum R_1^2=\frac{(\sqrt{4R^2+r^2}+r)^2(s^2-2r^2-2r\sqrt{4R^2+r^2})}{2s^2}, \tag{2.9}$$

$$\sum \frac{1}{R_1}=\frac{4}{r}, \tag{2.10}$$

$$\sum \frac{1}{R_1R_2}=\frac{4(s^2+2r^2+2r\sqrt{4R^2+r^2})}{r^2(\sqrt{4R^2+r^2}+r)^2}, \tag{2.11}$$

and

$$\sum \frac{1}{R_1R_2R_3}=\frac{16s^2}{r^2(\sqrt{4R^2+r^2}+r)^3} \tag{2.12}$$

hold.

*P*roof. It results from Theorem 2.1.

Theorem 2.3. If quadrilateral $ABCD$ is a cyclic and tangential quadrilateral, then the following inequalities

$$4r\le\sum R_1\le 2R\sqrt{2}, \tag{2.13}$$

$$6r^2 \le R^2+r^2+r\sqrt{4R^2+r^2} \le \sum R_1R_2 \le \frac{(5\sqrt{4R^2+r^2}-3r)(\sqrt{4R^2+r^2}+r)^3}{64R^2}$$

$$\le \frac{(5\sqrt{4R^2+r^2}-3r)R\sqrt{2}}{4} \le \frac{(10R\sqrt{2}-3r)R\sqrt{2}}{4}, \tag{2.14}$$

$$4r^3 \le \frac{r(\sqrt{4R^2+r^2}+r)^2}{4} \le \sum R_1R_2R_3 \le \frac{(\sqrt{4R^2+r^2}+r)^5}{128R^2} \le R^3\sqrt{2}, \tag{2.15}$$

$$r^4 \le \frac{r^2(\sqrt{4R^2+r^2}+r)^2}{16} \le R_1R_2R_2R_4 \le \frac{r(\sqrt{4R^2+r^2}+r)^5}{512R^2} \le \frac{R^4}{4}, \tag{2.16}$$

$$\frac{8r^4}{R^2} \le \frac{(\sqrt{4R^2+r^2}+r)^3 3(\sqrt{4R^2+r^2}-5r)}{32R^2} \le \sum R_1^2 \le 2R^2, \tag{2.17}$$

$$\frac{4\sqrt{2}}{R} \le \sum \frac{1}{R_1}, \tag{2.18}$$

$$\frac{12}{R^2} \le \frac{8(5\sqrt{4R^2+r^2}-3r)}{r(\sqrt{4R^2+r^2}+r)^2} \le \sum \frac{1}{R_1R_2} \le$$

$$\le \frac{16(R^2+r^2+r\sqrt{4R^2+r^2})}{r^2(\sqrt{4R^2+r^2}+r)^2} \le \frac{3R^2}{r^4} \tag{2.19}$$

and

$$\frac{8}{rR^2} \le \frac{512R^2}{r(\sqrt{4R^2+r^2}+r)^4} \le \sum \frac{1}{R_1R_2R_3} \le \frac{16}{r^2(\sqrt{4R^2+r^2}+r)} \le \frac{4}{r^3} \tag{2.20}$$

hold.

*P*roof. It results from Theorem 2.2 and from inequalities (1.2), (1.3) and (1.5).

IDENTITIES AND INEQUALITIES WITH r_1, r_2, r_3, r_4

In this section, we consider that $ABCD$ is a cyclic and tangential quadrilateral and we note with r_1, r_2, r_3, r_4 the radii of the inscribed circles to the triangles AMB, BMC, CMD and DMA.

Lemma 3.1. The following identities

$$AM = \frac{eda}{ab+cd}, \tag{3.1}$$

$$BM = \frac{eab}{ab+cd}, \tag{3.2}$$

$$CM = \frac{ebc}{ab+cd} \tag{3.3}$$

and

$$DM = \frac{ecd}{ab+cd} \tag{3.4}$$

hold.

*P*roof. From the similarity of triangles ABM and DCM, respectively ADM and BCM, we have that $\frac{AM}{DM} = \frac{BM}{CM} = \frac{AB}{DC} = \frac{a}{c}$ and $\frac{AM}{BM} = \frac{DM}{CM} = \frac{AD}{BC} = \frac{d}{b}$. From the equalities above, it results that $BM = \frac{b}{d}AM$, $CM = \frac{c}{a}BM = \frac{bc}{ad}AM$ and $DM = \frac{c}{a}AM$. If we choose BM and DM in the equality $BM + DM = e$, then we obtain $\frac{b}{d}AM + \frac{c}{a}AM = e$, from where relation (3.1) is obtained and after that, by replacing relations, relations (3.2)-(3.4) follow.

We note with s_1, T_1 the semiperimeter, respectively the area of the triangle AMB and $\alpha = \frac{sr^2(\sqrt{4R^2+r^2}+r)}{2R(2R+r+\sqrt{4R^2+r^2})}$.

Lemma 3.2. We have that

$$r_1 = \frac{\alpha}{c}. \tag{3.5}$$

*P*roof. By taking $s = a + c = b + d$, (3.1),(3.2) and (1.6)-(1.11) into account, we calculate

$$\begin{aligned}
s_1 &= \frac{1}{2}(AB + MB + MA) = \frac{1}{2}\left(a + \frac{eab}{ab+cd} + \frac{eda}{ab+cd}\right) \\
&= \frac{a}{2}\left(1 + \frac{e(b+d)}{ab+cd}\right) = \frac{a}{2}\left(1 + \frac{s}{ab+cd}\right) = \left(\sqrt{\frac{(ac+bd)(ab+cd)}{ad+bc}}\right) \\
&= \frac{a}{2}\left(1 + s\frac{ef}{\sqrt{(ab+cd)(ac+bd)(ad+bc)}}\right) = \frac{a}{2}\left(1 + s\frac{ef}{4R\Delta}\right) \\
&= \frac{a}{2}\left(1 + \frac{ef}{4Rr}\right) = \frac{a}{2}\left(1 + \frac{2r(\sqrt{4R^2+r^2}+r)}{4Rr}\right),
\end{aligned}$$

from where

$$s_1 = a\frac{2R + r + \sqrt{4R^2+r^2}}{4R}. \tag{3.6}$$

We have that $T_1 = \dfrac{MA \cdot MB \cdot \sin\varphi}{2}$ and by taking $\Delta = \dfrac{ef \sin\varphi}{2}$ and (3.1), (3.2), (1.10), (1.11), (1.6) into account, it results that

$$\begin{aligned}
T_1 &= \frac{1}{2}\frac{e^2a^2bd}{(ab+cd)^2}\frac{2\Delta}{ef} = \frac{e}{f}\frac{a^2bd\Delta}{(ab+cd)^2} = \frac{a^2bd\Delta}{(ab+cd)(ad+bc)} \\
&= \frac{a^2bd\Delta ef}{(ab+cd)(ac+bd)(ad+bc)} = \frac{a^2bd\Delta ef}{16R^2\Delta^2},
\end{aligned}$$

from where

$$T_1 = \frac{a^2bd(\sqrt{4R^2+r^2}+r)}{8sR^2}. \tag{3.7}$$

Because $abcd = \Delta^2 = s^2r^2$, from (3.6) and (3.7) it results that $r_1 = \dfrac{T_1}{s_1} = \dfrac{abcd(\sqrt{4R^2+r^2}+r)}{8sR^2}\dfrac{4R}{c(2R+r+\sqrt{4R^2+r^2})}$, and than (3.5) follows.

Remark 3.1. *Similarly, we obtain that* $r_2 = \frac{\alpha}{d}$, $r_3 = \frac{\alpha}{a}$ *and* $r_4 = \frac{\alpha}{b}$.

Remark 3.2. *Because* $R_1 = \frac{a \cdot MB \cdot MA}{4T_1}$, *with the help of relations (3.1), (3.2) and (3.7), we can calculate* R_1*, respectively* R_2*,* R_3 *and* R_4*.*

Theorem 3.1. If $ABCD$ is a cyclic and tangential quadrilateral, then r_1, r_2, r_3, r_4 are the solutions of the equation

$$16R^4(2R+r+\sqrt{4R^2+r^2})^4x^4 - 16rR^3(\sqrt{4R^2+r^2}+r)^2(2R+r+\sqrt{4R^2+r^2})^3x^3+$$

$$+4r^2R^2(s^2+2r^2+2r\sqrt{4R^2+r^2})(\sqrt{4R^2+r^2}+r)^2(2R+r+\sqrt{4R^2+r^2})^2x^2-$$

$$-4s^2r^4R(\sqrt{4R^2+r^2}+r)^3(2R+r+\sqrt{4R^2+r^2})x+$$

$$+s^2r^6(\sqrt{4R^2+r^2}+r)^4 = 0. \tag{3.8}$$

*P*roof. By taking c from (3.5) as a solution of the equation (1.1), by replacing c from (3.5) in (1.1), after calculus, we obtain that r_1 verifies equation (3.8).

Theorem 3.2. If quadrilateral $ABCD$ is a cyclic and tangential quadrilateral, then

$$\sum r_1 = \frac{r(\sqrt{4R^2+r^2}+r)^2}{R(2R+r+\sqrt{4R^2+r^2})}, \tag{3.9}$$

$$\sum r_1r_2 = \frac{r^2(s^2+2r^2+2r\sqrt{4R^2+r^2})(\sqrt{4R^2+r^2}+r)^2}{4R^2(2R+r+\sqrt{4R^2+r^2})^2}, \tag{3.10}$$

$$\sum r_1r_2r_2 = \frac{s^2r^4(\sqrt{4R^2+r^2}+r)^3}{4R^3(2R+r+\sqrt{4R^2+r^2})^3}, \tag{3.11}$$

$$r_1r_2r_2r_4 = \frac{s^2r^6(\sqrt{4R^2+r^2}+r)^4}{16R^4(2R+r+\sqrt{4R^2+r^2})^4}, \tag{3.12}$$

$$\sum r_1^2 = \frac{r^2(\sqrt{4R^2+r^2}+r)^2(8R^2+2r^2+2r\sqrt{4R^2+r^2}-s^2)}{2R^2(2R+r+\sqrt{4R^2+r^2})^2}, \tag{3.13}$$

$$\sum \frac{1}{r_1} = \frac{4R(2R + r + \sqrt{4R^2 + r^2})}{r^2(\sqrt{4R^2 + r^2} + r)}, \tag{3.14}$$

$$\sum \frac{1}{r_1 r_2} = \frac{4R^2(s^2 + 2r^2 + 2r\sqrt{4R^2 + r^2})(2R + r + \sqrt{4R^2 + r^2})^2}{s^2 r^4(\sqrt{4R^2 + r^2} + r)^2} \tag{3.15}$$

and

$$\sum \frac{1}{r_1 r_2 r_3} = \frac{16R^3(2R + r + \sqrt{4R^2 + r^2})^3}{s^2 r^5(\sqrt{4R^2 + r^2} + r)^2}. \tag{3.16}$$

*P*roof. It results from Theorem 3.1.

Theorem 3.3. If quadrilateral $ABCD$ is a cyclic and tangential quadrilateral, then the following inequalities

$$\frac{8r^3(\sqrt{2} - 1)}{R^2} \leq \sum r_1 \leq 2\sqrt{2}(\sqrt{2} - 1)R, \tag{3.17}$$

$$\frac{24r^6(\sqrt{2} - 1)^2}{R^4} \leq \sum r_1 r_2 \leq \frac{3R^4(\sqrt{2} - 1)^2}{2r^2}, \tag{3.18}$$

$$\frac{32r^9(\sqrt{2} - 1)^3}{R^6} \leq \sum r_1 r_2 r_3 \leq \frac{R^6(\sqrt{2} - 1)^3}{2r^3}, \tag{3.19}$$

$$\frac{64r^{12}(\sqrt{2} - 1)^4}{R^8} \leq r_1 r_2 r_2 r_4 \leq \frac{R^4(\sqrt{2} - 1)^4}{4} \tag{3.20}$$

and

$$\frac{4\sqrt{2}(\sqrt{2} + 1)}{R} \leq \sum \frac{1}{r_1} \leq \frac{2R^2(\sqrt{2} + 1)}{r^3} \tag{3.21}$$

hold.

*P*roof. It results from Theorem 3.2 and from inequalities (1.2), (1.3) and (1.5).

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An inequality for means of two arguments

József Sándor[33]

ABSTRACT. Let A and G denote the arithmetic, respectively geometric means of two positive real numbers, and let A_k denote the power mean of order k of these numbers. We will prove that $(pG+(1-p)A)\frac{G}{A} \geq A_{-p}$ for all $p \in \left(\frac{1}{2},1\right)$. Particularly, for $p=\frac{2}{3}$ we get $\frac{G+2H}{3} \geq A_{-2/3}$, where denotes the harmonic mean of the numbers.

MAIN RESULTS

Let $A_p(x,y)=\left(\frac{x^p+y^p}{2}\right)^{1/p}$ $(p \neq 0)$ $(x,y>0)$ and
$A_0(x,y)=\lim\limits_{p\to 0} A_p(x,y)=G(x,y)$. Clearly $A_1=A$ and G are the arithmetic, respectively geometric means of x,y. Put
$H(x,y)=\frac{2}{\left(\frac{1}{x}+\frac{1}{y}\right)}=\frac{2xy}{x+y}$ for the harmonic mean of x and y.

The aim of this note is to prove the following inequalities:

Theorem. For all $p \in \left(\frac{1}{2},1\right)$ one has

$$(pG+(1-p)A)\cdot\frac{G}{A}=pH+(1-p)G \geq A_{-p}, \tag{1}$$

with equality only for $x=y$.
Proof. The equality of (1) is based on the trivial remark that $H=\frac{G^2}{A}$. On the other hand, remark that as

$$\frac{1}{A_{-p}(x,y)}=\left(\frac{x^{-p}+y^{-p}}{2}\right)^{\frac{1}{p}}=\left(\frac{1}{2}\left(\frac{x^p+y^p}{x^py^p}\right)\right)^{\frac{1}{p}}=\left(\frac{x^p+y^p}{2}\right)^{\frac{1}{p}}\cdot\frac{1}{xy},$$

one has

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$$A_{-p} = \frac{G^2}{A_p}, \tag{2}$$

so the prove the inequality (1), it will be sufficient to show that

$$A_p \left[pG + (1-p)\,A\right] \geq A \cdot G \tag{3}$$

By an inequality of H. J. Seiffert (see [1]) one has

$$A_p \geq (1-p)\,G + pA \tag{4}$$

for all $p \in \left(\frac{1}{2}, 1\right)$ with equality only for $x > y$.
On the other hand, by the weighted arithmetic-geometric inequality we can write that (see [2])

$$pG + (1-p)\,A \geq G^p \cdot A^{1-p}; \;\; (1-p)\,G + pA \geq G^{1-p} \cdot A^p \tag{5}$$

for any $p \in (0,1)$. Since $\left(G^p A^{1-p}\right)\left(G^{1-p} A^p\right) = GA$, inequality (3) follows.
Particularly, as $\frac{2}{3} \in \left(\frac{1}{2}, 1\right)$; we can see that

$$\frac{2H+G}{3} \geq A_{-\frac{2}{3}}, \tag{6}$$

with equality only for $x = y$.

Remark 1. *For another application of (1), put* $p = \frac{\log 2}{\log 3}$*. Clearly,* $p < 1$ *and* $p > \frac{1}{2}$*, as*

$$2\log 2 = \log 4 > \log 3$$

Remark 2. *It can be shown that the result is best possible in the sense that the constant "$-p$" in the inequality (1) cannot be improved; i.e. for any* $k > 0$ *one can find* $a, b > 0$ *such that*

$$A_{-p+k}\,(a,b) > pH + (1-p)\,G \;\; \left(\text{for } p \in \left(\frac{1}{2}, 1\right)\right)$$

Put e.g. $b = 1$ *and* $a = 1 + x$*; and apply Tayler expanasion for the difference*

$$A_{-p+k}\,(1, 1+x) - (1-p)\,G\,(1, 1+x) + pH\,(1, 1+x);$$

etc.

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A better lower bound for $\sigma(n)$

József Sándor[34]

ABSTRACT. We prove that $\sigma(n) \geq n + 1 + (d(n) - 2)\sqrt{n}$ for all, $n \geq 2$ thus improving the result by M.Le [].

Let $\sigma(n)$ denote the sum of divisors of n. Further, let $\omega(n)$ denote the number of distinct prime factors of n; while $d(n)$ the number of distinct divisors of n.
Recently,by extending the result by W. Sierpinski, namely

$$\sigma(n) > n + \sqrt{n} \tag{1}$$

for $n =$ composite, we prove that [1]

$$\sigma(n) > n + (\Omega(n) - 1)\sqrt{n} \tag{2}$$

M. Le has further improved (2) as follows (see [2])

$$\sigma(n) \geq n + 1 + \left[\frac{d(n) - 1}{2}\right]\sqrt{n} \tag{3}$$

We note that, we used the simple relation (2) in the solutions of some open questions. In what follows, in place of the slightly coplicated inequality (3) we will prove a simpler, but stronger inequality, namely:

Theorem. For $n \geq 2$ one has

$$\sigma(n) \geq n + 1 + (d(n) - 2)\sqrt{n} \tag{4}$$

with equality only for $n =$prime or $=$ (prime)^2 .
Proof. If $n =$prime, then $d(n) = 2$ and $\sigma(n) = n + 1$, so (4) is satisfied with equality.

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Suppose first that n is not a perfect square. Then for any divisor $d \neq 1$ or $d \neq n$ of n one has also $\frac{n}{d} \neq 1$, $\frac{n}{d} \neq n$ and $\frac{n}{d} \neq d$. Thus, we can write the equality

$$\sigma(n) = n + 1 + \frac{1}{2} \sum_{\substack{d \neq 1, n \\ d \mid n}} \left(d + \frac{n}{d} \right) \tag{5}$$

By the arithmetic-geometric mean inequality one has $d + \frac{n}{d} > 2\sqrt{d \cdot \frac{n}{d}} = 2\sqrt{n}$; so from (5) we get (4), as in the sum there are $d(n) - 2$ terms. As $d \neq \frac{n}{d}$, the inequality is strict.
If n is a perfect square, then $d = \frac{n}{d}$ only for $d = \sqrt{n}$; but for other terms we have a similar identity, so inequality (4) follows again at once. One has equality only when $d = \frac{n}{d}$ for all divisors d; and this happens only when $n = p^2$ $(p = \text{prime})$.
We note that $d(n) \geq 3$; so when n is composite (i.e., not prime), relation (4) is stronger than (3).

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The difference of the median and altitude of a triangle

József Sándor[35]

ABSTRACT. We offer a simple proof of the inequality $m_a - h_a \geq \frac{(b-c)^2}{2a}$, which appears in [1]

Let m_a and h_a denote the lenght of median, and altitude corresponding to the vertex A of the triangle ABC. Recently, M. Minculete and M. Bencze [1] proved the inequality

$$m_a - h_a \geq \frac{(b-c)^2}{2a} \tag{1}$$

The proof of (1) is based on geometrical constructions, and an application of Ptolemy‘s inequality in quadrilaterals. We note that, all other results of [1] are essentially based on relation (1).

Our aim in what follows is a simple proof of inequality (1). Our proof will offer also the condition of equality in (1); namely when the triangle ABC is right-angled in A.

Our proof is based on equality

$$m_a^2 - h_a^2 = \left(\frac{b^2 - c^2}{2a}\right)^2 \tag{2}$$

Now, by (2) we get $m_a - h_a = \sqrt{h_a^2 + \left(\frac{b^2-c^2}{2a}\right)^2} - h_a$, so in order to prove (1) we must show that

$$\sqrt{h_a^2 + \left(\frac{b^2 - c^2}{2a}\right)^2} \geq h_a + \frac{(b-c)^2}{2a} \tag{3}$$

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After some easy transformations, it is immediate that (3) becomes

$$h_a \leq \frac{(b+c)^2}{4a} - \frac{(b-c)^2}{4a} = \frac{b+c}{a}, \text{ i.e.} \tag{4}$$

Inequality (4) is well known, as $ah_a = 2\triangle = bc \cdot \sin A \leq bc$, with equality when $\sin A = 1$.
This proves the assertions made above.

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An improvement of the Bagchi-Gupta arithmetic inequality

József Sándor[36]

ABSTRACT. In 1954 Bagchi and Gupta proved that $\sigma(n) \geq \varphi(n) + d(n)$, where σ, φ and d are the well-known classical arithmetic functions representing the sum of divisors, Euler‘s totient, and the number of divisors, respectivelly. We prove that
$\sigma(n) \geq n + 1 \geq \phi(n) + d(n) \geq 2$ for all $n \geq 2$.

Let $\sigma(n)$ be the sum of divisors of n. Then clearly

$$\sigma(n) \geq n + 1, n \geq 2 \tag{1}$$

with equality only if n is prime.

MAIN RESULTS

The main aim of this note is to prove that the inequality
$\varphi(n) + d(n) \leq \sigma(n)$, by H.D. Bagchi and M. Gupta ([1], [2]) may be improved, as follows:

Theorem. For all $n \geq 2$ one has

$$\varphi(n) + d(n) \leq n + 1 \leq \sigma(n) \tag{2}$$

Proof. In view of (1), it is sufficient to prove the first inequality of (2).
For the proof of this inequality, let us remark first that, when $d > 1$ is a divisor of n, then clearly $(d, n) > 1$; so d cannot be a "totative" of n; i.e. a number such that $(d, n) = 1$. Therefore, the set of divisors and the set of totatives of n has a single common element, namely 1.
When n is prie, then any $1 \leq k < n$ is a totative, and there are only two divisors: 1, and n, so $\varphi(n) + d(n) = n + 1$.

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This is true also when $n = 4$, as any number in the set

$$1 < 2 < 3 < 4$$

or is a divisor of 4, or is a totative of n.

Lemma. Assume that $n \neq$ prime and $n \neq 4$. Then there exists an $a \in \{1, 2, ..., n\}$, such that $a \nmid n$, but $(a, n) > 1$ (where $a \nmid n$ means that a doesn't divide n)
Proof of the Lemma. If n is composite, and n odd; let $n = N \cdot M$, where $N, M > 1$. Put $a = N \cdot m$, where $m < M$ and $(m, M) = 1$. Then clearly $a \nmid n$, but $n|a$, $N|n$, thus $(n, a) > 1$.
If n is even, remark that $n - 2$ is even, too, and $n - 2 \nmid n$, if $n \nmid 4$. Thus $a = n - 2$ is acceptable, as $2|a$, $2|n$.
This finishes the proof of the Lemma.
By this lemma, thenumber of divisors, and the number of totative cannot be in sum greather than n, provind that

$$\varphi(n) + d(n) \leq n \text{ for } n \nmid \text{ prime}, \ n \nmid 4 \tag{3}$$

Therefore, for all n, the left side of (2) holds true, with equality only when n= prime, or $n = 4$.

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Solutions of József Wildt International Mathematical Competition

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Mihály Bencze and Zhao Chang Jian[37]

W1. Consider the function $f:(0,1)\to R$ defined by

$$f(x)=\frac{1}{x}\ln\left(\frac{1+x}{1-x}\right)$$

Since $f(x)=2\sum\limits_{k=0}^{\infty}\frac{x^{2k}}{2k+1}$ for $|x|<1$, then $f'(x)=4\sum\limits_{k=0}^{\infty}\frac{kx^{2k-1}}{2k+1}$ $(|x|<1)$ and $f''(x)=4\sum\limits_{k=0}^{\infty}\frac{k(2k-1)x^{2k-2}}{2k+1}$ $(|x|<1)$.. Therefore, $f'(x)>0$ and $f''(x)>0$ for all $x\in(0,1)$, and f is increasing and convex.

Applying Jensen‘s inequality, we have $f\left(\frac{a+b+c}{3}\right)\leq\frac{f(a)+f(b)+f(c)}{3}$ or equivalently

$$\frac{3}{a+b+c}\ln\left(\frac{3+(a+b+c)}{3-(a+b+c)}\right)\leq$$

$$\leq\frac{1}{3}\left[\ln\left(\frac{1+a}{1-a}\right)^{1/a}+\ln\left(\frac{1+b}{1-b}\right)^{1/b}+\ln\left(\frac{1+c}{1-c}\right)^{1/c}\right]$$

Taking into account that $a+b+c=1$ and the properties of logarithms, we get

$$\sqrt[3]{\left(\frac{1+a}{1-a}\right)^{1/a}\ln\left(\frac{1+b}{1-b}\right)^{1/b}\ln\left(\frac{1+c}{1-c}\right)^{1/c}}\geq 8 \qquad (1)$$

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WLOG we can assume that $a \geq b \geq c$. We have, $\frac{1}{a} \leq \frac{1}{b} \leq \frac{1}{c}$ and $g(a) \geq g(b) \geq g(c)$, where g is the increasing function defined by $g(x) = \ln\left(\frac{1+x}{1-x}\right)$. Applying rearragement's inequality, we get

$$\frac{1}{b}g(a) + \frac{1}{c}g(b) + \frac{1}{a}g(c) \geq \frac{1}{a}g(a) + \frac{1}{b}g(b) + \frac{1}{c}g(c) \text{ or}$$

$$\left(\frac{1+a}{1-a}\right)^{1/b}\left(\frac{1+b}{1-b}\right)^{1/c}\left(\frac{1+c}{1-c}\right)^{1/a} \geq \left(\frac{1+a}{1-a}\right)^{1/a}\left(\frac{1+b}{1-b}\right)^{1/b}\left(\frac{1+c}{1-c}\right)^{1/c}$$

From the preceeding and (1) we obtain

$$\sqrt[3]{\left(\frac{1+a}{b+c}\right)^{1/a}\left(\frac{1+b}{c+a}\right)^{1/b}\left(\frac{1+c}{a+b}\right)^{1/c}} =$$

$$= \sqrt[3]{\left(\frac{1+a}{1-a}\right)^{1/b}\left(\frac{1+b}{1-b}\right)^{1/c}\left(\frac{1+c}{1-c}\right)^{1/a}} \geq \sqrt[3]{\left(\frac{1+a}{1-a}\right)^{1/a}\left(\frac{1+b}{1-b}\right)^{1/b}\left(\frac{1+c}{1-c}\right)^{1/c}} \geq 8$$

Likewise, applying rearrangement's inequality again, we get

$$\frac{1}{c}g(a) + \frac{1}{a}g(b) + \frac{1}{b}g(c) \geq \frac{1}{a}g(a) + \frac{1}{b}g(b) + \frac{1}{c}g(c) \text{ and}$$

$$\sqrt[3]{\left(\frac{1+a}{b+c}\right)^{1/c}\left(\frac{1+b}{c+a}\right)^{1/a}\left(\frac{1+c}{a+b}\right)^{1/b}} = \sqrt[3]{\left(\frac{1+a}{1-a}\right)^{1/c}\left(\frac{1+b}{1-b}\right)^{1/a}\left(\frac{1+c}{1-c}\right)^{1/b}} \geq$$

$$\geq \sqrt[3]{\left(\frac{1+a}{1-a}\right)^{1/a}\left(\frac{1+b}{1-b}\right)^{1/b}\left(\frac{1+c}{1-c}\right)^{1/c}} \geq 8$$

Multiplying up the preceeding inequalities yields,

$$\sqrt[3]{\left(\frac{1+a}{b+c}\right)^{\frac{1}{b}+\frac{1}{c}}\left(\frac{1+b}{c+a}\right)^{\frac{1}{c}+\frac{1}{a}}\left(\frac{1+c}{a+b}\right)^{\frac{1}{a}+\frac{1}{b}}} \geq 64$$

from which the statement follows. Equality holds when $a = b = c = 1/3$, and we are done.

W2. The function $f(x)$ can be written in the form

$$f(x)=\begin{vmatrix} 1 & 1 & 1 & 1 \\ \ln x & \ln x^2 & \ln x^3 & \ln x^4 \\ (\ln x)^2 & \left(\ln x^2\right)^2 & \left(\ln x^3\right)^2 & \left(\ln x^4\right)^2 \\ (\ln x)^3 & \left(\ln x^2\right)^3 & \left(\ln x^3\right)^3 & \left(\ln x^3\right)^3 \end{vmatrix}$$

Developing the Vandermonde‘s determinant, we get

$$f(x)=(\ln x)\,2(\ln x)\,3(\ln x)(\ln x)\,2(\ln x)(\ln x)=12(\ln x)^6$$

Let $I_n=\int(\ln x)^n\,dx,\ (n\geq 1)$. We have to compute I_6. We will argue integrating by parts. Let

$$\left.\begin{array}{cc} u=(\ln x)^6 & du=\frac{6}{x}(\ln x)^5\,dx \\ v=x & dv=dx \end{array}\right\}$$

Then,

$$I_6=x(\ln x)^6-6\int(\ln x)^5\,dx=x(\ln x)^6-6I_5$$

Likewise, $I_5=x(\ln x)^5-5I_4$, $I_4=x(\ln x)^4-4I_3$, $I_3=x(\ln x)^3-3I_2$, $I_2=x(\ln x)^2-2I_1$ and $I_1=x\ln x-x+K$. Therefore,

$$Area\,(A)=\int_1^e f(x)\,dx=12\int_1^e(\ln x)^6\,dx=$$

$$=\Big[12x(\ln x)^6-72x(\ln x)^5+360x(\ln x)^4-1440x(\ln x)^3+4320x(\ln x)^2$$

$$-8640(\ln x)+8640x\Big]_1^e\simeq 4.13$$

and we are done.

W3. We have

$$\frac{\varphi(n)}{n}=\prod_{p|n}\left(1-\frac{1}{p}\right)\text{ and }\frac{\Psi(n)}{n}=\prod_{p|n}\left(1+\frac{1}{p}\right),$$

where p are the prime divisors of n. This implies a well-known property, namely that for $n\geq 3$, $\varphi(n)$ and $\Psi(n)$ are even.

1). Let now n be odd; $n \geq 3$.
Then as $\varphi(n)$ is even and $n+\Psi(n)$ odd; $\varphi(n)$ divides $n+\Psi(n)$ is impossible.
For $n=1$ the property is true, as $\varphi(1)=1$. Therefore, $n=1$ is the single odd solution to the problem.
2). Let n be even, and put $n=2^kN$, where N is odd. Then

$$\varphi(n)=2^{k-1}\cdot\varphi(N) \text{ and } n+\Psi(n)=2^kN+2^{k-1}\cdot 3\cdot\Psi(N)=2^{k-1}\left[2N+3\Psi(N)\right]$$

Thus we should have also:

$$\varphi(N) \mid \left[2N+3\Psi(N)\right] \tag{$*$}$$

If $N=1$, then (*) holds true. Let $n>1$, $N=\prod\limits_{i=1}^{r} p_i^{a_i}$ (prime representation). Then from (*) we get

$$\prod_{i=1}^{r} p_i^{a_i-1}(p_1-1)$$

divides

$$\prod_{i=1}^{r} p_i^{a_i-1}\left[2\prod_{i=1}^{r} p_i+3\prod_{i=1}^{r}(p_i+1)\right],$$

which means that

$$\prod_{i=1}^{r}(p_i-1) \mid \left[2\prod_{i=1}^{r} p_i+3\prod_{i=1}^{r}(p_i+1)\right] \tag{$**$}$$

(thus, it is sufficient that in (*), N to be squarefree). Let $r \geq 2$. Then the left side of (**) is divisible by 4, but the left side is not (divisible only by 2). Thus $r=1$, which gives, with $p_1=p$:

$$p-1 | \left[2p+3(p+1)\right]=5(p-1)+8$$

Therefore, $p-1|8$. This is possible only for $p=3$ and 5 implying that $N=3^a$ or $N=5^a$.
This means that $n=2^k1$ or 2^k3^a or 2^k5^a.
In conclusion, all solutions to the problem are $n=1$ or $n=2^k$, or $n=2^k3^a$, or $n=2^k\cdot 5^a$, where $k\geq 1$, $a\geq 1$ are arbitrary positive integers.

W4. i). It is well-known that $a|b \Rightarrow \frac{\varphi(a)}{a} \leq \frac{\varphi(b)}{b}$. Let p be a prime. Since 2^p+1 is always multiple of 3, we get

$$\frac{\varphi\left(2^p+1\right)}{2^p+1} \geq \frac{\varphi(3)}{3} = \frac{2}{3}$$

Thus $\varphi\left(2^p+1\right) \geq \frac{2}{3}\left(2^p+1\right) > 2^{p-1}$, as $2 \cdot 2^p + 2 > 3 \cdot 2^{p-1}$ or equivalently $4 \cdot 2^{p-1} + 2 > 3 \cdot 2^{p-1}$. Thus for $m = p$, the second assertion follows.
ii). Let $p_1 < p_2 < ... < p_n < ...$ be all primes $p \equiv 3 \pmod{8}$.
Note that for such primes p we have

$$p | 2^{p-1} - 1 = \left(2^{\frac{p-1}{2}} - 1\right)\left(2^{\frac{p-1}{2}} + 1\right),$$

and p divides the second paranthesis and not the first (indeed, if p divides the first, then since $\frac{p-1}{2}$ is odd, we would get that 2 is a quadratic residue mod p, false for $p \equiv 3 \pmod{8}$ [see textbooks of Number Theory]).
Thus p divides $2^{\frac{p-1}{2}} + 1$.
Now let n be the smallest number such that

$$\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) ... \left(1 - \frac{1}{p_n}\right) < \frac{1}{3}$$

(this is true, since the left side tends to 0 as $n \to \infty$).
Put now

$$k = l.c.m.\left[\frac{p_1 - 1}{2}, ..., \frac{p_n - 1}{2}\right]$$

This number k is odd as has the property that

$$p_1 p_2 ... p_n | 2^k + 1,$$

so

$$\frac{\varphi\left(2^k+1\right)}{2^k+1} \leq \frac{\varphi\left(p_1 ... p_n\right)}{p_1 ... p_n} = \left(1 - \frac{1}{p_1}\right) ... \left(1 - \frac{1}{p_n}\right) < \frac{1}{3}$$

Thus $\varphi\left(2^k+1\right) < \frac{2^k+1}{2} < 2^{k-1}$ for $k \geq k_0$.
This proves the first assertion of the problem.

W5. From the given assumption the case when $p_1 = 2$ and $p_2 = 2$ is not considered. Thus we let $p_2 = 2x + 1$ for $x \in N$. Let us assume that the given equation accept at least one integer solution.
Then the equation

$$\left(\frac{p_2-1}{2}\right)^{p_1}+\left(\frac{p_2-1}{2}\right)^{p_1}=a^n$$

is written in the form

$$x^{p_1}+(x+1)^{p_1}=a^n \tag{1}$$

Since p_1 is an odd integer, it follows that

$$a^n=x^{p_1}+(x+1)^{p_1}=[x+(x+1)]\,A=(2x+1)\,A,\ \ A\in N$$

Thus $(2x+1)\,|a^n$. However, $2x+1$ is a prime number, therefore

$$(2x+1)\,|a$$

Therefore

$$(2x+1)^2\,|a^2$$

But $n>1$, therefore

$$(2x+1)^2\,|a^n \text{ or } (2x+1)^2\,|x^{p_1}+(x+1)^{p_1} \tag{2}$$

We obtain

$$x^{p_1}+(x+1)^{p_1}=x^{p_1}+[(2x+1)-x]^{p_1}=x^{p_1}+(2x+1)^{p_1}+$$

$$+\binom{p_1}{1}(2x+1)^{p_1-1}(-x)+...\binom{p_1}{p_1-1}(2x+1)(-x)^{p_1-1}+(-x)^{p_1}=$$

$$=x^{p_1}+(2x+1)^2B+\binom{p_1}{p_1-1}(2x+1)\,x^{p_1-1}-x^{p_1}=$$

$$=\binom{p_1}{p_1-1}(2x+1)\,x^{p_1-1}+(2x+1)^2B,\ \ B\in Z$$

From the relation (2), it follows that

$$(2x+1)^2\,|\binom{p_1}{p_1-1}(2x+1)\,x^{p_1-1} \text{ or}$$

$$(2x+1)\,|\binom{p_1}{p_1-1}x^{p_1-1} \text{ or } (2x+1)\,|p_1x^{p_1-1}$$

Since the number $2x+1$ is prime,it is implied that

$$(2x+1)\,|p_1 \text{ or } (2x+1)\,|x.$$

The second case is not allowed since

$$2x+1>x\ k \text{ as } (2x+1,x)=1$$

Finally $(2x+1)\,|p_1$, i.e. $p_1=2x+1$, since p_1 is a prime number.
Hence $p_1=p_2$. However, we have assumed that the equation does not accept integer solutions for $p_1=p_2$, therefore the equation (1) and thus the equation

$$\left(\frac{p_2-1}{2}\right)^{p_1}+\left(\frac{p_2-1}{2}\right)^{p_1}=a^n$$

does not accept integer solutions.

W6. It is evident that

$$p\,(n)=p'_1\,(n)+p'_2\,(n)+...+p'_n\,(n)=1+p'_2\,(n)+...+p'_{n-1}\,(n)+1 \qquad (1)$$

However for $\left\lfloor\frac{n}{2}\right\rfloor\le m\le n$ it holds.

$$P'_m\,(n)=p\,(n-m)\,, \qquad (2)$$

as it easily follows from the Ferrer‘s diagrams.
From (1) and (2) we get

$$p\,(n)=2+\left[p'_2\,(n)+...+p'_{\left\lfloor\frac{n}{e}\right\rfloor-1}\,(n)\right]+\left[p'_{\left\lfloor\frac{n}{2}\right\rfloor}\,(n)+...+p'_{n-1}\,(n)\right]=$$

$$=2+\left[p'_2\,(n)+...+p'_{\left\lfloor\frac{n}{e}\right\rfloor-1}\,(n)\right]+\left[p\left(n-\left\lfloor\frac{n}{2}\right\rfloor\right)+...+p\,(1)\right] \qquad (3)$$

If n is an even integer, then

$$n-\left\lfloor\frac{n}{2}\right\rfloor=2k-\left\lfloor\frac{2k}{2}\right\rfloor=k=\left\lfloor\frac{n}{2}\right\rfloor$$

for some $k\in N$.
In the case n is an odd integer, it follows

$$n-\left\lfloor\frac{n}{2}\right\rfloor=(2k+1)-\left\lfloor\frac{2k+1}{2}\right\rfloor=(2k+1)-k=k+1=\left\lfloor\frac{n}{2}\right\rfloor+1,$$

which means that

$$n-\left\lfloor \frac{n}{2}\right\rfloor =\left\lfloor \frac{n}{2}\right\rfloor +\chi_1\left(n\right),$$

where $\chi_1\left(n\right)$ denotes the principal character Dirichlet modulo 2.
This is valid since

$$\chi_1\left(n\right)=\begin{cases} 1, & \text{if} \quad \left(n,2\right)=1\\ 0, & \text{if} \quad \left(n,2\right)=0\end{cases}$$

Therefore from relation (3) we obtain the equality

$$p\left(n\right)=2+\left[p_2'\left(n\right)+...+p'_{\left\lfloor \frac{n}{2}\right\rfloor -1}\left(n\right)\right]+\left[p\left(\left\lfloor \frac{n}{2}\right\rfloor +\chi_1\left(n\right)\right)+...+p\left(1\right)\right],$$

which coincides with the equality that we wished to prove.

W7. If $x=\frac{ab}{t}, dx=-\frac{ab}{t^2}dt$, and after then we substitute $t\rightarrow x$ we give

$$I=\int\limits_a^b\frac{\left(x^2-ab\right)\ln\frac{x}{a}\ln\frac{x}{b}dx}{\left(x^2+a^2\right)\left(x^2+b^2\right)}=\int\limits_a^b\frac{\left(\left(\frac{ab}{x}\right)^2-ab\right)\ln\frac{b}{x}\ln\frac{a}{x}}{\left(\left(\frac{ab}{x}\right)^2+a^2\right)\left(\left(\frac{ab}{x}\right)^2+b^2\right)}\frac{abdx}{x^2}=$$

$$\int\limits_a^b\frac{\left(ab-x^2\right)\ln\frac{x}{a}\ln\frac{x}{b}dx}{\left(x^2+a^2\right)\left(x^2+b^2\right)}=-I,\text{ therefore } I=0$$

but $x^2-\left(\frac{a+b}{2}\right)^2<x^2-ab$ which prove that

$$\int\limits_a^b\frac{\left(x^2-\left(\frac{a+b}{x}\right)^2\right)\ln\frac{x}{a}\ln\frac{x}{b}dx}{\left(x^2+a^2\right)\left(x^2+b^2\right)}>\int\limits_a^b\frac{\left(x^2-ab\right)\ln\frac{x}{a}\ln\frac{x}{b}dx}{\left(x^2+a^2\right)\left(x^2+b^2\right)}=0$$

W8. The identity can be written in following form:

$$a_n=\sum_{k=0}^{\left[\frac{n}{2}\right]}\left(-1\right)^k\binom{pn}{k}\binom{\left(q-p\right)n}{n-2k}=\frac{\binom{\left(p+q\right)n}{n}}{\binom{\left(p+q\right)n}{pn}}\sum_{k=0}^{n}\left(-1\right)^k\binom{n}{k}\binom{\left(p+q-1\right)n}{pn-k}=b_n$$

From this we give

$$b_n = \sum_{k=0}^{n} (-1)^k \binom{pn}{k} \binom{qn}{n-k}$$

Denote α the coefficient of x^n in the expression $f(x) = (1-x)^{pn}(1+x)^{qn}$, and

$$f(x) = \left(\binom{pn}{0} - x\binom{pn}{1} + \binom{pn}{2}x^2 - ...\right)\left(\binom{qn}{0} + x\binom{qn}{1} + x^2\binom{qn}{2} + ...\right)$$

therefore

$$\alpha = \binom{pn}{0}\binom{qn}{n} - \binom{pn}{1}\binom{qn}{qn-1} + \binom{pn}{2}\binom{qn}{n-2} - ... =$$

$$= \sum_{k=0}^{n} (-1)^k \binom{pn}{k}\binom{qn}{n-k} = b_n$$

In another way we obtain $f(x) = \left(1-x^2\right)^{pn}(1-x)^{(q-p)n}$, and

$$\alpha = \binom{pn}{0}\binom{(q-p)n}{n} - \binom{pn}{1}\binom{(q-p)n}{n-2} + ... =$$

$$= \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \binom{pn}{k}\binom{(q-p)n}{n-2k} = a_n.$$

W9. If we consider the module series:

$$\sum_{n\geq 0} \frac{|(1-x)(1-2x)...(1-nx)|}{n!},$$

and if we apply it Cauchy-d1Alembert criterion, we find:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|1-(n+1)x|}{n+1} = |x|$$

If $|x| < 1$ the initial series is convergent. Also, for $x = 1$, this series becomes $\sum_{n\geq 0} 0$, so it is convergent to.
If $x = -1$, we obtain the series $\sum_{n\geq 0}(n+1)$, which is divergent.
We conclude that convergence set for the initial series is $C = (-1, 1]$.

In the following, we apply the binomial series for $(1+x)^{1/x}$, when $\alpha = \frac{1}{x}$, $x \neq 0$ and $x \in (-1,1)$. We have:

$$(1+x)^{1/x} = 1 + \sum_{n \geq 1} \frac{\frac{1}{x}\left(\frac{1}{x}-1\right)\dots\left(\frac{1}{x}-(n-1)\right)}{n!} n^x =$$

$$= 1 + \sum_{n \geq 1} \frac{(1-x)(1-2x)\dots(1-(n-1)x)}{n!} = \sum_{n \geq 0} \frac{(1-x)(1-2x)\dots(1-nx)}{n!}$$

Now, we conclude,

$$\sum_{n \geq 0} \frac{(1-x)(1-2x)\dots(1-nx)}{n!} = \begin{cases} (1+x)^{1/x}, & x \in (-1,1) \setminus \{0\} \\ e, & x = 0 \\ 0, & x = 1 \end{cases}$$

Surely $\lim\limits_{(n,x) \to (\infty,0)} = e$.

W10. i). Eliminating k fixed points from $1,2,\dots,n$ remain $n-k$ points. So, we have $(n-k)^{n-k}$ functions $g : \{1,2,\dots,n-k\} \to \{1,2,\dots,n-k\}$. Because k points from n, can be choosen in $\binom{n}{k}$ posibilities, it occurs:

$$|F| = (n-k)^{n-k} \binom{n}{k}$$

ii). We shall use Stirling formula of the factorial

$$n! = \left(\frac{n}{e}\right)^n \sqrt{2\pi n} e^{\theta/12n}, \ \theta \in (0,1)$$

If $n = 2k$, we have:

$$\binom{2k}{k} = \frac{(2k)!}{(k!)^2} = \frac{\left(\frac{2k}{e}\right)^{2k} \sqrt{4\pi k} e^{\theta_1/24k}}{\left[\frac{k}{e}\sqrt{2\pi k} e^{\theta_2/12k}\right]^2},$$

where $\theta_1, \theta_2 \in (0,1)$.
Therefore

$$\binom{2k}{k} = 2^{2k} \frac{e^{\frac{1}{6k}\left(\frac{\theta_1}{4}-\theta_2\right)}}{\sqrt{\pi k}} < 2^{2k} e^{1/24k},$$

because $\frac{\theta_1}{4} - \theta_2 < \frac{1}{4}$ and $\frac{1}{\sqrt{\pi k}} < 1$.

More, because $e^{1/24k} < e$, it occurs:

$$\binom{2k}{k} < e \cdot 2^{2k}$$

Finally,

$$|F| = k^k \binom{2k}{k} < e\,(4k)^k$$

iii). For $n = 2k$, the equation $|F| = 540$ becomes:

$$k^k \binom{2k}{k} = 540.$$

We notice that $k = 3$ is a unique solution, because by induction, $k^k\binom{2k}{k} > 540$, $(\forall)\, k \geq 4$. So, $n = 6$.

W11. The idea is to order the given polynomial with respect the powers of m. It obtains the quadratic equation with the discriminant $\triangle = x^4\,(x-2)^2$:

$$x^4m^2 + 3x^3m + \left(2x^2 + x - 1\right) = 0 \text{ or}$$

$$\left(mx^2 + x + 1\right)\left(mx^2 + 2x - 1\right) = 0$$

Thus

$$x_{12} = \frac{-1 \pm \sqrt{m+1}}{m}, \; x_{34} = \frac{-1 \pm \sqrt{1-4m}}{m}$$

Case 1.

$$\frac{1-m}{2m} = \frac{-1+\sqrt{m+1}}{m} \Rightarrow m_1 = 5 - 2\sqrt{5}$$

Case 2.

$$\frac{1-m}{2m} = \frac{-1-\sqrt{m+1}}{m} \Rightarrow m_2 = 5 + 2\sqrt{5}$$

Case 3.

$$\frac{1-m}{2m} = \frac{-1+\sqrt{1-4m}}{2m} \Rightarrow m_{34} = \pm i\sqrt{3} \text{ are not real}$$

Case 4.

$$\frac{1-m}{2m}=\frac{-1-\sqrt{1-4m}}{2m},\ \text{no real solutions.}$$

W12. The function $g(x)=\ln f(x)$ satisfies:
a). $g(ax)\le ag(x)$
b). $g(x+y)\le g(x)+g(y)$
Then, with the aid of a), we deduce that

$$g(x)=g\left(\frac{x}{x+y}(x+y)\right)\le\frac{x}{x+y}g(x+y)$$

and

$$g(y)=g\left(\frac{y}{x+y}(x+y)\right)\le\frac{y}{x+y}g(x+y)$$

By addition, $g(x)+g(y)\le g(x+y)$ and comparing with b) , we have

$$g(x)+g(y)=g(x+y)$$

By a standard procedure, $g(x)=\lambda x$, where $\lambda\in(0,\infty)\cap Q$.
Finally, $f(x)=e^{\lambda x}$.
Remark that if continuiuty is a part of the contest programme, then $(0,\infty)\cap Q$ can be replaced by $(0,\infty)$.

W13. If $a_1=a_2=...=a_n$ then the equality holds. Suppose that between $a_k, k=\overline{1,n}$, are two different numbers. The function $f:R\to R$ defined by

$$f(x)=\left(\frac{1}{n}\sum_{k=1}^{n}a_k^x\right)^{\frac{1}{x}}$$

is strictly increasing in this case. (see: Pólya-Szegő)
This implies that

$$\left(\frac{1}{n}\sum_{k=1}^{n}a_k^5\right)^{\frac{1}{5}}>\left(\frac{1}{n}\sum_{k=1}^{n}a_k^4\right)^{\frac{1}{4}}$$

which is equivalent to

$$\left(\frac{1}{n}\sum_{k=1}^{n}a_k^5\right)^{4}>\left(\frac{1}{n}\sum_{k=1}^{n}a_k^4\right)^{5}\tag{1}$$

On the other hand the inequality holds:

$$\sum_{k=1}^{n} a_k^4 > \frac{2}{n-1} \sum_{1\leq q<l\leq n} a_q^2 a_l^2 \tag{2}$$

because it is equivalent to

$$\sum_{1\leq q<l\leq n} \left(a_q^2 - a_l^2\right) > 0.$$

Now inequalities (1) and (2) imply the desired inequality (**). We have proved the inequality and we shown that equality holds if and only if $a_1 = a_2 = ... = a_n$.

W14. We use the substitution $x = t^{\frac{a+1}{a+2}}$ and we get the equalities:

$$\int_0^1 \frac{x^{a+1}}{f(x)\, dx} = \frac{a+1}{a+2} \int_0^1 \frac{t^a}{f\left(t^{\frac{a+1}{a+2}}\right)} dt = \frac{a+1}{a+2} \int_0^1 \frac{x^a}{f\left(x^{\frac{a+1}{a+2}}\right)} dx \tag{3}$$

Since f is an increasing function and $x^{\frac{a+1}{a+2}} \geq 0, (\forall)\, x \in [0,1]$ the inequality follows: $f\left(x^{\frac{a+1}{a+2}}\right) \geq f(x)\,, (\forall)\, x \in [0,1]\,.$
Thus the following inequality holds;

$$\frac{a+1}{a+2} \int_0^1 \frac{x^a}{f\left(x^{\frac{a+1}{a+2}}\right)} dx \leq \frac{a+1}{a+2} \int_0^1 \frac{x^a}{f(x)} dx \tag{4}$$

Now (3) and (4) imply the result.

W15. Since

$$b + c = 2R(\sin B + \sin C) = 4R \sin \frac{B+C}{2} \cos \frac{B-C}{2} =$$

$$= 4R \cos \frac{A}{2} \cos \frac{B-C}{2} \leq 4R \cos \frac{A}{2},$$

it follows that

$$b + c \leq 4R \cos \frac{A}{2}$$

In an analogous way, we have $c + a \leq 4R \cos \frac{B}{2}$; $a + b \leq 4R \cos \frac{C}{2}$. Therefore

$$x^n \cos\frac{A}{2}+y^n \cos\frac{B}{2}+z^n \cos\frac{C}{2}\ge \frac{1}{4R}\left[x^n(b+c)+y^n(c+a)+z^n(a+b)\right]=$$

$$=\frac{1}{4R}\left[b(x^n+z^n)+c(y^n+x^n)+a(z^n+y^n)\right]\ge$$

$$\ge\frac{1}{4R}\left(2b(xz)^{\frac{n}{2}}+2c(xy)^{\frac{n}{2}}+2a(yz)^{\frac{n}{2}}\right)=(xy)^{\frac{n}{2}}\sin C+(yz)^{\frac{n}{2}}\sin A+(zx)^{\frac{n}{2}}\sin B.$$

Consequently, we obtain

$$x^n \cos\frac{A}{2}+y^n \cos\frac{B}{2}+z^n \cos\frac{C}{2}\ge (xy)^{\frac{n}{2}}\sin C+(yz)^{\frac{n}{2}}\sin A+(zx)^{\frac{n}{2}}\sin B.$$

Remark. *In same way we obtain the following stronger inequality*

$$x^n \cos\frac{A}{2}+y^n \cos\frac{B}{2}+z^n \cos\frac{C}{2}\ge\left(\frac{y+z}{2}\right)^n\sin A+\left(\frac{z+x}{2}\right)^n\sin B+\left(\frac{x+y}{2}\right)^n\sin C$$

W16. We check the inequality for $n=1$, so $\frac{1}{\tau(1)}=1>\sqrt{2}-1$, which is true. We assume that the inequality holds for n. Hence we prove that the inequality is true and for $n+1$, thus:

$$\sum_{k=1}^{n+1}\frac{1}{\tau(k)}=\sum_{k=1}^{n}\frac{1}{\tau(k)}+\frac{1}{\tau(n+1)}>\sqrt{n+1}-1+\frac{1}{\tau(n+1)},$$

but $\tau(n)<2\sqrt{n}$, for any $n\ge 1$, so $\frac{1}{\tau(n+1)}>\frac{1}{2\sqrt{n+1}}$, which means that

$$\sum_{k=1}^{n+1}\frac{1}{\tau(k)}>\sqrt{n+1}-1+\frac{1}{2\sqrt{n+1}}>\sqrt{n+2}-1$$

By the mathematical induction, the inequality is true for any $n\ge 1$.

W17. If $a=b=c=1$ then we have equality. We consider $c=\max\{a,b,c\}$ and then $c\ge 1$. We show that

$$(a+1)(b+1)(c+1)\ge\min\{f(b),g(b)\}$$

and similar result that

$$(a+1)(b+1)(c+1) \geq \min\{f(a), f(b)\}$$

We have $a = \frac{1}{bc}$ and

$$(a+1)(b+1)(c+1) = \left(\frac{1}{bc}+1\right)(b+c)(c+1) = 2 + \frac{1}{bc} + bc + b + c + \frac{1}{b} + \frac{1}{c}$$

If $h(c) = 2 + \frac{1}{bc} + bc + b + c + \frac{1}{b} + \frac{1}{c}$, then

$$h'(c) = \frac{(b+1)\left(bc^2-1\right)}{bc^2}$$

evidently

$$h : [1, +\infty) \to (0, +\infty).$$

If $b < 1$, then

c	1		$\frac{1}{\sqrt{b}}$		$+\infty$
h'	$- - -$	$-$	0	$+ + +$	$+$
h	$\frac{2(b+1)^2}{b}$	$\searrow$	$(b+1)\left(\frac{2}{\sqrt{b}}+1\right)$		$\nearrow$

therefore $h(c) \geq g(b)$. If $b \geq 1$, then

c	1		$+\infty$
h'	$+ + +$	$++$	$++$
h	$\frac{2(b+1)^2}{b}$		$\nearrow$

therefore $h(c) \geq f(b)$.

W18. We have

$$a^{2n+3} + b^{2n+3} \geq a^{2n+2}b + ab^{2n+2} \Leftrightarrow (a-b)\left(a^{2n+1} - b^{2n+1}\right) \geq 0$$

By induction we get

$$a^{2n+1} + b^{2n+1} \geq (a+b)(ab)^n$$

therefore

$$\frac{ab}{a^{2n+3} + b^{2n+3} + ab} \leq \frac{ab}{a^{2n+2}b + ab^{2n+2} + ab} = \frac{1}{a^{2n+1} + b^{2n+1} + 1} \leq$$

$$\leq \frac{1}{(a+b)(ab)^n+1} = \frac{1}{(a+b)(ab)^n+(abc)^n} = \frac{1}{(ab)^n(a+b+c^n)} =$$

$$= \frac{c^n}{a+b+c^n}$$

or

$$\frac{abc}{a^{2n+3}+b^{2n+3}+ab} \leq \frac{c^{n+1}}{a+b+c^n} \Leftrightarrow \frac{a+b+c^n}{a^{2n+3}+b^{2n+3}+ab} \leq c^{n+1} \Leftrightarrow$$

$$\Leftrightarrow \sum \frac{a+b+c^n}{a^{2n+3}+b^{2n+3}+ab} \leq \sum a^{n+1}$$

W19. We have

$$\frac{x_1^2}{\left(1+x_1^2\right)^2} \leq \frac{x_1^2}{1+x_1^2} = 1 - \frac{1}{1+x_1^2}$$

and for $k \geq 2$

$$\left(\frac{x_k}{1+x_1^2+...+x_k^2}\right)^2 \leq \frac{x_k^2}{\left(1+x_1^2+...+x_{k-1}^2\right)\left(1+x_1^2+...+x_k^2\right)} =$$

$$= \frac{1}{1+x_1^2+...+x_{k+1}^2} - \frac{1}{1+x_1^2+...+x_k^2}$$

so

$$\sum_{k=1}^{n}\left(\frac{x_k}{1+x_1^2+...+x_k^2}\right)^2 \leq \left(\frac{x_1}{1+x_1^2}\right)^2 +$$

$$+\sum_{k=2}^{n}\left(\frac{1}{1+x_1^2+...+x_{k-1}^2} - \frac{1}{1+x_1^2+...+x_k^2}\right) \leq 1 - \frac{1}{1+x_1^2} + \frac{1}{1+x_1^2} -$$

$$-\frac{1}{1+\sum\limits_{k=1}^{n} x_k^2} = \frac{\sum\limits_{k=1}^{n} x_k^2}{1+\sum\limits_{k=1}^{n} x_k^2}$$

W20. 1). If $z = \cos 2x + i \sin 2x$, then

$$\sum_{0 \le j < k \le n} \sin\left(2\left(j+k\right)x\right) = Im\left(\sum_{0 \le j < k \le n} z^{j+k}\right) = \frac{1}{2}\left[\left(\sum_{j=0}^{n} z^{j}\right)^{2} - \sum_{j=0}^{n} z^{2j}\right] =$$

$$= \frac{1}{2} Im\left[\left(\frac{z^{n+1}}{z-1}\right)^{2} - \frac{z^{2n+2}-1}{z^{2}-1}\right]$$

2).

$$\sum_{0 \le j < k \le n} \cos\left(2\left(j+k\right)x\right) = \frac{1}{2} Re\left[\left(\frac{z^{n+1}-1}{z-1}\right)^{2} - \frac{z^{2n+2}-1}{z^{2}-1}\right]$$

but

$$\frac{z^{n+1}-1}{z-1} = \frac{\sin\left(n+1\right)x}{\sin x}\left(\cos nx + i \sin nx\right)$$

so

$$\sum_{0 \le j < k \le n} \sin\left(2\left(j+k\right)x\right) = \frac{\sin nx \sin\left(n+1\right)x \sin 2nx}{\sin x \sin 2x}$$

and

$$\sum_{0 \le j < k \le n} \cos\left(2\left(j+k\right)x\right) = \frac{\sin nx \sin\left(n+1\right)x \cos 2nx}{\sin x \sin 2x}$$

and

$$\left(\sum_{0 \le j < k \le n} \sin\left(2\left(j+k\right)x\right)\right)^{2} + \left(\sum_{0 \le j < k \le n} \cos\left(2\left(j+k\right)x\right)\right)^{2} =$$

$$= \frac{\sin^{2} nx \sin^{2}\left(n+1\right)x}{\sin^{2} x \sin^{2} 2x}$$

W21. We prove by induction the inequality

$$\sum_{k=1}^{n} \frac{x_k}{1+x_k} \ge \frac{\sum\limits_{k=1}^{n} x_k}{1+\sum\limits_{k=1}^{n} x_k}$$

It is true for $n=1$, for $n=2$ after computation we have

$$(x_1+x_2)\,x_1x_2+2x_1x_2\geq 0$$

true. We suppose true for n and we prove for $n+1$.

$$\sum_{k=1}^{n+1}\frac{x_k}{1+x_k}=\sum_{k=1}^{n}\frac{x_k}{1+x_k}+\frac{x_{n+1}}{1+x_{n+1}}\geq\frac{\sum\limits_{k=1}^{n}x_k}{1+\sum\limits_{k=1}^{n}}+\frac{x_{n+1}}{1+x_{n+1}}\geq\frac{\sum\limits_{k=1}^{n+1}x_k}{1+\sum\limits_{k=1}^{n+1}x_k}$$

the last step is the case for $n=2$.

If $x_k=\frac{1}{k^s}$, then $\sum\limits_{k=1}^{n}\frac{1}{k^s+1}\geq\frac{\sum\limits_{k=1}^{n}\frac{1}{k^s}}{1+\sum\limits_{k=1}^{n}\frac{1}{k^s}}$. In this we take $n\to\infty$, and we obtain the desired inequality.

W22. For $k\in\{0,1\}$ we have equality. Using the weighted AM-GM inequality we get

$$\left(\frac{a_1}{a_2}\right)^k+\frac{k-1}{k+n-1}\left(\left(\frac{a_2}{a_3}\right)^k+\ldots+\left(\frac{a_n}{a_1}\right)^k\right)\geq\frac{nk}{k+n-1}\cdot\frac{a_1}{a_2}$$

therefore

$$\frac{nk}{k+n-1}\sum\left(\frac{a_1}{a_2}\right)^k=\sum_{cyclic}\left(\left(\frac{a_1}{a_2}\right)^k+\frac{k-1}{k+n-1}\left(\left(\frac{a_2}{a_3}\right)^k+\ldots+\left(\frac{a_n}{a_1}\right)^k\right)\right)$$

$$\geq\frac{nk}{k+n-1}\sum\frac{a_1}{a_2}$$

therefore

$$\sum\left(\frac{a_1}{a_2}\right)^k\geq\sum\frac{a_1}{a_2}$$

W23. If $x,y\in R$ then

$$\left(x^2+xy+y^2\right)^m\leq\frac{3^m}{2}\left(x^{2m}+y^{2m}\right)\qquad(1)$$

For $m = 1$ is true. We suppose true for m, and we prove for $m + 1$.

$$\left(x^2 - xy + y^2\right)^{m+1} \le \frac{3^m}{2}\left(x^{2m} + y^{2m}\right)\left(x^2 - xy + y^2\right) \le$$

$$\le \frac{3^{m+1}}{2}\left(x^{2m+2} + y^{2m+2}\right)$$

We have

$$\left(x^{2m} + y^{2m}\right)\left(x^2 - xy + y^2\right) \le 3\left(x^{2m+2} + y^{2m+2}\right) \Leftrightarrow$$

$$\Leftrightarrow \left(x^{2m+1} + y^{2m+1}\right)(x + y) + \left(x^{2m} - y^{2m}\right)\left(x^2 - y^2\right) \ge 0$$

1). $\sum\limits_{cyclic}\left(x_1^2 - x_1x_2 + x_2^2\right)^m \le \sum\limits_{cyclic}\frac{3^m}{2}\left(x_1^{2m} + x_2^{2m}\right) = 3^m\sum\limits_{k=1}^{n} x_k^{2m}$

2). $\prod\limits_{cyclic}\left(x_1^2 - x_1x_2 + x_2^2\right)^m \le \left(\frac{3^m}{2}\right)^n \prod\limits_{cyclic}\left(x_1^{2m} + x_2^{2m}\right) \le$

$\left(\frac{3^m}{2}\right)^n\left(\frac{1}{n}\sum\limits_{cyclic}\left(x_1^{2m} + x_2^{2m}\right)\right)^n = \left(\frac{3^m}{2}\right)^n\left(\frac{2}{n}\sum\limits_{k=1}^{n} x_k^{2m}\right)^n = \left(\frac{3^m}{n}\right)^n\left(\sum\limits_{k=1}^{n} x_k^{2m}\right)^n$

W24. If $A(a), B(b), C(c), P(z)$, $K\left(\frac{a+b}{2}\right)$, $L\left(\frac{b+c}{2}\right)$, $M\left(\frac{c+a}{2}\right)$ and

$$\sum\left|\frac{z_1 - z_2}{z_1 + z_2}\right| \ge \left|\frac{z_1 - z_2}{z_1 + z_2} + \frac{z_2 - z_3}{z_2 + z_3} + \frac{z_3 - z_1}{z_3 + z_1}\right| =$$

$$= \left|\frac{z_1 - z_2}{z_1 + z_2} + \frac{z_2 - z_3}{z_2 + z_3} - \frac{z_1 - z_2}{z_3 + z_1} - \frac{z_2 - z_3}{z_3 + z_1}\right| =$$

$$= \left|\frac{(z_1 - z_2)(z_3 - z_2)}{(z_1 + z_2)(z_3 + z_1)} + \frac{(z_1 - z_2)(z_2 - z_3)}{(z_2 + z_3)(z_3 + z_1)}\right| = \left|\frac{(z_1 - z_2)(z_2 - z_3)(z_3 - z_1)}{(z_1 + z_2)(z_2 + z_3)(z_3 + z_1)}\right|$$

If $z_1 = z - a$, $z_2 = z - b$, $z_3 = z - c$, then

$$\sum\frac{1}{2}\left|\frac{a - b}{z - \frac{a+b}{2}}\right| \ge \frac{1}{8}\left|\prod\frac{a - b}{z - \frac{a+b}{2}}\right| \Rightarrow \sum\frac{AB}{PK} \ge \frac{1}{4}\prod\frac{AB}{PK}$$

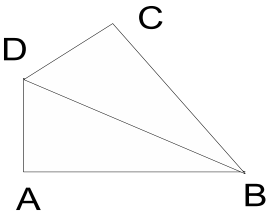
W25. If $x, y > 0$ and $x^2 + y^2 = 1$, then

$$x + y + \frac{1}{xy} \ge 2 + \sqrt{2} \qquad (*)$$

Proof. We have

$$E=x+y+\frac{1}{xy}=(x+y)\sqrt{x^2+y^2}+\frac{1}{xy}\geq 2\sqrt{xy}\sqrt{2xy}+\frac{1}{xy}=2\sqrt{2}xy+\frac{1}{xy}.$$

But $1=x^2+y^2\geq 2xy$, $xy=t\Rightarrow t\leq\frac{1}{2}$ and $t\leq\frac{1}{\sqrt{2}}\Rightarrow E=2\sqrt{2}t+\frac{1}{t}\geq 2+\sqrt{2}\Leftrightarrow 2\sqrt{2}t^2-\left(2+\sqrt{2}\right)t+1\geq 0\Leftrightarrow\left(t-\frac{1}{2}\right)\left(t-\frac{1}{\sqrt{2}}\right)\geq 0$



In triangle ABD we take
(1)

$$x=\frac{AD}{BD},y=\frac{AB}{BD}\overset{(*)}{\Rightarrow}\frac{AD+AB}{BD}+\frac{BD^2}{AD\cdot AB}\geq 2+\sqrt{2}$$

In triangle BCD we take
(2)

$$x=\frac{BC}{BD},y=\frac{DC}{BD}\overset{(*)}{\Rightarrow}\frac{BC+CD}{BD}+\frac{BD^2}{BC\cdot CD}\geq 2+\sqrt{2}$$

Adding (1) and (2) we get

$$\frac{1}{BD}\left(AB+BC+CD+DA\right)+BD^2\left(\frac{1}{AB\cdot AD}+\frac{1}{CB\cdot CD}\right)\geq 2\left(2+\sqrt{2}\right)$$

W26. We have $1=\sum\limits_{i=1}^{n}a_i^k\geq n\sqrt[n]{\prod\limits_{i=1}^{n}a_i^k}$ if
$\prod\limits_{i=1}^{n}a_i=t\Rightarrow 1\geq n\sqrt[n]{t^k}\Rightarrow t\leq n^{-\frac{n}{k}}\leq n^{-1+\frac{1}{k}}$, because $k\leq n+1$, therefore

$$\sum_{i=1}^{n}a_i+\frac{1}{\prod\limits_{i=1}^{n}a_i}=\left(\sum_{i=1}^{n}a_i\right)\left(\sum_{i=1}^{n}a_i^k\right)^{\frac{n-1}{k}}+\frac{1}{\prod\limits_{i=1}^{n}a_i}\geq$$

$$\geq n\sqrt[n]{\prod_{i=1}^{n}\left(n\sqrt[n]{\prod_{i=1}^{n}a_i^k}\right)^{\frac{n-1}{k}}}+\frac{1}{\prod\limits_{i=1}^{n}a_i}=n^{\frac{n-1}{k}+1}\prod_{i=1}^{n}a_i+\frac{1}{\prod\limits_{i=1}^{n}a_i}=n^{\frac{n-1}{k}+1}t+\frac{1}{t}\geq$$

$$\geq n^{1-\frac{1}{k}}+n^{\frac{n}{k}} \Leftrightarrow n^{\frac{n-1}{k}+1}t^2-\left(n^{1-\frac{1}{k}}+n^{\frac{n}{k}}\right)t+1 \geq 0 \Leftrightarrow \left(n^{\frac{n}{k}}t-1\right)\left(n^{1-\frac{1}{k}}t-1\right) \geq 0$$

true

W27. It is a known fact that for every perfect number m it holds

$$\sum_{d|m} \frac{1}{d} = 2.$$

Therefore

$$\sum_{d|a^n} \frac{1}{d} = 2.$$

Assume that

$$a^n = p_1^{k_1} p_2^{k_2} ... p_\mu^{k_\mu}$$

is the unique prime factorization of the number a^n. It is obvious that

$$\frac{1}{p_1^{k_1}} + \frac{1}{p_2^{k_2}} + ... + \frac{1}{p_\mu^{k_\mu}} < 2 \qquad (1)$$

From Cauchy‘s arithmetic-geometric mean inequality it follows

$$\sqrt[\mu]{p_1^{k_1} p_2^{k_2} ... p_\mu^{k_\mu}} \geq \frac{\mu}{\frac{1}{p_1^{k_1}} + \frac{1}{p_2^{k_2}} + ... + \frac{1}{p_\mu^{k_\mu}}} \Leftrightarrow \frac{1}{p_2^{k_2}} + ... + \frac{1}{p_\mu^{k_\mu}} \geq \frac{\mu}{\sqrt[\mu]{p_1^{k_1} p_2^{k_2} ... p_\mu^{k_\mu}}}$$

Therefore

$$\frac{1}{p_1^{k_1}} + \frac{1}{p_2^{k_2}} + ... + \frac{1}{p_\mu^{k_\mu}} \geq \frac{\mu}{a^{n/\mu}} \qquad (2)$$

From (1) and (2) it follows that

$$2 > \frac{\mu}{a^{n/\mu}} \Leftrightarrow a^{n/\mu} > \frac{\mu}{2}$$

W28. Set $y = 0$ in (1). Then

$$\left\|2f\left(\frac{x}{2}\right) - f(x)\right\|_Y \leq \theta \left\|x\right\|_X^p$$

for all $x \in X$, since $x \perp 0$. Thus

$$\left\| f(x) - \frac{1}{2} f(2x) \right\|_Y \leq \frac{2^p \cdot \theta}{2} \|x\|_X^p$$

for all $x \in X$. Therefore

$$\left\| \frac{1}{2^n} f(2^n x) - \frac{1}{2^m} f(2^m x) \right\|_Y \leq \frac{2^p \cdot \theta}{2} \sum_{k=n}^{m-1} \frac{2^{pk}}{2^k} \|x\|_X^p$$

for all nonnegative integers n, m satisfying $n < m$. Hence

$$\left\{ \frac{1}{2^n} f(2^n x) \right\}$$

is a Cauchy sequence in Y. Because of the fact Y is complete, there exists a mapping $T : X \to Y$ defined by

$$T(x) = \lim_{n \to +\infty} \frac{1}{2^n} f(2^n x)$$

for all $x \in X$. Setting $n = 0$ and $m \to +\infty$ in (4), we get the inequality (2). It follows from (1) that

$$\left\| 2T\left(\frac{x+y}{2}\right) - T(x) - T(y) \right\|_Y =$$

$$= \lim_{n \to +\infty} \frac{1}{2^n} \left\| 2f\left(2^{n-1}(x+y)\right) - f(2^n x) - f(2^n y) \right\|_Y \leq$$

$$\leq \lim_{n \to +\infty} \frac{2^{pn} \cdot \theta}{2^n} \left(\|x\|_X^p + \|y\|_X^p \right) = 0$$

for all $x, y \in X$ with $x \perp y$. Thus

$$2T\left(\frac{x+y}{2}\right) - T(x) - T(y) = 0$$

for all $x, y \in X$ with $x \perp y$. Therefore $T : X \to Y$ is an orthogonally Jensen additive mapping.

Let $L : X \to Y$ be another orthogonally Jensen additive mapping satisfying (2).

It follows that

$$\|T(x) - L(x)\|_Y = \frac{1}{2^n} \|T(2^n x) - L(2^n x)\|_Y \leq$$

$$\leq \frac{1}{2^n}\left(\left\|f\left(2^n x\right)-T\left(2^n x\right)\right\|_Y+\left\|f\left(2^n x\right)-L\left(2^n x\right)\right\|_Y\right) \leq \frac{2^{p+1}\cdot\theta}{2-2^p}\cdot\frac{2^{pn}}{2^n}\|x\|_X^p$$

which approaces zero for all $x \in X$. Thus

$$T(x)=L(x) \text{ for all } x \in X,$$

which proves the uniqueness of the mapping T.

W29. We have

$$2\left(1-x+x^2\right)\left(1-y+y^2\right)=1+x^2y^2+(x-y)^2+(1-x)^2(1-y)^2 \geq 1+x^2y^2$$

If $x=\sqrt{\sqrt{3}tg\frac{A}{2}}, y=\sqrt{\sqrt{3}tg\frac{B}{2}}$, then

$$2\sum\left(1-\sqrt{\sqrt{3}tg\frac{A}{2}}+\sqrt{3}tg\frac{A}{2}\right)\left(1-\sqrt{\sqrt{3}tg\frac{B}{2}}+\sqrt{3}tg\frac{B}{2}\right) \geq$$

$$\geq \sum\left(1+3tg\frac{A}{2}tg\frac{B}{2}\right)=6 \Rightarrow$$

$$\Rightarrow \sum\left(1-\sqrt{\sqrt{3}tg\frac{A}{2}}+\sqrt{3}tg\frac{A}{2}\right)\cdot\left(1-\sqrt{\sqrt{3}tg\frac{B}{2}}+\sqrt{3}tg\frac{B}{2}\right) \geq 3$$

Equality holds if and only if $A=B=C=\frac{\pi}{3}$.

W30. We have the following identity:

$$2\sum_{0\leq i<j\leq n} x^{i+j}\binom{n}{i}\binom{n}{j}=(1+x)^{2n}-\sum_{i=0}^{n} x^{2i}\binom{n}{i}^2$$

and after derivation we get

$$\sum_{0\leq i<j\leq n}(i+j)\,x^{i+j-1}\binom{n}{i}\binom{n}{j}=n\,(1+x)^{2n-1}-\sum_{i=0}^{n} ix^{2i-1}\binom{n}{i}^2$$

but

$$\sum_{i=0}^{n} i\binom{n}{i}^2=n\binom{2n}{n}$$

and after then we take $x = 1$, therefore

$$\sum_{0 \le i < j \le n} (i+j) \binom{n}{i} \binom{n}{j} = n \left(2^{2n-1} - \binom{2n}{n} \right)$$

RESULTS

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Proposed problems

PP. 15870. [38] If $a, b \in N^*$ and $\left|\frac{a}{b} - \sqrt{7}\right| < \frac{4}{18081}$, then $\left|\frac{4a+7b}{a+4b} - \sqrt{7}\right| < \frac{1}{2009}$.

Mihály Bencze

PP. 15871. Prove that:

1). $\sum\limits_{k=1}^{n} \sqrt{F_{2k} L_{2k}} \le \sqrt{\left(F_{2n+1} - 1\right)\left(L_{2n+1} - 2\right)}$
2). $\sum\limits_{k=1}^{n} \sqrt{F_{2k-1} F_{2k}} \le \sqrt{F_{2n}\left(F_{2n+1} - 1\right)}$

Mihály Bencze

PP. 15872. If $f(k) = \left\{1 + \frac{1}{2} + ... + \frac{1}{k}\right\}$ where $k \in N^*$ and $\{\cdot\}$ denote the fractional part, then solve the following system: $\begin{cases} f(a) = f(b) \\ f(b) = f(c) \\ f(c) = f(a) \end{cases}$, where $a, b, c \in N^*$.

Mihály Bencze

PP. 15873. If $x_k, a, b > 0$ $(k = 1, 2, ..., n)$, then

$$\left(\frac{1}{a} + \frac{1}{b}\right) \sum_{k=1}^{n} x_k^2 \ge \frac{1}{a+b} \sum_{cyclic} (x_1 + x_2)^2 + \frac{1}{nab(a+b)} \left(\sum_{cyclic} |bx_1 - ax_2|\right)^2.$$

Mihály Bencze

PP. 15874. If $a, b \in N^*$ and $\left|\frac{a}{b} - \sqrt{3}\right| < \frac{1}{502}$, then $\left|\frac{2a+3b}{a+2b} - \sqrt{3}\right| < \frac{1}{2008}$.

Mihály Bencze

PP. 15875. In all triangle ABC holds:

1). $2 \le \frac{r_a + r_c}{r_a + r_b} + \frac{r_a + r_b}{r_a + r_c} \le \frac{R}{r}$
2). $2 \le \frac{a+c}{a+b} + \frac{a+b}{a+c} \le \frac{s^2 + r^2 + 2Rr}{2Rr}$

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[38] *Solution should be mailed to editor until* 30.12.2010. *No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new in sights on past problems.*

3). $2 \le \frac{h_a+hc}{h_a+h_b} + \frac{h_a+h_b}{h_a+h_c} \le \frac{\left(s^2+r^2+4Rr\right)^2+8s^2Rr}{32s^2Rr}$

4). $2 \le \frac{\sin^2\frac{A}{2}+\sin^2\frac{C}{2}}{\sin^2\frac{A}{2}+\sin^2\frac{B}{2}} + \frac{\sin^2\frac{A}{2}+\sin^2\frac{B}{2}}{\sin^2\frac{A}{2}+\sin^2\frac{C}{2}} \le \frac{(2R-r)\left(s^2+r^2-8Rr\right)-2Rr^2}{8Rr^2}$

5). $2 \le \frac{\cos^2\frac{A}{2}+\cos^2\frac{C}{2}}{\cos^2\frac{A}{2}+\cos^2\frac{B}{2}} + \frac{\cos^2\frac{A}{2}+\cos^2\frac{B}{2}}{\cos^2\frac{A}{2}+\cos^2\frac{C}{2}} \le \frac{(4R+r)^3+s^2(2R+r)}{8Rs^2}$

Mihály Bencze

PP. 15876. Solve the following system: $\begin{cases} xy - \frac{1}{xy} = x + y - \frac{1}{x} - \frac{1}{y} \\ yz - \frac{1}{yz} = y + z - \frac{1}{y} - \frac{1}{z} \\ zx - \frac{1}{zx} = z + x - \frac{1}{z} - \frac{1}{x} \end{cases}$.

Mihály Bencze

PP. 15877. If $x, y \in R$, then

$$\left(\sin^2 x \sin^2 y + \cos^2 x \cos^2 y\right)\left(1 + \frac{1}{\sin^2 x \sin^2 y \cos^2 x \cos^2 y}\right) \ge$$
$$\ge \frac{1}{\sin^2 x \cos^2 x} + \frac{1}{\sin^2 y \cos^2 y} - 2.$$

Mihály Bencze

PP. 15878. If $a, b \in N^*$, then $\left|\frac{3a+4b}{2a+3b} - \sqrt{2}\right| < \frac{1}{4}\left|\frac{a+2b}{a+b} - \sqrt{2}\right| < \frac{1}{16}\left|\frac{a}{b} - \sqrt{2}\right|$.

Mihály Bencze

PP. 15879. Compute

1). $\lim\limits_{n\to\infty} \sqrt[4]{n!} \prod\limits_{k=1}^{n} \left((k+1)^{\frac{3}{4}} - k^{\frac{3}{4}}\right)$ 2). $\lim\limits_{n\to\infty} n! \prod\limits_{k=1}^{n} \ln\left(1 + \frac{1}{k}\right)$

Mihály Bencze and Bálint Bartha

PP. 15880. Prove that $\sum\limits_{k=1}^{n} \prod\limits_{p=1}^{k} \left(1 + \frac{1}{2p}\right) > \frac{n(n+4)}{3}$.

Mihály Bencze

PP. 15881. If $A_k = \int\limits_0^1 \frac{x^k}{3x+2}dx$ and $B_k = \int\limits_0^1 \frac{x^k}{x^2+1}dx$, then $\frac{10n\left(n^2-1\right)}{3} \leq \sum\limits_{k=1}^{n} \frac{1}{A_k B_k} \leq \frac{5n\left(2n^2+9n+13\right)}{3}$.

Mihály Bencze

PP. 15882. Let $a_k\,(k=1,2,...,n)$ denote the sides of a convex n-gon, and $S = \sum\limits_{k=1}^{n} a_k$. Prove that $\prod\limits_{cyclic} (S-2x_1)^{x_2} \leq \prod\limits_{k=1}^{n} x_k^{x_k}$.

Mihály Bencze

PP. 15883. In all triangle ABC holds
$\sum m_a^2 m_b^2 \geq \sum r_a^2 w_a^2 + \frac{1}{16} \sum\limits_{cyclic} \left(2\left(s^2-r^2-4Rr\right)-3ab\right)^2$.

Mihály Bencze

PP. 15884. In all triangle ABC holds
$\prod \left(\cos\frac{B-C}{2} - \sin\frac{3A}{2}\right) + 8\prod \sin^2\frac{A-B}{2} = 0$.

Mihály Bencze

PP. 15885. If $a,b,c,d>0$, then $\left(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}\right)\sqrt[3]{d}+\sqrt[3]{ab}+\sqrt[3]{bc}+\sqrt[3]{ca} \leq$

$\leq \sqrt[3]{a+b+c}\left(\sqrt[3]{a+b+d}+\sqrt[3]{b+c+d}+\sqrt[3]{c+a+d}\right)$.

Mihály Bencze

PP. 15886. If $x_k \in R\ (k=1,2,...,n)$, then
$\left(1-\prod\limits_{k=1}^{n} \sin^2 x_k\right)^m + \left(1-\prod\limits_{k=1}^{n} \cos^2 x_k\right)^m \geq 1$, when $n,m \in N^*$.

Mihály Bencze

PP. 15887. Solve in R the following system: $\begin{cases} e^{x^3+2y^2+3z} = \frac{1}{x+1} \\ e^{y^3+2z^2+3x} = \frac{1}{y+1} \\ e^{z^3+2x^2+3y} = \frac{1}{z+1} \end{cases}$.

Mihály Bencze

PP. 15888. If $x \in \left[0, \frac{\pi}{4}\right]$, then
$\sin^2 x \cos x \left(\sin^3 x + \cos^4 x + \sin^5 x\right) + (\sin x \cos x)^6 \le 1.$

Mihály Bencze

PP. 15889. If $x \in R$, then $\frac{ch^4 x}{chx+\sqrt{2+ch^2 x}} - \frac{sh^2 x}{shx+\sqrt{1+ch^2 x}} \ge \frac{1}{2} shx.$

Mihály Bencze

PP. 15890. Prove that $\sum\limits_{k=1}^{n} \left(\frac{\left(k^2+1\right)^2}{\sqrt{k^2+1}+\sqrt{k^2+3}} - \frac{k^4}{k+\sqrt{k^2+2}} \right) > \frac{n(n+1)}{4}.$

Mihály Bencze

PP. 15891. Prove that $\sum\limits_{n=2}^{\infty} \frac{\left(n^2+3n+2\right)^{2n-2}}{9^n (n!)^4} > \frac{6\pi^2-49}{36}.$

Mihály Bencze

PP. 15892. Prove that $\prod\limits_{k=0}^{n} \binom{n}{k}^{4(k+1)} \le \left(\frac{2\binom{2n-1}{n-1}}{n+1} \right)^{(n+1)(n+2)}.$

Mihály Bencze

PP. 15893. In all acute triangle ABC holds $\sum \frac{\cos^2 A \cos^2 B}{\cos C} \ge$

$\ge \frac{1}{2} \sum (\cos A - \cos B)^2 \cos^2 C + \frac{(R+r)\left(s^2+r^2-4R^2\right)}{R^3} - \frac{3\left(s^2-(2R+r)^2\right)}{2R^2}.$

Mihály Bencze

PP. 15894. If $x_k > 0$ $(k = 1, 2, ..., n)$, then $\sum\limits_{k=1}^{n} \frac{x_k}{\alpha+x_k} + \frac{1}{\alpha+\sum\limits_{k=1}^{n} x_k} \ge \frac{1}{\alpha}$ for all $\alpha \ge 1$.

Mihály Bencze

PP. 15895. If in quadrilateral $ABCD$ we have $\widehat{A} = \widehat{C} = 90^\circ$, then
$\sqrt{(BD-AB)\,AD} + \sqrt{(BD-AD)\,AB} + \sqrt{(BD-BC)\,DC} +$
$+\sqrt{(BD-DC)\,BC} \le 2BD.$

Mihály Bencze

PP. 15896. If F_k denote the k-th Fibonacci number, then

$$\prod_{k=1}^{n}\left(n-1+F_k^2\right)\geq n^{n-2}\left(F_{n+2}-1\right)^2.$$

Mihály Bencze

PP. 15897. Let $ABCD$ be a quadrilateral in which $\widehat{A}=\widehat{C}=90^\circ$. Prove that $\frac{1}{BD}\left(AB+BC+CD+DA\right)+BD^2\left(\frac{1}{AB\cdot AD}+\frac{1}{CB\cdot CD}\right)\geq 4+2\sqrt{2}$.

Mihály Bencze

PP. 15898. Let $ABCDA_1B_1C_1D_1$ be a rectangle paralelipipedon with sides a,b,c and diagonal d. Prove that $\frac{a+b+c}{d}+\frac{d^3}{abc}\geq 4\sqrt{3}$.

Mihály Bencze

PP. 15899. 1). Let be $a_k,b_k\in R$ $(k=1,2,...,n)$ such that $b_p=\frac{a_1+a_2+...+a_p}{p}$ $(p=1,2,...,n)$. Prove that $(a_n)_{n\geq 1}$ is arithmetical progression if and only if $(b_n)_{n\geq 1}$ is arithmetical progression.

2). Let be $a_k,b_k,\lambda_k\in R$ $(k=1,2,...,n)$ such that $b_p=\frac{\lambda_1a_1+\lambda_2a_2+...+\lambda_pa_p}{\lambda_1+\lambda_2+...+\lambda_p}$ $(p=1,2,...,n)$. Determine all $\lambda_k\in R$ $(k=1,2,...,n)$ for which $(a_n)_{n\geq 1}$ is arithmetical progression if and only if $(b_n)_{n\geq 1}$ is arithmetical progression.

Mihály Bencze

PP. 15900. Let be $E(x)=2+\sin 2x+\left(1+\frac{3}{2}\sin 2x\right)(\sin x+\cos x)$, where $x\in R$.

1). Prove that $\max E(x)=3\left(1+\sqrt{2}\right)$ 2). Determine $\min E(x)$.

Mihály Bencze

PP. 15901. If F_k denote the k-th Fibonacci number, then

$$\frac{F_{n+2}-1}{\sqrt{F_nF_{n+1}}}+\frac{\left(\sqrt{F_nF_{n+1}}\right)^n}{\prod\limits_{k=1}^{n}F_k}\geq\sqrt{n}+\sqrt{n^n}.$$

Mihály Bencze

PP. 15902. If $a_i > 0$ $(i = 1, 2, ..., n)$ and $\sum\limits_{i=1}^{n} a_i^k = 1$, where $1 \leq k \leq n + 1$, then $\sum\limits_{i=1}^{n} a_i + \frac{1}{\prod\limits_{i=1}^{n} a_i} \geq n^{1-\frac{1}{k}} + n^{\frac{n}{k}}$.

Mihály Bencze

PP. 15903. If $a_i > 0$ $(i = 1, 2, ..., n)$ and $k \geq 2$ $(k \in N)$, then

$\sum\limits_{i=1}^{n} a_i^k \geq a_1 a_2 ... a_k + a_2 a_3 ... a_{k+1} + ... + a_n a_1 ... a_{k-1}$.

Mihály Bencze

PP. 15904. In all acute triangle ABC holds $r \sum tgA \geq R \sum \sin A$.

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PP. 15905. In all triangle ABC holds

1). $3r^2 + 4Rr + 4R^2 \geq s^2$
2). $r\left(s^2 - r^2 - 4Rr\right)^2 \leq 2s^2 (R - r)\left(s^2 + r^2 - 2Rr\right)$
3). $\left(s^2 - (2R + r)^2\right)\left(s^2 - r^2 - 4Rr\right) \geq 9$

Mihály Bencze

PP. 15906. In all acute triangle ABC holds

$\left(\sum a^2\right)^3 \left(\sum \frac{1}{a^2+b^2-c^2}\right) \geq 27 \left(4 \sum a^2 b^2 - \left(\sum a^2\right)^2\right)$.

Mihály Bencze

PP. 15907. In all acute triangle ABC holds

1). $\left(\sum \sqrt{\cos A}\right)^2 \leq \frac{s^2+r^2+4Rr}{2R^2}$
2). $(2R + r)^2 + \frac{8s^2r^2}{27R^2} \geq s^2$
3). $\sum \cos A \sin B \sin C \geq \frac{(R+r)^2}{2R^2}$

Mihály Bencze

PP. 15908. Prove that

$$\cos\frac{2\pi}{19}+\cos\frac{6\pi}{19}+\cos\frac{10\pi}{19}+\cos\frac{14\pi}{19}+\cos\frac{18\pi}{19}+\cos\frac{22\pi}{19}+\cos\frac{30\pi}{19}+\cos\frac{34\pi}{19}\leq\frac{161}{18}.$$

Mihály Bencze

PP. 15909. In all acute triangle ABC holds

1). $\left(\sum a\sqrt{\cos A\cos B}\right)^2\leq\frac{2sr}{R}\sum a\cos B$

2). $\left(\sum a\sqrt{\cos A\sin B}\right)^2\leq\frac{2sr}{R}\sum a\sin B$

Mihály Bencze

PP. 15910. In all triangle ABC holds:

1). $\sum\sin^2 AtgA\geq\frac{\left(s^2-r^2-4Rr\right)^2}{4sR^2r}$

2). $\sum\sin^4 AtgA\geq\frac{s^2\left(s^2-3r^2-6Rr\right)^2}{16sR^4r}$ 3). $\sum tgA\sin^2 B\geq\frac{\left(s^2+r^2+4Rr\right)^2}{16R^2sr}$

Mihály Bencze

PP. 15911. In all triangle ABC holds:

1). $\sum\sin A\cos A\leq\frac{sr}{R^2}$ 2). $\sum tgA\cos^2 B\geq\frac{R^2}{sr}\left(\sum\sin A\cos B\right)^2$

Mihály Bencze

PP. 15912. Prove that:

1). $\prod\limits_{k=1}^{n}(k!)^k\leq\left(\frac{2((n+1)!-1)}{n(n+1)}\right)^{\frac{n(n+1)}{2}}$

2). $\prod\limits_{k=1}^{n}(k!)^{(k+1)^2}\leq\left(\frac{6((n+2)!-2)}{n(n+1)(2n+1)-6}\right)^{\frac{n(n+1)(2n+1)}{6}-1}$

3). $\prod\limits_{k=1}^{n}(k!)^{(k-1)^2}\leq\left(\frac{6((n-2)(n+1)!+2)}{(n-1)n(2n-1)}\right)^{\frac{(n-1)n(2n-1)}{6}}$

4). $\prod\limits_{k=1}^{n}(k!)^{k^2+1}\leq\left(\frac{6(n+1)!}{2n^2+3n+7}\right)^{\frac{n\left(2n^2+3n+7\right)}{6}}$

Mihály Bencze

PP. 15913. In all triangle ABC holds:
1). $\left(\sum a\sqrt{a}\right)^2 \leq 16s\left(R-r\right)\left(4R+r\right)$ 2). $\left(\sum a\sqrt{b}\right)^2 \leq \frac{8s(R-r)\left(s^2-r^2-4Rr\right)}{r}$

Mihály Bencze

PP. 15914. Prove that $\prod\limits_{k=1}^{n} \binom{n}{k}^{\binom{n}{k-1}} \leq \left(\frac{n\binom{2n}{n}}{(n+1)(2^n-1)}\right)^{2^n-1}$.

Mihály Bencze

PP. 15915. Prove that $\sum\limits_{k=1}^{n} 3\cdot 5\cdot \ldots \cdot (2k+1) \geq 2\left(\sum\limits_{k=1}^{n}\sqrt{k}\right)^2$.

Mihály Bencze

PP. 15916. 1). Prove that $\sum\limits_{k=1}^{n} \frac{1}{\sin\frac{(6k-5)\pi}{6n}} \leq 2n\sqrt{n}$
2). Determine the maximum of $\sum\limits_{k=1}^{n} \frac{1}{\cos\frac{(6k-5)\pi}{6n}}$

Mihály Bencze

PP. 15917. In all acute triangle ABC holds

$2\sum \frac{1}{9R^2-2a^2} \leq \frac{1}{s^2-(2R+r)^2} \leq \sum \frac{1}{18r^2-a^2}$.

Mihály Bencze

PP. 15918. In all acute triangle ABC holds $\sum \frac{1}{a^2+b^2-c^2} \geq \frac{1}{4r^2}$.

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PP. 15919. In all acute triangle ABC holds
$\sum \frac{1}{4R^2+2r^2-a^2} \leq \frac{1}{s^2-(2R+r)^2} \leq \sum \frac{1}{12Rr-6r^2-a^2}$.

Mihály Bencze

PP. 15920. In all acute triangle ABC holds
$\frac{1}{4\left(2Rr-R^2-r^2-(R-2r)\sqrt{R^2-2Rr}\right)} \leq \sum \frac{1}{a^2+b^2-c^2} \leq \frac{1}{4\left(3Rr-R^2-r^2+(R-2r)\sqrt{R^2-2Rr}\right)}$.

Mihály Bencze

PP. 15921. In all triangle ABC holds $\frac{3r(2R-r)}{(R+r)^2} \leq \frac{\sum a^2}{\sum ab} \leq \frac{2R^2+r^2}{r(5R-r)}$.

Mihály Bencze

PP. 15922. In all acute triangle ABC holds $\frac{1}{4r^2} \leq \frac{1}{2\left(s^2-(2R+r)^2\right)} \leq \frac{R^2}{16r^4}$.

Mihály Bencze

PP. 15923. In all triangle ABC holds
1). $2s^2R\left(2R-r\right) \geq \left(s^2-r^2-4Rr\right)^2$ 2). $\frac{9}{2R(R+r)} \leq \left(\frac{s^2+r^2+4Rr}{4sRr}\right)^2 \leq \frac{3}{4r^2}$

Mihály Bencze

PP. 15924. Let ABC be a triangle. Determine all $\alpha \in R$ for which
$\left(\frac{2(5R-r)}{R}\right)^{\alpha} \leq \left(\sum h_a^{\alpha}\right)\left(\sum \frac{1}{h_a^{\alpha}}\right) \leq \left(\frac{2(R+r)^2}{Rr}\right)^{\alpha}$.

Mihály Bencze

PP. 15925. In all triangle ABC holds
1). $\sum \sqrt{\left(r_a-2r\right)r_b} \leq s$ 2). $\sum \sqrt{\left(h_a-2r\right)h_b} \leq s\sqrt{\frac{2r}{R}}$

Mihály Bencze

PP. 15926. In all triangle ABC holds $\frac{81r}{2s} \leq \left(\sum \frac{1}{a}\right)\left(\sum m_a\right) \leq \frac{81R^2}{8sr}$.

Mihály Bencze

PP. 15927. 1). Let $A_1A_2...A_{2n}$ be a convex polygon. Determine the geometrical locus of points M in the plane of the polygon, for which
$\sum\limits_{k=1}^{n} MA_{2k-1}^2 = \sum\limits_{k=1}^{n} A_{2k}^2$
2). What is the geometrical locus if M is in space?

Mihály Bencze

PP. 15928. If
$P\left(n,\Gamma\left(n\right)\right) = \binom{n}{0}\Gamma\left(n\right) + \frac{1}{2}\binom{n}{1}\Gamma^2\left(n\right) + \frac{1}{3}\binom{n}{2}\Gamma^3\left(n\right) + ... + \frac{1}{n+1}\binom{n}{n}\Gamma^{n+1}\left(n\right)$,
then $\left(n, P\left(n,\Gamma\left(n\right)\right)\right) = 1$ for all prime n.

Laurenţiu Modan

PP. 15929. Let consider the bidimensional random variable $z = (x, y)$, given by the following table:

$y \backslash x$	-1	2
0	α^2	$...$
1	$...$	$\frac{\alpha}{6}$
	$\frac{2}{3}$	$\frac{1}{3}$

, where $\alpha \in [0, 1]$.

1). Find α peritting the existence of z, as random bidimensional variable
2). Compute $P(x < 2|y > 0)$
3). Study the independence of the unidimensional random variables X and Y, using the covariance $cov(X, Y)$.

Laurenţiu Modan

PP. 15930. For a selection $x_1, x_2, ..., x_n$, of the volume n, we consider the first centred moment of selection $\overline{\mu_1} = \frac{1}{n} \sum\limits_{k=1}^{n} (x_k - \overline{x})$, where $\overline{x} = \frac{1}{n} \sum\limits_{k=1}^{n} x_k$ is the selection average. Find the correct answer from the following 5 possibilities:
1). $P\left(\overline{\mu_1} \leq \frac{1}{2}\right) = 0;$ 2). $P\left(\overline{\mu_1} \geq \frac{1}{4}\right) = \frac{1}{4};$
3). $P\left(\frac{1}{4} \leq \overline{\mu_1} \leq \frac{1}{2}\right) = 0;$ 4). $P\left(|\overline{\mu_1}| > \frac{1}{4}\right) = \frac{1}{2};$
5). $P\left(\frac{1}{4} < \overline{\mu_1} < \frac{1}{2}\right) = \frac{1}{4}.$

Laurenţiu Modan

PP. 15931. Let the series $\sum\limits_{n \geq 1} \sin \frac{x}{n}$, $x \in (0, 1)$ be. Explain the correctness of the following statement: "In acoordance to the limit criterion of coparison for positive series we find: $\lim\limits_{n \to \infty} \frac{\left|\sin \frac{x}{n}\right|}{\frac{1}{n}} = \lim\limits_{n \to \infty} \left|\frac{x}{n}\right| \cdot n = |x|$. Because $|x| \in (0, 1)$, it occurs that $\sum\limits_{n \geq 1} \sin \frac{x}{n}$ has the same behavior of the series $\sum\limits_{n \geq 1} \frac{1}{n}$, being a divergent series.

Laurenţiu Modan

PP. 15932. For $S(n) = \sum\limits_{k=1}^{n} \frac{\binom{2n-1}{k}}{\binom{2n+1}{2n-k}}$ find the limit $\lim\limits_{n \to \infty} \frac{S(n)}{n}$. (In link with PP. 13474).

Laurenţiu Modan

PP. 15933. If $a, b, c > 0$, then $\sum \frac{a}{a+mb+c} \geq \frac{3}{m+2}$.

Gheorghe Gherasim

PP. 15934. If $a > 1$, $0 < \alpha < \beta$, then
$\int\limits_{a^{\alpha}-1}^{a^{\beta}-1} \frac{1}{x+1} arctg \frac{x}{\sqrt{a^{\alpha+\beta}-1}} dx = \left(\frac{\beta-\alpha}{2} arctg \sqrt{a^{\alpha+\beta}-1} \right) \ln a.$

György Szöllősy

PP. 15935. Prove that

$\sum\limits_{1 \leq i < j \leq n} (-1)^{i+j-1} ij \left(i^2+j^2\right) \binom{n}{i} \binom{n}{j} = n^2 (n-1)^2 \binom{2n-4}{n-2} + n^3 \binom{2n-2}{n-1}$ for all $n \geq 2$.

György Szöllősy

PP. 15936. 1). Prove that exist $n \in N$, $1 \leq n \leq 89$ such that
$1 + \sec 20^\circ = \sqrt{3} tgn^\circ$
2). Prove that $ctg 12^\circ - 4 \sin 12^\circ = \sqrt{15}$
3). Determine $m \in N$ such that $3\sqrt{3} + \sqrt{15} + \sqrt{2\left(25+11\sqrt{5}\right)} = 2tgm^\circ$

György Szöllősy

PP. 15937. If $0 < a < b$ then $\int\limits_a^b \frac{x \ln x dx}{(x^2+a^2)(x^2+b^2)} = \frac{\ln ab \ln \frac{a^2+b^2}{2ab}}{2(b^2-a^2)}$.

György Szöllősy

PP. 15938. Let $ABCD$ be a convex quadrilateral in which $AB = AD$, $DAC\measuredangle = \alpha^\circ$, $BAC\measuredangle = 3\alpha^\circ$, $DCA\measuredangle = 30^\circ$ $(0 < \alpha < 45)$. Prove not trigonometrically that $DBC\measuredangle$ not depond from α.

György Szöllősy

PP. 15939. Determine all $\lambda > 0$ such that $\cos A + \lambda^3 \sin \frac{A}{2} \cos \frac{B-C}{2} \leq \lambda^2$ in all triangle ABC.

Mihály Bencze

PP. 15940. In all triangle ABC holds $\sum \sin \frac{A}{2} \cos \frac{B-C}{2} \leq \frac{\left(4+3\lambda^2\right)R-2r}{2\lambda R}$, for all $\lambda > 0$.

Mihály Bencze

PP. 15941. In all triangle ABC holds $54r^3(4R+r)^2(16R-5r) \leq 6s^6 \leq$
$\leq (4R+r)^2\left(2(2R+r)^2+R^2\right)\left((2R+r)^2+2r^2\right)$.

Mihály Bencze

PP. 15942. In all triangle ABC holds $\prod(4R-r_a) \geq 3\sqrt{3}sr^2$.

Mihály Bencze

PP. 15943. In all triangle ABC holds $\frac{9\sum w_a^2}{8R(R+r)} \leq \left(\left(\sum a\right)\left(\sum \frac{1}{a}\right)\right)^2 \leq \frac{3\sum m_a^2}{4r^2}$.

Mihály Bencze

PP. 15944. If $x \in R$ and $n \in N^*$, then
$n\left(1+x^2+x^4+...+x^{2n}\right) \geq (n+1)\left(x+x^3+...+x^{2n-1}\right)$.

György Szöllősy

PP. 15945. If $a_0 = 0$ and $a_{n+1}\left(1+\sqrt{1+a_n}\right) = 1$ for all $n \geq 1$, then prove that the sequence $(a_n)_{n\geq 1}$ is convergent and compute:

1). $\alpha = \lim\limits_{n\to\infty} a_n$ 2). $\lim\limits_{n\to\infty} n(a_n-\alpha)$

György Szöllősy

PP. 15946. If $x \geq 1$, then:

1). $\left(\frac{2x+1}{x+1}\right)^{\frac{1}{x}} \leq 1+\left(\frac{x}{x+1}\right)^x$
2). $\prod\limits_{k=1}^{n}\left(\left(\frac{2k+1}{k+1}\right)^{\frac{1}{k}}-1\right)^{\frac{1}{k}} \leq \frac{1}{n+1}$
3). Compute $\lim\limits_{n\to\infty}\prod\limits_{k=1}^{n}\left(\left(\frac{2k+1}{k+1}\right)^{\frac{1}{k}}-1\right)^{\frac{1}{k}}$

György Szöllősy and Mihály Bencze

PP. 15947. Solve the following equations:
1). $x^5-x^4-1=0$
2). $x^5-x^4+2x^3-2x^2+2x-1=0$.

György Szöllősy

PP. 15948. If $x_n = \prod\limits_{k=1}^{n} \frac{an+kb-c}{an+kb-c+1}$, where $a, b, c > 0$, $b \geq c$, then $\lim\limits_{n\to\infty} x_n = \left(\frac{a}{a+b}\right)^{\frac{1}{b}}$.

György Szöllősy

PP. 15949. Compute

1). $\int\limits_0^a \arcsin(\cos x)\,dx$; 2). $\int\limits_0^a \arccos(\sin x)\,dx$;

3). $\int\limits_0^a arctg\,(ctgx)\,dx$; 4). $\int\limits_0^a arcctg\,(tgx)\,dx$, where $a \in \left[0, \frac{\pi}{2}\right]$.

György Szöllősy and Mihály Bencze

PP. 15950. The triangle ABC have the sides in geometrical progression if and only if $abc\sum a^5 b = \sum a^5 b^4$ or if and only if $abc\sum a^3 = \sum a^3 b^3$.

György Szöllősy and Mihály Bencze

PP. 15951. Solve the system: $\left\{\begin{array}{l} 4\left(36x^2+y^2\right) = 9\,(x+y) \\ 2\,(6x-y) = 3\left(x^2+y^2\right) \end{array}\right.$.

György Szöllősy

PP. 15952. Prove that $\int\limits_0^{\frac{\pi}{4}} \frac{x}{\sqrt{2}+\sin 2x+\cos 2x}dx = \frac{\left(2-\sqrt{2}\right)\pi}{16}$.

György Szöllősy

PP. 15953. Solve in $(0, +\infty)$ the following system: $\left\{\begin{array}{l} e^{x_1^a} = x_2^{ae} \\ e^{x_2^a} = x_3^{ae} \\ - - -- \\ e^{x_n^a} = x_1^{ae} \end{array}\right.$, where $a > 0$.

György Szöllősy and Mihály Bencze

PP. 15954. Prove that $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \frac{2n+1-2k}{\sqrt{(2n+1)^4+k}+\sqrt{(2n+1)^4+2n+1-k}} \sin\frac{2k\pi}{2n+1} = \frac{1}{4\pi}$.

György Szöllősy

PP. 15955. Prove that $\int\limits_{1}^{e^4} \sqrt{\ln x}dx < 2e^4 - \frac{118}{15}$.

Mihály Bencze and György Szöllősy

PP. 15956. If $a \in (0,1) \cup (1,+\infty)$, then
$2e\left(a^{\frac{\pi}{2}} - 1\right) < \pi\left(a^e - 1\right) < e\left(a^{\pi} - 1\right).$

György Szöllősy and Mihály Bencze

PP. 15957. In all triangle ABC holds:
1). $24r \leq \sum \frac{b^2+c^2}{m_a} \leq 12R$ 2). $\frac{1}{R} \leq \sum \frac{\sin\frac{A}{2}}{w_a} \leq \frac{1}{2r}$

György Szöllősy and Mihály Bencze

PP. 15958. In all triangle ABC holds
$\frac{\sin B}{\sin\left(C+\frac{A}{3}\right)} + \frac{\sin C}{\sin\left(B+\frac{A}{3}\right)} > \frac{2}{3}\left(1 + 2\cos\frac{2A}{3}\right).$

György Szöllősy

PP. 15959. If $a_k > 0$ $(k = 1, 2, ..., n+1)$, $\prod\limits_{k=1}^{n+1} a_k = 1$, $x \geq 1$, then
$\sum \frac{a_2+a_3+...+a_{n+1}}{a_1+a_2^x+...+a_{n+1}^x} \leq n.$

György Szöllősy

PP. 15960. In all triangle ABC holds $\sum \frac{b^2+c^2-a^2}{(b+c)m_a} \leq \sqrt{3}$.

György Szöllősy

PP. 15961. Prove that:
1). $\sum\limits_{k=1}^{\infty} \frac{k^2-k+1}{k^3} > \frac{16\sqrt{3}}{9}\left(\frac{\pi^2}{6} - 1\right)$
2). Determine all $x, y > 0$ such that
$\frac{x^2-x+1}{y^3} + \frac{y^2-y+1}{x^3} \geq \frac{16\sqrt{3}}{9}\left(\frac{1}{(x+1)^2} + \frac{1}{(y+1)^2}\right).$

Mihály Bencze and György Szőllösy

PP. 15962. If $n \in N$, $n \geq 2$, then in all triangle ABC holds:
1). $\sum \sin A\left(1 - \sin^n A\right) \leq \frac{3n}{\sqrt[n]{(n+1)^{n+1}}}$ 2). $\sum \frac{1}{1-\sin^n A} \geq \frac{3\sqrt[n]{(n+1)^{n+1}}s}{nR}$

György Szöllősy and Mihály Bencze

PP. 15963. In all triangle ABC holds
$\prod\left(tg\frac{A}{2}\right)^{\frac{1}{b}+\frac{1}{c}}\leq\left(\frac{2sR}{s^2+r^2+4Rr}\right)^{\frac{4sRr}{s^2+r^2+4Rr}}$.

Mihály Bencze and György Szöllősy

PP. 15964. Let ABC be a triangle. Determine all $n\in N^*$ for which
$\sum\cos\frac{A}{n}=\sqrt{n}\prod\left(\cos\frac{A}{2n}+\sin\frac{A}{2n}\right)$.

Mihály Bencze and György Szöllősy

PP. 15965. In all triangle ABC holds $\sum\frac{bc}{(b+c)m_a}\geq\frac{27\sqrt{3}Rr}{2(R+r)^2}\sqrt{\frac{r}{4R+r}}$.

György Szöllősy

PP. 15966. Prove that $\int\limits_1^8\frac{\ln(x+1)dx}{x^2+17}=\frac{\ln 18}{2\sqrt{17}}arctg\frac{7\sqrt{17}}{25}$.

György Szöllősy

PP. 15967. Prove that $\sum\limits_{k=0}^{2n-1}\sec^2\frac{\alpha+k\pi}{2n}=4n^2\cos ec^2\alpha$, where $\alpha\in(0,\pi)$.

György Szöllősy

PP. 15968. If $x_{n+1}=-x_n^2-\frac{1}{4}$ for all $n\geq 1$ and $x_0\in R$, then $(x_n)_{n\geq 1}$ is convergent, $\lim\limits_{n\to\infty}x_n=-\frac{1}{2}$ and compute $\lim\limits_{n\to\infty}n\left(x_n+\frac{1}{2}\right)$.

György Szöllősy

PP. 15969. In convex quadrilateral $ABCD$, where a,b,c and d are the lengths of the sides of quadrilateral and Δ is the area. Prove that
$\sum(3a+b+c+d)^n\geq 2^{\frac{n+8}{2}}\cdot 3^n\sqrt{\Delta}$ for all $n\geq 1$.

Nicuşor Minculete

PP. 15970. In a tetrahedron $ABCD$, we consider F_A,F_B,F_C,F_D the areas of its faces BCD,ACD,ABD,ABC, F is it total area, R is the circumradius and r is the inradius. Prove that:
1). $\sum\frac{1}{\sqrt{F_A}}\leq\frac{8R}{3r\sqrt{F}}$ 2). $16\leq\left(\sum\sqrt{F_A}\right)\left(\sum\frac{1}{\sqrt{F_A}}\right)\leq\frac{16R}{3r}$

Nicuşor Minculete

PP. 15971. If $x_k \in (0,1)$ and $p_k > 0$ $(k = 1, 2, ..., n)$, then

$$1). \prod_{k=1}^{n} \left(\frac{1+x_k}{1-x_k}\right)^{\frac{p_k}{x_k}} \geq \left(\frac{\sum\limits_{k=1}^{n} p_k(1+x_k)}{\sum\limits_{k=1}^{n} p_k(1-x_k)}\right)^{\frac{\left(\sum\limits_{k=1}^{n} p_k\right)^2}{\sum\limits_{k=1}^{n} p_k x_k}}$$

$$2). \prod_{1\leq i_1<...<i_p\leq n} \left(\frac{\lambda_{i_1}\left(1+x_{i_1}\right)+...+\lambda_{i_p}\left(1+x_{i_p}\right)}{\lambda_{i_1}(1-x_i)+...+\lambda_{i_p}\left(1-x_{i_p}\right)}\right)^{\frac{\left(\lambda_{i_1}+...+\lambda_{i_p}\right)^2}{\lambda_{i_1}x_{i_1}+...+\lambda_{i_p}x_{i_p}}} \leq$$

$$\leq \left(\prod_{i=1}^{n} \left(\frac{1+x_i}{1-x_i}\right)^{\binom{n-p}{p-1}\frac{\lambda_i}{x_i}} \left(\frac{\sum\limits_{i=1}^{n} \lambda_i(1+x_i)}{\sum\limits_{i=1}^{n} \lambda_i(1-x_i)}\right)^{\frac{\left(\sum\limits_{i=1}^{n} \lambda_i\right)^2}{\sum\limits_{i=1}^{n} \lambda_i x_i}}\right)^{\binom{n-2}{p-2}}.$$

Mihály Bencze and José Luis Diaz-Barrero

PP. 15972. If $a_i > 0$ $(i = 1, 2, ..., n)$ and $2 \leq k \leq n-1$, then
$\sum\limits_{cyclic} \frac{a_1^k+a_2a_3...a_{k+1}}{a_2+a_3+...+a_{k+2}} \geq \frac{2}{k}\sum\limits_{i=1}^{n} a_i^{k-1}$.

Mihály Bencze

PP. 15973. If $a_i > 0$ $(i = 1, 2, ..., n)$, then

1). $\sum\limits_{cyclic} \frac{a_1^{\alpha+\beta}}{a_1^\alpha+a_2^\alpha} \geq \frac{1}{2}\sum\limits_{i=1}^{n} a_i^\beta$, for all $\alpha \geq \beta \geq 1$

2). $\sum\limits_{cyclic} \frac{a_1^{\alpha+\beta}}{a_1^\alpha+a_2^\alpha+...+a_k^\alpha} \geq \frac{1}{k}\sum\limits_{i=1}^{n} a_i^\beta$, for all $\alpha \geq \beta \geq 1$ and $2 \leq k \leq n$.

Mihály Bencze

PP. 15974. If $x_k \in R$ $(k = 1, 2, ..., n)$ and $m \in N$, then

1). $\sum\limits_{cyclic} \left(x_1^2 - x_1x_2 + x_2\right)^m \leq 3^m \sum\limits_{k=1}^{n} x_k^{2m}$

2). $\prod\limits_{cyclic} \left(x_1^2 - x_1x_2 + x_2^2\right)^m \leq \left(\frac{3^m}{n}\right)^n \left(\sum\limits_{k=1}^{n} x_k^{2m}\right)^n$

Mihály Bencze

PP. 15975. If K, L, M denote the midpoints of sides AB, BC, CA in triangle ABC then for all points P in the plane of triangle we have:

$\frac{AB}{PK}+\frac{BC}{PL}+\frac{CA}{PM}\geq\frac{AB\cdot BC\cdot CA}{4PK\cdot PL\cdot PM}$.

Mihály Bencze

PP. 15976. If $x\in\left[0,\frac{\pi}{2}\right]$, then $\frac{1}{\sin x+\cos x}+\frac{1+\sin^2 x}{1+\sin x}+\frac{1+\cos^2 x}{1+\cos x}\leq\frac{6}{1+\sin x+\cos x}$.

Mihály Bencze

PP. 15977. If $x\in R$, then $\frac{2\sin^4 x+\cos^2 x}{1+2\cos^2 x}+\frac{2\cos^4 x+\sin^2 x}{1+2\sin^2 x}+\sin^2 x\cos^2 x\geq\frac{3}{4}$.

Mihály Bencze

PP. 15978. If $x\in R$, then $\frac{\sin^2 x}{\sqrt{1+\sin^2 x}}+\frac{\cos^2 x}{\sqrt{1+\cos^2 x}}+\frac{1}{\sqrt{\left(1+\sin^2 x\right)\left(1+\cos^2 x\right)}}\leq\frac{3}{2}$.

Mihály Bencze

PP. 15979. If $\lambda>0$ and $x\in\left[0,\frac{\pi}{2}\right]$, then

$\left(\lambda+\sqrt{\sin x}+\sqrt{\cos x}\right)\sqrt{\sin x\cos x}\leq\frac{\lambda^4+1}{\lambda}$.

Mihály Bencze

PP. 15980. If $\lambda>0$ and $x\in R$, then

$\lambda\left(1+\lambda\right)\sin^2 x\cos^2 x\leq\lambda^3\sin^2 x+\lambda\cos^6 x+\sin^6 x\cos^2 x$.

Mihály Bencze

PP. 15981. If $\lambda\in\left(0,\frac{3}{2}\right)$ and $x\in R$, then $\frac{\sin^2 x}{\lambda+\cos^2 x}+\frac{\cos^2 x}{\lambda+\sin^2 x}\geq\frac{3}{2}-\lambda$.

Mihály Bencze

PP. 15982. If $\lambda>0$ and $x\in R$, then

$\frac{1}{4}\left(\frac{1}{\sin^2 x\cos^2 x}+\frac{10\lambda+1}{\lambda(\lambda+1)}\right)\geq 1+\frac{2\lambda+1}{\left(\lambda+\sin^2 x\right)\left(\lambda+\cos^2 x\right)}$.

Mihály Bencze

PP. 15983. If $x, y, z \in R$, then:
1). $\left(1-\sqrt[3]{\sin^2 x \sin^2 y \sin^2 z}\right)^3+\left(1-\sqrt[3]{\cos^2 x \cos^2 y \cos^2 z}\right)^3 \geq$
$\geq \sin^2 x \sin^2 y \sin^2 z+\cos^2 x \cos^2 y \cos^2 z.$
2). $\left(\sqrt[3]{ch^2 x ch^2 y ch^2 z}-1\right)^3+\left(\sqrt[3]{\left(1+ch^2 x\right)\left(1+ch^2 y\right)\left(1+ch^2 z\right)}-1\right)^3 \geq$
$\geq sh^2 x sh^2 y sh^2 z+ch^2 x ch^2 y ch^2 z.$

Mihály Bencze

PP. 15984. If $x_k \in\left[0, \sqrt[n]{\lambda}\right]$ $(k=1,2,\ldots,n)$, where $\lambda>0$, then
$\left(\lambda-\prod_{k=1}^n x_k\right)^n \geq \prod_{k=1}^n\left(\lambda-x_k^n\right).$

Mihály Bencze

PP. 15985. In all triangle ABC holds:
1). $\prod\left(\sin \frac{A}{2}+\sin \frac{B}{2}\right)^2 \leq \frac{\left((2R-r)\left(s^2+r^2-8Rr\right)-2Rr^2\right)^3}{1024 R^5 r^4}$
2). $\prod\left(\cos \frac{A}{2}+\cos \frac{B}{2}\right)^2 \leq \frac{\left((4R+r)^3+s^2(2R+r)\right)^3}{1024 R^5 r^4}$

Mihály Bencze

PP. 15986. If $x_k>0$ $(k=1,2,\ldots,n)$, then $\prod_{cyclic} \frac{\left(x_1+x_2\right)^3}{\left(\sqrt{x_1}+\sqrt{x_2}\right)^2} \geq 2^n \prod_{k=1}^n x_k^2.$

Mihály Bencze

PP. 15987. If F_k denote the k-th Fibonacci number, then
$\frac{F_1}{F_2}+\frac{F_3}{F_4}+\frac{F_5}{F_6}+\ldots+\frac{F_{2n-1}}{F_{2n}} \geq 1.$

Mihály Bencze

PP. 15988. Let n be a positive integer. Prove that $\sum_{k=1}^n \sin \frac{k}{n}>\frac{(n+1)(11n-1)}{24n}.$

José Luis Díaz-Barrero

PP. 15989. Let n be a nonnegative integer. Prove that
$F_{n+2}+\sum_{k=0}^n L_k 3^{F_{2k}}>1+\sum_{k=0}^n F_{2k} 3^{L_k}$, where F_n and L_n represents the $n-$th Fibonacci and Lucas number respectively.

José Luis Díaz-Barrero

PP. 15990. Let a, b, c be positive real numbers. Prove that

$\frac{(b+c)^2}{bc} + \frac{(c+a)^2}{ca} + \frac{(a+b)^2}{ab} \geq 12$

José Luis Díaz-Barrero

PP. 15991. Let r be the inradius of a triangle ABC with sides a, b, c and semi-perimeter s. Prove that $\frac{(s-a)^4}{c(s-b)} + \frac{(s-b)^4}{a(s-c)} + \frac{(s-c)^4}{b(s-a)} \geq \frac{3r}{4}\sqrt[3]{rs^2}$.

José Luis Díaz-Barrero

PP. 15992. Compute $\lim\limits_{n\to\infty} \sum\limits_{1\leq i<j\leq n} \dfrac{1}{2009^{i+j}}$.

José Luis Díaz-Barrero

PP. 15993. Compute the following sum $\sum\limits_{n\geq 0} \dfrac{(n!)^2}{(2n+1)!}$.

José Luis Díaz-Barrero

PP. 15994. Find all triplets (x, y, z) of real numbers such that
$36x^2 - 24y + 1 = 0$
$16y^2 - 20z + 9 = 0$
$4z^2 - 12x + 25 = 0$

José Luis Díaz-Barrero

PP. 15995. Assume that any element x in a finite ring R is idempotent, then prove that R has a multiplicative identity.

José Luis Díaz-Barrero

PP. 15996. If a, b, c are distinct real numbers, then
$\left(\sum \left|\frac{a}{b-c}\right|\right)^2 \leq 6 + 3\left(\sum \frac{a}{b-c}\right)^2$.

Mihály Bencze

PP. 15997. If $a, b, c > 0$ then $\sum \frac{a}{\sqrt[4]{a^4+a^2(b+c)^2+(b+c)^4}} \geq 1$.

Mihály Bencze

PP. 15998. If $x_k > 0$ $(k = 1, 2, ..., n)$ and $\sum\limits_{cyclic} x_1 x_2 = n$, then
$\sum\limits_{k=1}^{n} \frac{1}{(1+x_k)^2} \geq \frac{n}{4}$.

Mihály Bencze

PP. 15999. If $a_k > 0$ $(k = 1, 2, ..., n)$, then
$\prod\limits_{cyclic} \left(a_1^2 + a_1 a_2 + a_2^2\right)^2 \geq \left(\frac{3}{4}\sqrt[n]{\prod\limits_{cyclic} (a_1 + a_2)^2} + \frac{1}{4}\sqrt[n]{\prod\limits_{cyclic} (a_1 - a_2)^2}\right)^n$.

Mihály Bencze

PP. 16000. If $x_k > 0$ $(k = 1, 2, ..., n)$ and

$\sum\limits_{cyclic} \frac{\left(x_1^2 - x_1 x_2 + x_2^2\right)\left(x_2^2 - x_2 x_3 + x_3^2\right)}{x_2^2} \geq \sum\limits_{k=1}^{n} x_k^2$.

Mihály Bencze

PP. 16001. If $a, b, c > 0$ and $k \in N^*$ $(k \geq 3)$ such that $a^k + b^k + c^k = 3$, then $a^{k+1}b^{k+1} + b^{k+1}c^{k+1} + c^{k+1}a^{k+1} \leq \frac{9(k+1)}{4k} + \frac{3(k-3)}{4k} a^k b^k c^k$.

Mihály Bencze

PP. 16002. If $x, y, z \geq \frac{3}{4}$, then
$3xyz \geq 1 + \frac{1}{8}\prod\left(1 + \sqrt{4x-3}\right) + \frac{1}{8}\prod\left(2x - 1 + \sqrt{4x-3}\right)$.

Mihály Bencze

PP. 16003. If $a_k > 0$ $(k = 1, 2, ..., n)$, then
$\prod\left(1 - a_k + a_k^2\right) \geq \left(\frac{1 + \prod\limits_{k=1}^{n} a_k^{\frac{4}{n}}}{2}\right)^{\frac{n}{2}}$.

Mihály Bencze

PP. 16004. In all triangle ABC holds

$\sum \left(1 - \sqrt{\sqrt{3} tg\frac{A}{2}} + \sqrt{3} tg\frac{A}{2}\right)\left(1 - \sqrt{\sqrt{3} tg\frac{B}{2}} + \sqrt{3} tg\frac{B}{2}\right) \geq 3$.

Mihály Bencze

PP. 16005. If $a_k > 0$ $(k = 1, 2, ..., n)$ and $\sum\limits_{cyclic} a_1^2 a_2^2 = n$, then

$$\sum_{cyclic} \left(1 - a_1 + a_1^2\right)\left(1 - a_2 + a_2^2\right) \geq n + \frac{1}{4} \sum_{cyclic} \left(|a_1 - a_2| + |(1 - a_1)(1 - a_2)|\right)^2 .$$

Mihály Bencze

PP. 16006. If $x_k > 0$ $(k = 1, 2, ..., n)$, then

$$\sum_{cyclic} \frac{\left(x_1^2 + x_2^2\right)\left(x_2^2 + x_3^2\right)}{x_2 (x_1 + x_2)(x_2 + x_3)} \geq \frac{1}{4\sqrt[3]{4}} \sum_{cyclic} \frac{\sqrt[3]{\left(x_1^3 + x_2^3\right)\left(x_2^3 + x_3^3\right)}}{x_2} \geq \frac{1}{n} \left(\sum_{k=1}^{n} \sqrt{x_k}\right)^2 .$$

Mihály Bencze

PP. 16007. If $a_k > 0$ $(k = 1, 2, ..., n)$, then

1). $\prod\limits_{k=1}^{n} \left(1 - a_k + a_k^2\right) \geq \frac{1}{3} \left(1 + \prod\limits_{k=1}^{n} a_k + \prod\limits_{k=1}^{n} a_k^2\right) \geq \prod\limits_{k=1}^{n} a_k$

2). $\prod\limits_{k=1}^{n} \left(1 - a_k + a_k^2\right) \geq \frac{1}{2} \left(1 + \prod\limits_{k=1}^{n} a_k^2\right) \geq \prod\limits_{k=1}^{n} a_k$

Mihály Bencze

PP. 16008. If $a_k > 0$ $(k = 1, 2, ..., n)$ and $\sum\limits_{k=1}^{n} a_k^3 = n$, then $\sum\limits_{k=1}^{n} \left(\frac{a_k^2 + 1}{a_k + 1}\right)^3 \geq n$.

Mihály Bencze

PP. 16009. If $a_k > 0$ $(k = 1, 2, ..., n)$ and $\sum\limits_{k=1}^{n} a_k = n$, then $\sum\limits_{k=1}^{n} \frac{\left(a_k^2 + 1\right)^3}{a_k^3 + 1} \geq 4n$.

Mihály Bencze

PP. 16010. If $x_k > 0$ $(k = 1, 2, ..., n)$, then
$\sum\limits_{cyclic} \frac{x_1^2 + x_2^2}{x_1 + x_2} \geq \sum\limits_{cyclic} \sqrt[3]{\frac{x_1^3 + x_2^3}{2}} \geq \sum\limits_{k=1}^{n} x_k$.

Mihály Bencze

PP. 16011. If $x, y, z > 0$, then $\sum \sqrt{\frac{x + y}{z}} + 3\sqrt{2} \geq 2 \sum \sqrt{\frac{x + y + 2z}{x + y}} \geq 6\sqrt{2}$.

Mihály Bencze

PP. 16012. If a, b, c, then
1). $\min\left\{\sum \frac{1}{3a+b}; \sum \frac{1}{3b+a}\right\} \geq \sum \frac{1}{2a+b+c}$ 2). $\frac{a+b}{(3a+b)(3b+a)} \geq \frac{1}{4} \sum \frac{1}{2a+b+c}$

Mihály Bencze

PP. 16013. Let ABC be a triangle in which
$2(4R+r) = s\left(1+\sqrt{1+4s^2}\right)$. Prove that $\sum tg^2 \frac{A}{2} tg^2 \frac{B}{2} \leq 1$.

Mihály Bencze and Shanhe Wu

PP. 16014. If $a, b, c > 0$, then $\sum \frac{1}{a+1} + \max\left\{\sum \frac{1}{a^2+b}; \sum \frac{1}{b^2+a}\right\} \leq \sum \frac{1}{a}$.

Mihály Bencze

PP. 16015. If $x, y, z > 0$, then $\frac{3}{2} + \sum \frac{x}{y+z} \geq 2 \sum \frac{x+y}{x+y+2z} \geq 3$ (A refinement of Nesbitt‘s inequality).

Mihály Bencze

PP. 16016. If $a, b, c > 0$, then
$\min\left\{\prod\left(1+a^3b\right); \prod\left(1+ab^3\right)\right\} \geq \prod\left(1+a^2bc\right)$.

Mihály Bencze

PP. 16017. If $a, b, c > 0$, then
1). $\max\left\{\sum \frac{ab^2}{a+b}; \sum \frac{a^2b}{a+b}\right\} \leq \frac{1}{2} \sum a^2$ 2). $\min\left\{\sum \frac{ab^2}{a+b}; \sum \frac{a^2b}{a+b}\right\} \geq \frac{9abc}{2\sum a}$
3). $\sum ab \leq \sum a^2$ 4). $\left(\sum a\right)\left(\sum ab\right) \geq 9abc$
5). Prove 1) $\Rightarrow$ 3) 6). Prove 2) $\Rightarrow$ 4)
7). It‘s possible 3) $\Rightarrow$ 1) ? 8). It‘s possible 4) $\Rightarrow$ 2) ?

Mihály Bencze

PP. 16018. If $a, b, c, x, y > 0$, then
$\frac{ya^2+xb^2-(x+y)c^2}{a^2+b} + \frac{-(x+y)a^2+yb^2+xc^2}{b+c} + \frac{xa^2-(x+y)b^2+yc^2}{c+a} \leq 0$.

Mihály Bencze

PP. 16019. If $a, b, c > 0$, then
1). $\sum a^3b^3\left(a^3+b^3\right) \geq a^2b^2c^2 \sum ab(a+b)$
2). $\left(\sum a^3b^6\right)\left(\sum a^6b^3\right) \geq a^4b^4c^4\left(\sum ab^2\right)\left(\sum a^2b\right)$

Mihály Bencze

PP. 16020. If $a, b, c > 0$, then
1). $\sum \frac{a}{b+c} \geq \max \left\{ \sum \frac{a}{a+b}, \sum \frac{b}{a+b} \right\}$
2). $\sum \frac{a}{b+c} \geq \frac{3}{2}$
3). Prove that 1) $\Rightarrow$ 2)
4). Is true the implication 2) $\Rightarrow$ 1) ?

Mihály Bencze

PP. 16021. If $a_k > 0$ $(k = 1, 2, ..., n)$, then

$$\left(a_1 + \sqrt{a_1 a_2} + a_2\right)\left(a_2 + 2\sqrt{a_2 a_3} + a_3\right) ... \left(a_n + n\sqrt{a_n a_1} + a_1\right) \geq \frac{(n+2)!}{2} \prod_{k=1}^{n} a_k.$$

Mihály Bencze

PP. 16022. If $0 \leq a_k \leq 1$ $(k = 1, 2, ..., n)$, then

$$\sum_{k=1}^{n} a_k + \sum_{cyclic} \frac{1}{1+a_1+a_2} \leq n + \frac{1}{3} \sum_{cyclic} a_1 a_2.$$

Mihály Bencze

PP. 16023. In all acute triangle ABC holds $\sum \sqrt{1 - tg^2 \frac{A}{2} tg^2 \frac{B}{2}} \leq 2\sqrt{2}$.

Mihály Bencze and Yu-Dong Wu

PP. 16024. If $x_i \in [-1, 1]$ $(i = 1, 2, ..., n)$, $k \in \{2, 3, ..., n\}$, then
1). $\sqrt{1 - x_1^2} + \sqrt{1 - x_2^2} + 2\sqrt{1 - x_3^2} + 4\sqrt{1 - x_4^2} + ... + 2^{n-2}\sqrt{1 - x_n^2} \leq$
$\leq \sqrt{4^{n-1} - \left(x_1 + x_2 + 2x_3 + 4x_4 + ... + 2^{n-2}x_n\right)^2}$
2). $\sum_{i=1}^{n} \sqrt{1 - x_i^2} \leq \sum_{cyclic} \sqrt{1 - \left(\frac{x_1+x_2+2x_3+4x_4+...+2^{k-2}x_k}{2}\right)^2}$.

Mihály Bencze

PP. 16025. If $x_k \in [-1, 1]$, $p_k > 0$ $(k = 1, 2, ..., n)$, then

$$\sum_{k=1}^{n} p_k \sqrt{1 - x_k^2} \leq \sqrt{\left(\sum_{k=1}^{n} p_k\right)^2 - \sum_{k=1}^{n} \left(p_k x_k\right)^2}.$$

Mihály Bencze

PP. 16026. Determine all numbers $m \in N$ which can be expressed
$m = \frac{1}{a_1} + \frac{2^2}{a_1+a_2} + \frac{3^3}{a_1+a_2+a_3} + ... + \frac{n^n}{a_1+a_2+...+a_n}$, where $a_i \in N^*$ $(i = 1, 2, ..., n)$.

Mihály Bencze

PP. 16027. If $S_n = 1 + \frac{1}{2} + ... + \frac{1}{n}$, then $\prod\limits_{k=2}^{n} \frac{S_k}{S_{k+1}-1} \geq \frac{n+1}{2}$.

Mihály Bencze

PP. 16028. If $a, b, c > 0$, then $\left(\sum \frac{a}{b}\right)^2 + \left(\sum \frac{b}{a}\right)^2 \geq 3 \sum \frac{a+b}{c}$.

Mihály Bencze and Zhao Changjian

PP. 16029. If $a, b, c > 0$, then $\left(\sum a\right)\left(\sum \frac{b^2+c^2}{a}\right) \geq 3 \sum a^2$.

Mihály Bencze

PP. 16030. In all triangle ABC holds $\prod \left(stg\frac{A}{2} - 2r\right) \leq 1$.

Mihály Bencze

PP. 16031. In all triangle ABC holds $\sum \sqrt{tg\frac{A}{2}} \geq 2 \sum \sqrt{ctg\frac{A}{2}} - \frac{5}{2}\sqrt{\frac{s}{r}}$.

Mihály Bencze

PP. 16032. In all triangle ABC holds $\sum \frac{\sin\frac{A-C}{2} \sin^2\frac{B}{2}}{\cos\frac{B}{2}\left(2\sin^2\frac{A}{2}\sin^2\frac{B}{2} + \cos^2\frac{A}{2}\cos^2\frac{B}{2}\right)} \leq 1$

Mihály Bencze

PP. 16033. In all triangle ABC holds

1). $\sum m_a^2 tgA \geq 9sr$
2). $\prod(-a+b+c) \leq \frac{54R^2r^2}{s}$

Mihály Bencze

PP. 16034. The triangle ABC is equilateral if and only if
$s^3 \geq s\left((4R+r)^2 - s^2\right) \geq \sqrt{3}\left((4R+r)^3 - 12s^2R\right)$.

Mihály Bencze

PP. 16035. If $x, y, z > 0$, then $\sum \left(\frac{x}{y} - \frac{y}{z} + \frac{z}{x}\right)\left(\frac{x}{y} + \frac{y}{z} - \frac{z}{x}\right) \leq 3$.

Mihály Bencze

PP. 16036. If $x, y > 0$ then
$x^{2n} - x^{2n-1}y + x^{2n-2}y^2 - x^{2n-3}y^3 + ... - xy^{2n-1} + y^{2n} \geq x^n y^n$.

Mihály Bencze

PP. 16037. If $x_k > 0$ $(k = 1, 2, ..., n)$, $n \geq 3$, $m \in N$ then
$3 \sqrt[2m+1]{(2m)^{2m}} \sum\limits_{k=1}^{n} x_k^{2m+1} \geq (2m+1) \sum\limits_{cyclic} x_1 \sqrt[2m+1]{x_2^{2m^2} x_3^{2m^2} (x_2 + x_3)^{2m}}$.

Mihály Bencze

PP. 16038. If $a_k > 0$ $(k = 1, 2, ..., n)$, $n \geq 3$, then
$\sqrt[3]{4} \sum\limits_{k=1}^{n} a_k^3 \geq \sum\limits_{cyclic} a_1 (a_2 a_3 (a_2 + a_3))^{\frac{2}{3}}$.

Mihály Bencze

PP. 16039. If $a_k > 0$ $(k = 1, 2, ..., n)$, $n \geq 3$, then
$\sqrt[5]{256} \sum\limits_{k=1}^{n} a_k^5 \geq \frac{5}{3} \sum\limits_{cyclic} a_1 \sqrt[5]{a_2^8 a_3^8 (a_2 + a_3)^4}$.

Mihály Bencze

PP. 16040. Let p, q be two prime. Prove that exist $n_0 \in N$ such that for all $n \in N$, $n \geq n_0$ exist $a, b \in Z$ such that $n = \left[a\sqrt{p}\right] + \left[b\sqrt{q}\right]$, where $[\cdot]$ denote the integer part.

Mihály Bencze

PP. 16041. If $a, b, c > 0$, then
1). $\sum a^4 + \frac{7}{3} \left(\sum ab\right)^2 + abc \sum a \geq \left(\sum a\right)^2 \left(\sum ab\right)$
2). $2 \sum a^4 + \left(3\sqrt{3} - 2\right) abc \sum a + 2\sqrt{3} \left(\sum ab\right)^2 \geq \sqrt{3} \left(\sum ab\right) \left(\sum a\right)^2$

Mihály Bencze

PP. 16042. In all acute triangle ABC holds

1). $\sum \sin(\sin A) \leq \frac{s}{R} \leq \sum \cos(\cos A)$
2). $\sum \sin(\cos A) \leq 1 + \frac{r}{R} \leq \sum \cos(\sin A)$
3). $\sum \sin(\sin A) \sin(\cos A) \leq \frac{sr}{R^2} \leq \sum \cos(\sin A) \cos(\cos A)$

Mihály Bencze

PP. 16043. In all triangle ABC holds
$\left(s^2+r^2+4Rr\right)\left(s^2+r^2+2Rr\right) \geq 12Rr\left(s^2-r^2-Rr\right)$.

Mihály Bencze

PP. 16044. In all triangle ABC holds $\sum \frac{tg^2\frac{A}{2}\left(tg^2\frac{B}{2}+tg^2\frac{C}{2}\right)}{r+stg^3\frac{A}{2}} \geq \frac{1}{s}$.

Mihály Bencze

PP. 16045. In all triangle ABC holds
1). $1+\sum \frac{\sin^4\frac{A}{2}}{\sin^2\frac{B}{2}} \geq \frac{r}{2R}+\frac{3\left(8R^2+r^2-s^2\right)}{2R(2R-r)}$
2). $2+\frac{r}{2R}+\sum \frac{\cos^4\frac{A}{2}}{\cos^2\frac{B}{2}} \geq \frac{3\left((4R+r)^2-s^2\right)}{2R(4R+r)}$

Mihály Bencze

PP. 16046. In all triangle ABC holds $\sum \sqrt[4]{tg\frac{A}{2}ctg^2\frac{C}{2}} \geq \sqrt[4]{\frac{27s}{r}}$.

Mihály Bencze

PP. 16047. In all triangle ABC holds $\sum \frac{\sqrt{tg\frac{A}{2}}}{s+rctg\frac{A}{2}} \leq \frac{3}{4\sqrt{sr}}$.

Mihály Bencze

PP. 16048. If $x, y, z > 0$, then $\sum \frac{x^2+z^2}{y^2}+12 \geq 3\sum \frac{x^2+y^2}{xy}$.

Mihály Bencze

PP. 16049. In all triangle ABC holds $\sum \frac{1}{5\sqrt{s}-6\sqrt{rtg\frac{A}{2}}} \leq 1$.

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PP. 16050. In all triangle ABC holds $\sum \frac{tg\frac{A}{2}}{s^2+4r^2ctg^2\frac{A}{2}} \leq \frac{1}{4sr}$.

Mihály Bencze

PP. 16051. In all triangle ABC holds
$\sum \frac{ctg^3\frac{A}{2}}{\sin^2 A} \geq \frac{4sR(R+3r)}{r^3}-\frac{3s}{r}\left(\frac{s^2+r^2+Rr}{2sr}\right)^2$.

Mihály Bencze

PP. 16052. In all triangle ABC holds $\sum\sqrt{\left(tg^3\frac{A}{2}+\frac{3r}{s}\right)tg\frac{A}{2}}\geq 2$.

Mihály Bencze

PP. 16053. In all triangle ABC holds $\sum\sqrt{tg\frac{A}{2}+\frac{r}{s}ctg^2\frac{A}{2}}\leq\frac{3}{2}\sqrt{\frac{s}{r}}$.

Mihály Bencze

PP. 16054. In all triangle ABC holds
1). $\sum\sqrt{tg\frac{A}{2}}\leq\frac{1}{13}\left(7\sqrt{\frac{s}{r}}+\frac{18\sqrt{3}r}{s}\right)$
2). $\sum\sqrt{tg\frac{A}{2}}\leq\frac{6\sqrt{s}-3\sqrt[3]{s\sqrt{3}}}{3\sqrt{r}}$

Mihály Bencze

PP. 16055. In all triangle ABC holds
$\left(\sum\sqrt{tg\frac{A}{2}ctg\frac{C}{2}}\right)\left(\sum\sqrt{tg\frac{A}{2}}\right)\geq 3\sqrt{\frac{3s}{r}}$.

Mihály Bencze

PP. 16056. In all triangle ABC holds $\sum\frac{\sin^2\frac{A}{2}\cos^3\frac{A}{2}}{s\sin^3\frac{A}{2}+2r\cos^3\frac{A}{2}}\geq\frac{(4R+r)^2-s^2}{4R\left((4R+r)^2-2s^2\right)}$.

Mihály Bencze

PP. 16057. In all triangle ABC holds $\sum\frac{\sin^2\frac{A}{2}}{tg^2\frac{A}{2}+2tg\frac{B}{2}tg\frac{C}{2}}\geq\frac{24R}{r}$.

Mihály Bencze

PP. 16058. Prove that $\sum\left(\sqrt{\frac{x}{y+3z}}+\sqrt{\frac{z}{x+3z}}\right)\sqrt{\frac{z}{y}}\geq 3$, for all $x,y,z>0$.

Mihály Bencze

PP. 16059. If $a,b,c>0$, then $\sum\sqrt{\frac{a}{xa+yb}}\leq\frac{3}{\sqrt{x+y}}$, for all $x,y>0$.

Mihály Bencze

PP. 16060. In all triangle ABC holds $9+\sum tg\frac{A}{2}ctg\frac{C}{2}\geq\frac{2\left(s^2+r^2+4Rr\right)}{sr}$.

Mihály Bencze and Shanhe Wu

PP. 16061. If $a, b, c > 0$, then $\left(\sum \sqrt{a}\right)^2 \left(\sum a\right)^2 \geq 8\left(2\sum a^3 + 3abc\right)$.

Mihály Bencze and Zhao Changjian

PP. 16062. In all triangle ABC holds $\sum \frac{5s-27rtg\frac{A}{2}}{s+3rctg\frac{A}{2}} \geq 1$.

Mihály Bencze

PP. 16063. If $x, y, z > 0$, then $\sum \left(\sqrt{x} + \sqrt{z}\right) \sqrt{\frac{x^2+yz}{xz(x+y)}} \geq 6$.

Mihály Bencze

PP. 16064. If $a, b, c > 0$, then $2\sum \frac{a^2}{b+c} \geq \sum a + \frac{\left(\sum a\right)\left(\sum |a-b|\sqrt{a+b}\right)^2}{3\prod(a+b)}$.

Mihály Bencze

PP. 16065. If $a, b, c > 0$, then
$\prod (a-b)^2 + 2\sum a^3b^3 + 2abc\sum a^3 + 9a^2b^2s^2 \leq \left(\sum a^2\right)\left(\sum a^4\right)$.

Mihály Bencze

PP. 16066. If $a, b, c > 0$ and $a+b+c=1$, then determine all $x, y > 0$ such that $\sum \left(xa^2 + ybc\right)(1-a)^2 \leq 4$.

Mihály Bencze

PP. 16067. If $x, y, z > 0$, then
$\sum x^4 + 17\left(\sum xy\right)^2 \geq 3\left(\sum x\right)^2 \left(\sum xy\right) + 25xyz\sum x$.

Mihály Bencze

PP. 16068. If $a, b, c > 0$, then

$$\sum \left(\frac{1}{(b+c)^2} + \frac{1}{(c+a)^2}\right) a^2 + 4\sum \left(\frac{b}{b+c}\right)^2 + 2\sum \frac{a(a-b)}{(b+c)(c+a)} \geq 4\sum \frac{ab}{(b+c)^2}.$$

Mihály Bencze

PP. 16069. If $a, b, c > 0$, then
$5\sum a^4 + 2\sum a^3b \geq 3\sum a^2b^2 + 4\max\left\{\sum ab^3; abc\sum a\right\}$.

Mihály Bencze

PP. 16070. If $a, b, c, x, y \in R$, then $\left(2x^2+2xy+y^2\right)\sum a^4+2xyabc\sum a \geq$ $\left(2x^2+2xy-y^2\right)\sum a^2b^2+2y\left(x+y\right)\sum a^3c.$

Mihály Bencze

PP. 16071. If $a, b, c \in R$, then
$\sum a^6+6\sum a^4b^2+\sum a^2b^4+2\sum a^5b+2\sum a^3b^3 \geq 12a^2b^2c^2+4abc\sum ab\left(a+b\right).$

Mihály Bencze

PP. 16072. If $\alpha \in [1,3]$, then in all triangle ABC holds
$\sum\left(\left(tg\frac{A}{2}\right)^{4-\alpha}+\left(tg\frac{B}{2}\right)^{4-\alpha}\right)\left(tg\frac{C}{2}\right)^{4-2\alpha} \leq \frac{2}{3}\left(\frac{s}{r}\right)^{\alpha}.$

Mihály Bencze and Yu-Dong Wu

PP. 16073. If $a, b, c > 0$, then $\sum\frac{9a^2+14ab+9b^2}{(a+b)(3a+b)(a+3b)} \leq \frac{1}{2}\sum\frac{1}{a}.$

Mihály Bencze

PP. 16074. If $a, b, c > 0$, then
$3\left(\sum a\right)\left(\sum ab\right)+4\left(\sum ab\right)^2 \geq 9abc+2\left(\sum a\right)^2\left(\sum ab\right).$

Mihály Bencze

PP. 16075. In all triangle ABC holds

1). $2s^2-5r^2-20Rr \geq \frac{1}{36}\left(\sum\left|a+b-c\right|+2\sum\left|a-b\right|\right)^2$
2). $7s^2-20r^2-8Rr \geq \frac{1}{9}\left(\frac{1}{2}\sum\left|3a-b+3c\right|+2\sum\left|a-b\right|\right)^2$
3). $7\left(4R+r\right)^2-20s^2 \geq \frac{1}{9}\left(\sum\left|r_a+r_b-r_c\right|+2\sum\left|r_a-r_b\right|\right)^2$
4). $56R^2+24Rr+4r^2-10s^2 \geq$
$\geq \frac{8R^2}{9}\left(\sum\left|\sin^2\frac{A}{2}+\sin^2\frac{B}{2}-\sin^2\frac{C}{2}\right|+2\sum\left|\sin^2\frac{A}{2}-\sin^2\frac{B}{2}\right|\right)^2$
5). $2\left(4R+r\right)^2-7s^2 \geq$
$\geq \frac{4R^2}{9}\left(\sum\left|\cos^2\frac{A}{2}+\cos^2\frac{B}{2}-\cos^2\frac{C}{2}\right|+2\sum\left|\cos^2\frac{A}{2}-\cos^2\frac{B}{2}\right|\right)^2$

Mihály Bencze

PP. 16076. If $a, b, c > 0$, then
$\prod\left(a+b\right)^2 \geq 4\prod\left(b^2+ca\right)+4abc\sum bc\left(b+c\right)+\frac{1}{9}\left(\left|\prod\left(a-b\right)\right|+8abc\right)^2.$

Mihály Bencze

PP. 16077. In all triangle ABC holds

$12\left(s^2-r^2-Rr\right)\left(s^2-r^2-4Rr\right) \geq \left(s^2+r^2+2Rr\right)\left(11s^2-15r^2+44Rr\right).$

Mihály Bencze

PP. 16078. If $a, b, c > 0$, then

$9\sum a^2 \geq 2\sum(a+b)(b+c)+\frac{1}{9}\left(\sum|a+b-c|+2\sum|a-b|\right)^2.$

Mihály Bencze

PP. 16079. If $a, b, c > 0$, then

1). $\sum \frac{a^2\left(b^2+2ca\right)}{(2a^2+b^2)(2a^2+c^2)} \leq 1$ 2). $\sum \frac{a^2\left(a^3+3bc(b+c)+2abc\right)}{(2a^2+b^2)(2a^2+c^2)} \leq a+b+c$

Mihály Bencze

PP. 16080. If $a, b, c > 0$, then

1). $\sum\left(\frac{a}{a+c}\right)^2 \geq \frac{1}{2}\sum\frac{a}{a+b}$ 2). $\sum\frac{a^2+b^2}{(a+b)^2} \geq \sum\frac{bc}{a^2+bc}$

Mihály Bencze

PP. 16081. In all triangle ABC holds $\sum\frac{\sin\frac{A}{2}}{\cos^3\frac{A}{2}} \geq \frac{2\left(13(4R+r)^2-30s^2\right)}{3s\left((4R+r)^2-2s^2\right)}$.

Mihály Bencze

PP. 16082. If $a, b, c > 0$, then $9\prod(a+b)\left(b^2+bc+c^2\right) \geq 8\left(\sum a\right)\left(\sum ab\right)^4.$

Mihály Bencze

PP. 16083. If $a, b, c, d, e \in R$ and $\sum(b-c)(c-d)=\frac{1}{8}$ then
$5\left(\sum a\right)^2 = 1+\sum(a+b-3c+d+e)^2+24\sum ac.$

Mihály Bencze

PP. 16084. If $x, y, z > 0$, then $\prod\left(x^2+yz\right)+7x^2y^2z^2 \geq \frac{5}{2}\sum xy(x+y).$

Mihály Bencze

PP. 16085. If $x > 0$, then
1). $(x-1)^4\left(x^2+x+1\right)+\frac{1}{4}(x+1)^6 \leq$
$\leq 2\left(x^2+1\right)^3 \leq (x+1)^3\left(x^3+1\right)+(x-1)^4(x+1)^2$
2). $12\left(x^2+x+1\right) \geq (2(x+1)+|x-1|)^2$

Mihály Bencze

PP. 16086. If $a_k > 0$ $(k=1,2,...,n)$, then
$$2\sqrt{3}\sum_{cyclic}\sqrt{a_1^2+a_1a_2+a_2^2} \geq 4\sum_{k=1}^{n} a_k + \sum_{cyclic}|a_1-a_2|.$$

Mihály Bencze

PP. 16087. In all triangle ABC holds
1). $(2R-r)\left(s^2+r^2-8Rr\right) \geq 18Rr^2$
2). $(4R+r)^3 \geq (30R+7r)s^2$

Mihály Bencze

PP. 16088. If $a, b, c > 0$, then
$\left(7\sum a^4+10\sum a^3b\right)\left(7\sum \frac{1}{a^4}+10\sum\frac{1}{a^3b}\right) \geq 2601.$

Mihály Bencze

PP. 16089. If $a, b, c > 0$, then $7\sum a^4+5\sum ab\left(a^2+b^2\right) \geq \frac{17}{27}\left(\sum a\right)^4.$

Mihály Bencze

PP. 16090. If $a, b, c > 0$, then $\left(\sum a^2\right)^2\left(\sum\frac{1}{a^2}\right)^2 \geq 9\left(\sum a^3c^2\right)\left(\sum\frac{1}{a^3c^2}\right) \geq 81.$

Mihály Bencze

PP. 16091. If $a, b, c > 0$, then
$2\left(\sum a^2\right)^2 \geq \left(3+\sqrt{3}\right)\sum a^3b+\left(3-\sqrt{3}\right)abc\sum a+2\left(\sum ab\right)^2.$

Mihály Bencze

PP. 16092. If $x, y, z > 0$, then
1). $\sum\frac{y^2}{x^2-xy+y^2} \leq 3$ 2). $\sum\frac{y^2}{x^2+xy+y^2} \geq 1$

Mihály Bencze

PP. 16093. Determine the closed form of the expression
$\sum\limits_{k=1}^{n}\left(\left\{x+\frac{pk}{q}\right\}+\left\{x+\frac{qk}{p}\right\}\right)$, where p,q are two given primes, and $\{\cdot\}$ denote the fractional part.

Mihály Bencze

PP. 16094. If $a_i>0$ $(i=1,2,...,n)$ and $k\in\{2,...,n\}$, then
$\sum\limits_{cyclic}\frac{\sqrt[k]{a_1^2a_3...a_k}}{k-1+a_1a_2...a_k}\le\frac{1}{k}\sum\limits_{cyclic}\sqrt[k]{\frac{a_1}{a_2}}$.

Mihály Bencze

PP. 16095. If $x,y,z>0$ and $\sum\frac{1}{x}=2$, then $\sum\frac{1}{x+y}\le1$.

Mihály Bencze

PP. 16096. Solve the following system:

$$\begin{cases}\arccos\frac{1-x_1^2}{1+x_2^2}=\arcsin\frac{x_2}{\sqrt{1+x_3^2}}+arctgx_4\\ -------------------\\ \arccos\frac{1-x_n^2}{1+x_n^2}=\arcsin\frac{x_1}{\sqrt{1+x_2^2}}+arctgx_3\end{cases}$$

Mihály Bencze

PP. 16097. If $x,y\in(0,1)$, then
$\arccos\frac{1-x^2}{1+x^2}+\arccos\frac{1-y^2}{1+y^2}=\arcsin\frac{x+y}{\sqrt{(1+x^2)(1+y^2)}}+arctg\frac{x+y}{1-xy}$.

Mihály Bencze

PP. 16098. If $a,b,c>0$, then $\sum\frac{\sqrt{ab}}{a+b}\le\frac{3}{8}+\frac{1}{8}\left(\sum a\right)\left(\sum\frac{1}{a}\right)$.

Mihály Bencze

PP. 16099. If $x,y,z>0$, then $\sum\frac{ch^2x}{shy+shz}\ge3$.

Mihály Bencze

PP. 16100. If $a,b,c>1$ or $a,b,c\in(0,1)$, then determine all $k,p\in N$ such that $\sum\log_{a^kb^p}a\le\frac{3}{k+p}\le\sum\log_{a^pb^k}a$.

Mihály Bencze

PP. 16101. 1). In all acute triangle ABC holds $\sum \frac{\cos A}{\sin^2 A} \geq \frac{R}{r}$
2). Compute $\min \sum \frac{\sin A}{\cos^2 A}$

Mihály Bencze

PP. 16102. If $a, b \geq 0$, then $a^2b^2 + 2\left(3\sqrt{3} - 5\right)(a+b) \geq 6\left(2 - \sqrt{3}\right)ab$.

Mihály Bencze

PP. 16103. If $x, y, z > 0$, then $\sum \frac{xy\left(2x^2+7y^2\right)}{(x^2+2y^2)(x^2+5y^2)} \leq \frac{3}{2}$.

Mihály Bencze

PP. 16104. Solve the following system: $\begin{cases} x_1^3 + 1 = 2\sqrt[3]{2x_2 - 1} \\ x_2^3 + 1 = 2\sqrt[3]{2x_3 - 1} \\ ---------- \\ x_n^3 + 1 = 2\sqrt[3]{2x_1 - 1} \end{cases}$.

Mihály Bencze

PP. 16105. Prove that:
1). $\int\limits_{2-\sqrt{3}}^{\frac{\sqrt{3}}{3}} \arccos \frac{1-x^2}{1+x^2} dx = \frac{\left(5-2\sqrt{3}\right)\pi}{6\sqrt{3}} - \ln \frac{2+\sqrt{3}}{3}$
2). Compute $\int x^n \arccos \frac{1-x^2}{1+x^2} dx$.

Mihály Bencze

PP. 16106. In all triangle ABC holds $\sum tg^2 \frac{A}{2} tg^2 \frac{B}{2} \leq 1 - \frac{8r(R+r)}{s^2}$.

Mihály Bencze

PP. 16107. Let be $a, b, c \in C$ $(a \neq b \neq c)$ such that $\frac{a}{b-c} + \frac{b}{c-a} + \frac{c}{a-b} = 0$.

Compute $S_n = \frac{a}{(b-c)^n} + \frac{b}{(c-a)^n} + \frac{c}{(a-b)^n}$. We have $S_1 = 0$, $S_2 = \frac{a+b+c}{(a-b)(b-c)(c-a)}$.

Mihály Bencze

PP. 16108. Solve the following system:
$x_1 - \left(3^{\lg x_2} + 7\right)^{\lg 3} = x_2 - \left(3^{\lg x_3} + 7\right)^{\lg 3} = ... = x_n - \left(3^{\lg x_1} + 7\right)^{\lg 3} = 7.$

Mihály Bencze

PP. 16109. If $x, y, z > 0$, then

$$\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)(x+y+z)^{\frac{3}{2}} \geq 3\sqrt{3}\,(xy+yz+zx).$$

Mihály Bencze

PP. 16110. Prove that $\prod\limits_{k=0}^{n}\binom{2n}{k}^{\binom{n}{k}} \leq \left(\frac{\binom{3n}{n}}{2^n-1}\right)^{\frac{1}{2^n-1}}$.

Mihály Bencze

PP. 16111. 1). If $x \in \left(0, \frac{\pi}{2}\right)$, then $\frac{tg^2x}{tg^3x+ctgx} + \frac{ctg^2x}{ctg^3x+tghx} \geq \frac{2}{tg^2x+ctg^2x}$

2). Solve the following system: $\begin{cases} \frac{tg^2x_1}{tg^3x_1+ctgx_2} + \frac{ctg^2x_2}{ctg^3x_2+tgx_3} = \frac{2}{tg^2x_3+tg^2x_4} \\ ------------------ \\ \frac{tg^2x_n}{tg^3x_n+ctgx_1} + \frac{ctg^2x_1}{ctg^3x_1+tgx_2} = \frac{2}{tg^2x_2+tg^2x_3} \end{cases}$.

Mihály Bencze

PP. 16112. Determine all $\alpha > 1$ such that $\sum\limits_{k=1}^{n}\frac{1}{1+\alpha^k} \leq 1-\frac{1}{\alpha^n}$ for all $n \in N^*$.

Mihály Bencze

PP. 16113. If the equation $x^3+ax^2+bx+c=0$ have positive real roots

x_1, x_2, x_3, then $\left(4x_1^2+3\right)\left(4x_2^2+3\right)\left(4x_3^2+3\right) \geq 8\left(a^2-ab-2a+b+c+1\right)$.

Mihály Bencze

PP. 16114. In all triangle ABC holds:

1). $\sum\left(4a^2+3\right)\left(4b^2+3\right) \geq \frac{4}{3}(4s+3)^2$
2). $\sum\left(4h_a^2+3\right)\left(4h_b^2+3\right) \geq \frac{4}{3}\left(\frac{s^2+r^2+4Rr}{R}+3\right)^2$
3). $\sum\left(4r_a^2+3\right)\left(4r_b^2+3\right) \geq \frac{4}{3}(8R+4r+3)^2$
4). $\sum\left(4\sin^4\frac{A}{2}+3\right)\left(4\sin^4\frac{B}{2}+3\right) \geq \frac{4}{3}\left(5-\frac{r}{R}\right)^2$
5). $\sum\left(4\cos^4\frac{A}{2}+3\right)\left(4\cos^4\frac{B}{2}+3\right) \geq \frac{4}{3}\left(7+\frac{r}{R}\right)^2$

Mihály Bencze

PP. 16115. If $x_k > 0$ $(k = 1, 2, ..., n)$, then

1). $\prod\limits_{k=1}^{n} \left(x_k^2 + 1\right) \geq \frac{1}{3^n} \left(2 + \sqrt{3} \left(\prod\limits_{cyclic} (x_1 + x_2)^{\frac{1}{n}}\right)^{\frac{1}{n}}\right)^n$

2). $\sum\limits_{cyclic} \sqrt{\left(x_1^2 + 1\right)\left(x_2^2 + 1\right)} \geq \frac{2\sqrt{3}}{3} \sum\limits_{k=1}^{n} x_k + \frac{2n}{3}.$

Mihály Bencze

PP. 16116. If $a, b, c > 0$, then $abc \prod (4a + b + c) \leq (a + b + c)^3 \prod (a + b)$.

Mihály Bencze

PP. 16117. If $f(x) = \frac{1}{(x-1)^{n-1}} \begin{vmatrix} 1 & x & ... & x^{n-1} \\ 1 & 1 & ... & 1 \\ 1 & 2 & ... & n \\ - & - & -- & -- \\ 1^{n-2} & 2^{n-2} & ... & n^{n-2} \end{vmatrix}$, then solve the equations:

1). $f(x) = 0$ 2). $f'(x) = 0$ 3). $f(x) = f\left(\frac{1}{x}\right)$

4). $f(x+1) = f(x-1)$

Mihály Bencze

PP. 16118. In all triangle ABC holds

$$\begin{vmatrix} \sin(A+x) & \sin(B+x) & \sin(C+x) \\ \cos(A+x) & \cos(B+x) & \cos(C+x) \\ a & b & c \end{vmatrix} +$$

$$+ \begin{vmatrix} \sin(2A+x) & \sin(2B+x) & \sin(2C+x) \\ \cos(2A+x) & \cos(2B+x) & \cos(2C+x) \\ a & b & c \end{vmatrix} =$$

$$= 2R \begin{vmatrix} \sin 2A & \sin 2B & \sin 2C \\ \cos A & \cos B & \cos C \\ \sin A & \sin B & \sin C \end{vmatrix} \text{ for all } x \in R.$$

Mihály Bencze

PP. 16119. If $x > 0$, then

$$2n^2\left(\sum_{k=0}^{n} x^{2k}\right)\left(\sum_{k=0}^{n} \frac{x^{2k}}{2k+1}\right) \geq (n+1)^2 x^2 \left(\sum_{k=0}^{n-1} x^{2k}\right)\left(\sum_{k=1}^{n} \frac{x^{2k-2}}{k}\right).$$

Mihály Bencze

PP. 16120. Let $A_1A_2...A_n$ be a convex polygon with sides $a_1, a_2, ..., a_n$ and perimeter S. Prove that $\sum_{k=1}^{n} \frac{A_k^2}{S-a_k} \geq \frac{(n-2)^2\pi^2}{(n-1)S^2}$.

Mihály Bencze

PP. 16121. Solve in positive integers the equation $\sum_{k=1}^{n} x_k! = y^z$.

Mihály Bencze

PP. 16122. If $x_i \in R$ $(i = 1, 2, ..., n+1)$ such that $\sum_{i=1}^{n+1} x_i = an$, $\sum_{i=1}^{n+1} x_i^2 = bn$, where $0 < a < b$ and $(n+1)\,b \geq na^2$, then

$x_k \in \left[\frac{n\left(a-\sqrt{(n+1)b-na^2}\right)}{n+1}; \frac{n\left(a+\sqrt{(n+1)b-na^2}\right)}{n+1}\right]$ for all $k \in \{1, 2, ..., n+1\}$.

Mihály Bencze

PP. 16123. Let $p > q$ be two prime numbers such that p^n not dividing $((p-q)\,n)!$. Determine all n for which $((p-q)\,n)!$ are dividing by p^{n-1}.

Mihály Bencze

PP. 16124. 1). If $x \in R$, then $x^4 - 8x + 8 \geq 0$
2). Determine all $a, b \in R$ for which $\frac{x^6}{30} - \frac{x^3}{6} + \frac{x^2}{4} + ax + b \geq 0$ for all $x \in R$.

Mihály Bencze

PP. 16125. Determine all $x_k > 0$ $(k = 1, 2, ..., n)$ for which
$\left(\sum_{k=1}^{n} x_k\right)^2 + 1 \geq \sum_{cyclic} \frac{x_1x_2}{x_3} + \sum_{k=1}^{n} x_k$.

Mihály Bencze

PP. 16126. If $p, q \geq 2$ are prime, then determine all $a_k \in \{0, 1, ..., 9\}$ $(k = 1, 2, ..., n)$ such that $\overline{a_1 a_2 ... a_n} = p\left(\sum\limits_{k=1}^{n} a_k\right)^q + q\left(\sum\limits_{k=1}^{n} a_k\right)^p$.

Mihály Bencze

PP. 16127. If $x_i > 0$ $(i = 1, 2, ..., n)$ and $k \in \{1, 2, ..., n\}$, then $\frac{1}{2k}\binom{n-1}{k-1}\sum\limits_{i=1}^{n} x_i^k + \frac{1}{2}\binom{n}{k} \geq \sum\limits_{1 \leq i_1 < ... < i_k \leq n} \sqrt{x_{i_1} x_{i_2} ... x_{i_k}}$.

Mihály Bencze

PP. 16128. In all nonacute triangle ABC holds:
1). $r(4R + r) \leq \left(8\sqrt{2} - 11\right)s^2$
2). $s^2 + r^2 \geq 6\left(1 + \sqrt{2}\right)Rr$
3). $s^2 \geq 2\left(7 + 5\sqrt{2}\right)Rr$

Mihály Bencze

PP. 16129. If $n \geq 4$, then $\sum\limits_{k=1}^{n} \frac{1}{k^2(k+1)^2} \leq \frac{n\left(n^2 - n + 9\right)}{9(n+1)^2}$.

Mihály Bencze

PP. 16130. If $a, b, c > 0$, then $\sum (a+b)^2 (a+c)^2 \geq 12abc(a+b+c) + \frac{1}{3}\left(\sum |a(a+b+c) - bc|\right)^2$.

Mihály Bencze

PP. 16131. Determine all $x, y, z > 0$ such that $(x+y+z)^3\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) \geq 27\left(x^2 + y^2 + z^2\right)$.

Mihály Bencze

PP. 16132. In all tetrahedron $ABCD$ holds

1). $\frac{1}{4r}\sum h_a \geq 1 + 12\max\left\{\frac{\sum h_a^2}{\left(\sum h_a\right)^2}; r^2 \sum \frac{1}{h_a^2}\right\}$
2). $\frac{1}{2r}\sum r_a \geq 1 + 12\max\left\{\frac{\sum r_a^2}{\left(\sum r_a\right)^2}; \frac{r^2}{4}\sum \frac{1}{r_a^2}\right\}$

Mihály Bencze

PP. 16133. If $n \geq 4$, then

1). $\sum\limits_{k=1}^{n} \frac{1}{k} \geq \frac{2\left(n^2+2n-1\right)}{(n+1)^2}$ 2). $\sum\limits_{k=1}^{n} \frac{1}{k^2} \geq \frac{6\left(10n^3+39n^2+14n-33\right)}{5(n+1)^2(2n+1)^2}$

Mihály Bencze

PP. 16134. If F_k denote the k^{th} Fibonacci number, then

$\sum\limits_{k=1}^{n} \frac{1}{F_k} \geq \frac{n(n-3)}{F_{n+2}-1} + \frac{3n^2 F_n F_{n+1}}{(F_{n+2}-1)^3}$ for all $n \geq 4$.

Mihály Bencze

PP. 16135. In all simplex holds the following inequalities:

1). $\frac{1}{nr} \sum\limits_{k=1}^{n} h_k \geq n - 3 + 3n \max \left\{ \frac{\sum\limits_{k=1}^{n} h_k^2}{\left(\sum\limits_{k=1}^{n} h_k\right)^2}; r^2 \sum\limits_{k=1}^{n} \frac{1}{h_k^2} \right\}$

2). $\frac{n-2}{nr} \sum\limits_{k=1}^{n} r_k \geq n - 3 + 3n \max \left\{ \frac{\sum\limits_{k=1}^{n} r_k^2}{\left(\sum\limits_{k=1}^{n} r_k\right)^2}; \frac{r^2}{(n-2)^2} \sum\limits_{k=1}^{n} \frac{1}{r_k^2} \right\}$

Mihály Bencze

PP. 16136. Prove that $\sum\limits_{k=0}^{n} \frac{1}{\binom{n}{k}} \geq \frac{n+1}{2^n} \left(n - 2 + \frac{3(n+1)\binom{2n}{n}}{4^n} \right)$ for all $n \geq 3$.

Mihály Bencze

PP. 16137. In all triangle ABC holds $\sum \sin \frac{A}{2} + 2 \sum \cos \frac{A}{2} \cos \frac{B}{2} \leq \sum \frac{1}{\sin \frac{A}{2}}$.

Mihály Bencze

PP. 16138. In all triangle ABC holds $3 \cdot 4^{-\alpha} \leq \sum \left(\sin \frac{A}{2} \sin \frac{B}{2} \right)^{\alpha} \leq \frac{R+r}{2R}$ for all $\alpha \geq 1$.

Mihály Bencze

PP. 16139. If $a_k > 0$ $(k = 1, 2, ..., n)$ and $\sum\limits_{k=1}^{n} a_k = n$, then $\sum\limits_{cyclic} \frac{a_1}{1+a_1^2 a_2} \geq \frac{n}{2}$ (A generalization of problem 3345, Crux Mathematicorum).

Mihály Bencze

PP. 16140. In which triangle holds the following inequalities:

1). $s^2\left(s^2+r^2\right)+27Rr^2\left(4R+r\right) \geq 23s^2Rr$
2). $2s^2R+27r^2\left(4R+r\right) \geq 14s^2r$
3). $\left(s^2+r^2+4Rr\right)^3 \geq 54Rr\left(\left(s^2+r^2+4Rr\right)^2-16s^2Rr\right)$
4). $\left(4R+r\right)^3 \geq 27r\left(\left(4R+r\right)^2-2s^2\right)$
5). $\left(2R-r\right)^3\left(s^2+r^2-8Rr\right) \geq 27Rr^2\left(8R^2+r^2-s^2\right)$
6). $\left(4R+r\right)^3\left(s^2+\left(4R+r\right)^2\right) \geq 27Rs^2\left(\left(4R+r\right)^2-s^2\right)$

Mihály Bencze

PP. 16141. In all triangle ABC holds

1). $\prod \frac{a^2+b}{c+1} \geq 4sRr$
2). $\prod \frac{(s-a)^2+s-b}{s-c+1} \geq sr^2$
3). $\prod \frac{h_a^2+h_b}{h_c+1} \geq \frac{2s^2r}{R}$
4). $\prod \frac{r_a^2+r_b}{r_c+1} \geq s^2r$
5). $\prod \frac{\sin^2\frac{A}{2}+\sin\frac{B}{2}}{\sin\frac{C}{2}+1} \geq \frac{r}{4R}$
6). $\prod \frac{\cos^2\frac{A}{2}+\cos\frac{B}{2}}{\cos\frac{C}{2}+1} \geq \frac{s}{4R}$

Mihály Bencze

PP. 16142. In all triangle ABC holds $\sum \frac{\left(tg^2\frac{A}{2}+tg\frac{B}{2}\right)\left(tg^2\frac{B}{2}+tg\frac{A}{2}\right)}{\left(1+tg\frac{A}{2}\right)\left(1+tg\frac{B}{2}\right)} \geq 1.$

Mihály Bencze

PP. 16143. In all triangle ABC holds
1). $\sum \frac{(1+r_a)^2 r_b}{1+r_b} \geq s^2+4R+r$
2). $\sum \frac{\left(1+\sin^2\frac{A}{2}\right)^2\sin^2\frac{B}{2}}{1+\sin^2\frac{B}{2}} \geq \frac{16R^2-4Rr+r^2-s^2}{8R^2}$
3). $\sum \frac{\left(1+\cos^2\frac{A}{2}\right)^2\cos^2\frac{B}{2}}{1+\cos^2\frac{B}{2}} \geq \frac{32R^2+12Rr+r^2-s^2}{8R^2}$

Mihály Bencze

PP. 16144. In all triangle ABC holds
1). $2s \geq \sqrt{3\left(s^2+r^2+4Rr\right)} \geq 3\sqrt[3]{4sRr}$
2). $s \geq \sqrt{3r\left(4R+r\right)} \geq 3\sqrt[3]{sr^2}$
3). $s^2+r^2+4Rr \geq s\sqrt{6Rr} \geq 3\sqrt[3]{2s^2R^2r^2}$
4). $4R+r \geq s\sqrt{3} \geq 3\sqrt[3]{s^2r}$
5). $2R-r \geq \frac{1}{2}\sqrt{3\left(s^2+r^2-8Rr\right)} \geq 3\sqrt[3]{\frac{Rr^2}{2}}$
6). $4R+r \geq \frac{1}{2}\sqrt{3\left(s^2+\left(4R+r\right)^2\right)} \geq 3\sqrt[3]{\frac{Rs^2}{2}}$
(Refinements for classical triangle inequalities).

Mihály Bencze

PP. 16145. In all triangle ABC holds $\prod\left(5+2\cos A\right) \geq \frac{6\sqrt{3}s\left(s^2+r^2+2Rr\right)}{R^3}$.

Mihály Bencze

PP. 16146. If $a_k, \lambda_k, x, y > 0$ $\left(k=1,2,...,n\right)$, then $\left(\sum\limits_{k=1}^{n}\lambda_k a_k^x\right)\left(\sum\limits_{k=1}^{n}\lambda_k a_k^y\right) \leq \left(\sum\limits_{k=1}^{n}\lambda_k\right)\left(\sum\limits_{k=1}^{n}\lambda_k a_k^{x+y}\right)$. (A generalization of problem 26064, Gazeta Matematica, Bucuresti).

Mihály Bencze

PP. 16147. In all triangle ABC holds

1). $2\left(s^2-r^2-4Rr\right) \geq s^2+r^2+4Rr \geq 3\sqrt[3]{16s^2R^2r^2}$
2). $s^2-2r^2-8Rr \geq r\left(4R+r\right) \geq 3\sqrt[3]{\left(sr^2\right)^2}$
3). $\left(\frac{s^2+r^2+4Rr}{2R}\right)^2-\frac{4s^2r}{R} \geq \frac{2s^2r}{R} \geq 3\sqrt[3]{\left(\frac{2s^2r^2}{R}\right)^2}$
4). $\left(4R+r\right)^2-2s^2 \geq s^2 \geq 3\sqrt[3]{\left(s^2r\right)^2}$
5). $\frac{8R^2+r^2-s^2}{8R^2} \geq \frac{s^2+r^2-8Rr}{16R^2} \geq 3\sqrt[3]{\left(\frac{r^2}{16R^2}\right)^2}$
6). $\frac{\left(4R+r\right)^2-s^2}{8R^2} \geq \frac{s^2+\left(4R+r\right)^2}{16R^2} \geq 3\sqrt[3]{\left(\frac{s^2}{16R^2}\right)^2}$
(Refinements for classical triangle inequalities).

Mihály Bencze

PP. 16148. In all triangle ABC holds $\sum \frac{ctg^3\frac{A}{2}}{\left(tg\frac{B}{2}+tg\frac{C}{2}\right)^2} \geq \frac{s^2\sqrt{s}}{\sqrt{r}}\sum\frac{\sqrt{tg\frac{A}{2}}}{\left(s-rtg\frac{C}{2}\right)^2}$.

Mihály Bencze

PP. 16149. In all triangle ABC holds

1). $\sum\frac{1}{\cos^4 A}\geq\left(\frac{s^2+r^2-4R^2}{s^2-(2R+r)^2}\right)^2-\frac{8R(R+r)}{s^2-(2R+r)^2}+2sR^2r\sum\frac{\sin A}{(2R^2\sin A-sr)^2}$

2). If ABC is acute, then
$\sum\frac{1}{\sin^4 A}\geq\left(\frac{s^2+r^2+Rr}{2sr}\right)^2-\frac{4R}{r}+4R^2\left(s^2-(2R+r)^2\right)\sum\frac{\cos A}{\left(4R^2\cos A+(2R+r)^2-s^2\right)^2}$

Mihály Bencze

PP. 16150. If $x_k>0$ $(k=1,2,...,n)$ and $\sum\limits_{k=1}^{n}x_k=1$, then
$\sum\limits_{k=1}^{n}\frac{n-x_k}{x_k(1-x_k)}\geq\frac{n^3}{n-1}\sum\limits_{k=1}^{n}\sqrt[n]{1-x_k}$.

Mihály Bencze

PP. 16151. If $a,b>0$, then $\prod\limits_{k=1}^{n}\left(\frac{a^{\frac{k}{k+1}}+b^{\frac{k}{k+1}}}{a^{\frac{1}{k+1}}+b^{\frac{1}{k+1}}}\right)^{k+1}\geq\left(\frac{a+b}{2}\right)^{\frac{n(n-1)}{2}}$.

Mihály Bencze

PP. 16152. If $a_k,b_k>0$ $(k=1,2,...,n)$, then

$$\left(\frac{1}{2^n}\left(\sqrt[n]{\prod_{k=1}^{n}a_k}+\sqrt[n]{\prod_{k=1}^{n}b_k}\right)\right)^{m-1}\leq\prod_{k=1}^{n}\left(\frac{a_k^{\frac{m}{m+1}}+b_k^{\frac{m}{m+1}}}{a_k^{\frac{1}{m+1}}+b_k^{\frac{1}{m+1}}}\right)^{m+1}\text{, for all } m\in N$$

Mihály Bencze

PP. 16153. If $x_i\in(0,1)$ $(i=1,2,...,n)$ and $k\in\{1,2,...,n\}$, then
$\sum\limits_{i=1}^{n}\frac{x_i}{(1-x_i)^k}\geq\sum\limits_{cyclic}\frac{\sqrt[k]{x_1x_2...x_k}}{\left(1-\sqrt[k]{x_1x_2...x_k}\right)^k}$.

Mihály Bencze

PP. 16154. If $x_k > 0$ $(k = 1, 2, ..., n)$ and $\sum\limits_{k=1}^{n} x_k = 1$, then
$\sum\limits_{k=1}^{n} \frac{x_k(n-x_k)}{(1-x_k)^n} \geq \frac{n^{n+1}}{(n-1)^n} \left(\sum\limits_{k=1}^{n} \sqrt[n]{1-x_k} \right) \sqrt[n]{\prod\limits_{k=1}^{n} x_k}$.

Mihály Bencze

PP. 16155. If $e_n = \left(1 + \frac{1}{n}\right)^n$, $\overline{e}_n = \left(1 - \frac{1}{n}\right)^n$, $E_n = \left(1 + \frac{1}{n}\right)^{n+1}$,

$\overline{E}_n = \left(1 - \frac{1}{n}\right)^{n+1}$, then $\left(\frac{E_n + \overline{E}_n}{2}\right)^{n-1} \leq \left(\frac{e_n + \overline{e}_n}{2}\right)^{n+1}$ for all $n \in N^*$.

Mihály Bencze and Zhao Changjian

PP. 16156. Determine all $x, y \in [0, 2]$ such that
$4\left(x^{n+1} + (2-y)^{n+1}\right)^{n-1} + 4\left(y^{n+1} + (2-x)^{n+1}\right)^{n-1} \leq$
$\leq (x^n + (2-x)^n)^{n+1} + (y^n + (2-y)^n)^{n+1}$ for all $n \in N^*$.

Mihály Bencze and Zhao Changjian

PP. 16157. If $x \in [0, 2]$ and $\alpha \geq 1$, then

$x^{\frac{1}{\alpha+1}} + (2-x)^{\frac{1}{\alpha+1}} \leq x^{\frac{\alpha}{\alpha+1}} + (2-x)^{\frac{\alpha}{\alpha+1}}$.

Mihály Bencze

PP. 16158. If $a_k > 0$ $(k = 1, 2, ..., n)$ and $\sum\limits_{k=1}^{n} a_k = \sum\limits_{k=1}^{n} a_k^{\alpha}$, where $\alpha > 1$, then
$\sum\limits_{k=1}^{n} a_k^{\alpha+1} \leq n$ (A generalization of problem 249 GMA).

Mihály Bencze

PP. 16159. If $a, b > 0$, then $\left(\frac{a^{\frac{\alpha}{\alpha+1}} + b^{\frac{\alpha}{\alpha+1}}}{a^{\frac{1}{\alpha+1}} + b^{\frac{1}{\alpha+1}}}\right)^{\frac{2(\alpha+1)}{\alpha-1}} \leq \frac{a^2+b^2}{2}$ for all $\alpha > 1$.

Mihály Bencze

PP. 16160. If $x_k > 0$ $(k = 1, 2, ..., n)$, then

$$\left(\sum_{k=1}^{n} x_k^4\right)^2 \left(\sum_{k=1}^{n} x_k\right)^7 \leq n^3 \left(\sum_{k=1}^{n} x_k^2\right)^3 \left(\sum_{k=1}^{n} x_k^3\right)^3.$$

Mihály Bencze

PP. 16161. In all tetrahedron $ABCD$ holds

1). $\frac{\left(\sum \frac{1}{h_a^{\alpha+1}}\right)^{\alpha-1}}{\left(\sum \frac{1}{h_a^{\alpha}}\right)^{\alpha+1}} \leq 4^{\alpha-1} r^{\alpha+1}$

2). $\frac{\left(\sum \frac{1}{r_a^{\alpha+1}}\right)^{\alpha-1}}{\left(\sum \frac{1}{r_a^{\alpha}}\right)^{\alpha+1}} \leq 2^{\alpha-3} r^{\alpha+1}$, for all $\alpha > 1$.

Mihály Bencze and Zhao Changjian

PP. 16162. In all triangle ABC holds

1). $2s^4\left(s^2 - 3r^2 - 6Rr\right) \leq 3\left(s^2 - r^2 - 4Rr\right)^3$

2). $s^4\left(s^2 - 12Rr\right) \leq 3\left(s^2 - 2r^2 - 8Rr\right)^3$

3). $(4R + r)^3\left((4R + r)^3 - 12s^2R\right) \leq 3\left((4R + r)^2 - 2s^2\right)^3$

4). $2(2R - r)^3\left((2R - r)\left((4R + r)^2 - 3s^2\right) + 6Rr^2\right) \leq \left(8R^2 + r^2 - s^2\right)^3$

5). $2(4R + r)^3\left((4R + r)^3 - 3s^2(2R + r)\right) \leq \left((4R + r)^2 - s^2\right)^3$

Mihály Bencze and Zhao Changjian

PP. 16163. If $a_k > 0$ $(k = 1, 2, ..., n)$ and $\alpha > 1$, then
$\sum_{cyclic} \left(\frac{a_1^\alpha + a_2^\alpha}{a_1 + a_2}\right)^{\frac{\alpha+1}{\alpha-1}} \geq \sum_{k=1}^{n} a_k^{\alpha+1}$.

Mihály Bencze and Zhao Changjian

PP. 16164. In all tetrahedron $ABCD$ holds

1). $6\sum \frac{1}{h_a^3} \geq \frac{1}{r}\sum \frac{1}{h_a^2} + \left(\frac{1}{2r}\right)^3$ 2). $6\sum \frac{1}{r_a^3} \geq \frac{2}{r}\sum \frac{1}{r_a^2} + \left(\frac{1}{r}\right)^3$

Mihály Bencze

PP. 16165. If $x_i > 0$ $(i = 1, 2, ..., k)$ and $n \in N$, $n \geq 3$, then

$$\left(\sum_{i=1}^{k} x_i\right)^{\frac{(n-1)(n+4)}{2}} \left(\sum_{i=1}^{k} x_i^{n+1}\right)^{n-1} \leq k^{\frac{n^2+n-4}{2}} \left(\sum_{i=1}^{k} x_i^2\right)^3 \prod_{p=3}^{n} \left(\sum_{i=1}^{k} x_i^p\right)^2.$$

Mihály Bencze

PP. 16166. If $S_n^k = 1^k + 2^k + ... + n^k$, then

$n^2 (n+1)^k \left(S_n^{k+1}\right)^{k-1} \leq 2^{k+1} \left(S_n^k\right)^{k+1}$ for all $k \in N^*$.

Mihály Bencze

PP. 16167. In all triangle ABC holds

1). $\sum \left(\sin \sqrt{A}\right)^2 < \frac{s}{R}$ 2). $\prod \left(\cos \sqrt{A}\right)^2 > \left(\frac{2R+r-s}{2R}\right)^2$

Mihály Bencze

PP. 16168. If $x \in \left[0, \frac{\pi}{2}\right]$, then

1). $\sin^2(\sin x) + \sin^2(\cos x) \leq 2 \sin \frac{1}{2} \cos\left(\frac{1}{2} \cos 2x\right)$
2). $(\sin \sqrt{x})^4 + \left(\cos \sqrt{\frac{\pi}{2} - x}\right)^4 \leq 1$

Mihály Bencze

PP. 16169. If $p, q \geq 3$ are two prime numbers and $A_n = \{pn + q | n \in N\}$ and $B_n = \{qn + p | n \in N\}$, then
1). $A_n \cup B_n$ contains infinitely many twin primes
2). How many twin primes contains $A_n \cap B_n$?

Mihály Bencze

PP. 16170. In all tetrahedron $ABCD$ holds
1). $\sum \frac{h_b^2 h_c}{h_a\left(h_b^2 h_c^2 + 64r^3\right)} \geq \frac{1}{2r}$ 2). $\sum \frac{r_b^2 r_c}{r_a\left(r_b^2 r_c^2 + 8r^3\right)} \geq \frac{1}{r}$

Mihály Bencze

PP. 16171. If $a, b, c, d > 0$ and $S = a + b + c + d$, then
$\sum \frac{a}{n-1+b^n c} \geq s - \frac{s^2}{4n^2} - \frac{(n-1)s^2}{16n^2}$ for all $n \in N^*$. (A generalization of problem 3345, Crux Mathematicorum.)

Mihály Bencze

PP. 16172. If $a, b, c > 0$, then $\sum \frac{ab}{3ab+(a+b)(a+b+c)} \leq \frac{1}{3}$.

Mihály Bencze

PP. 16173. In all triangle ABC holds

1). $\sum \frac{tg\frac{A}{2}}{\sqrt{\sin A}} \leq \frac{\sqrt{2}(R+r)}{\sqrt{sr}}$
2). $\sum \left(\sqrt{s^2 - 4Rr - r^2 - 2R^2 \sin^2 A}\right) tg\frac{A}{2} \geq 2\left(R + r\right)$

Mihály Bencze

PP. 16174. If $a, b, c > 0$ and $ab + bc + ca = 1$, then

1). $\sum \frac{a^x}{1+b^y c^z} \geq \frac{3\left(\sqrt{3}\right)^{y+z-x}}{1+\left(\sqrt{3}\right)^{y+z}}$ 2). $\sum \frac{a^{x+y}}{1+a^x} \geq \frac{3}{\left(1+\left(\sqrt{3}\right)^x\right)\left(\sqrt{3}\right)^y}$
for all $x, y, z > 0$.

Mihály Bencze

PP. 16175. In triangle ABC denote R_a, R_b, R_c the distances from the incentre of triangle to the vertices A, B, C. Prove that

1). $\sum \frac{R_a}{\sqrt{s-a}} \leq 2\sqrt{s}$
2). $\sum R_a = \sqrt{s^2 + r^2 + 4Rr}$

3). $\max\left\{\sum \frac{R_a}{\sqrt{b}}, \sum \frac{R_a}{\sqrt{c}}\right\} \leq \frac{3\sqrt{s}}{2}$

Mihály Bencze

PP. 16176. Determine all $x, y, z > 0$ such that $\begin{cases} x^{1+2\sin y - \cos z} \leq 1 \\ y^{1+2\sin z - \cos x} \leq 1 \\ z^{1+2\sin x - \cos y} \leq 1 \end{cases}$.

Mihály Bencze

PP. 16177. In all triangle ABC holds

1). $\sum \frac{ctg\frac{A}{2}}{s+rtg\frac{A}{2}} \geq \frac{3\sqrt{3}}{4r}$ 2). $\sum \frac{ctg^2\frac{A}{2}}{s+rctg\frac{A}{2}} \geq \frac{s\sqrt{3}}{\left(1+\sqrt{3}\right)r^2}$

Mihály Bencze

PP. 16178. Let $P(x)$ be a polynomial with nonnegative coefficients.

Determine all $x_k > 0$ $(k = 1, 2, ..., n)$ for which
$P\left(\frac{1}{x_1}\right)P(x_2) + P\left(\frac{1}{x_2}\right)P(x_3) + ... + P\left(\frac{1}{x_n}\right)P(x_1) \geq n.$

Mihály Bencze

PP. 16179. If $x, y, z > 0$ and $n \in N$, then $\sum \frac{1}{x^n((x^n+y^n)z^n+y^{2n})} \leq \left(\frac{1}{xyz}\right)^n$.

Mihály Bencze

PP. 16180. If $x, y, z > 0$, then

1). $2\sum \frac{x^3z}{y} + \sum x^2z \geq \sum x^2y + 6xyz$
2). $2\sum x^2 + \sqrt[3]{xyz}\sum x \geq \sum xy + 6\sqrt[3]{x^2y^2z^2}$.

Mihály Bencze

PP. 16181. If $x_k > 0$ $(k = 1, 2, ..., n)$, then $\sum\limits_{cyclic} \frac{x_1^n-(n-1)x_1+n-1}{x_2+x_3+...+x_n} \geq \frac{n}{n-1}$.

Mihály Bencze

PP. 16182. Solve in Z the equation $\prod\limits_{k=1}^{n} x_k + \sum\limits_{k=1}^{n} \frac{1}{x_k} = \sum\limits_{k=1}^{n} x_k + \prod\limits_{k=1}^{n} \frac{1}{x_k}$.

Mihály Bencze

PP. 16183. Solve in N the equation $\binom{n}{k} = \sum\limits_{i=1}^{r} \binom{m_i}{p_i}$.

Mihály Bencze

PP. 16184. In all triangle ABC holds $\sqrt{6\sqrt{3}} \leq \sum \sqrt{tg\frac{A}{2} + tg\frac{B}{2}} \leq 3\sqrt{\frac{3R}{r}}$.

Mihály Bencze

PP. 16185. If $a, b, c, d \geq 0$, then
$\sqrt{a^4+b^4} + 2\sqrt{a^4+c^4} + \sqrt{a^4+d^4} + \sqrt{b^4+c^4} + 2\sqrt{b^4+d^4} + \sqrt{c^4+d^4} \geq$
$\geq 2\sqrt{2}\,(ab+bc+cd+da)$.

Mihály Bencze

PP. 16186. In all triangle ABC holds

1). $\prod\left(\frac{2s+a}{2s-a}\right)^{\frac{1}{a}} \geq 2^{\frac{9}{2s}}$

2). $\prod\left(\frac{b+c}{a}\right)^{\frac{1}{s-a}} \geq 2^{\frac{9}{s}}$

3). $\prod\left(\frac{4R+r+r_a}{4R+r-r_a}\right)^{\frac{1}{r_a}} \geq 2^{\frac{9}{4R+r}}$

4). $\prod\left(\frac{2R-r+2R\sin^2\frac{A}{2}}{2R-r-2R\sin^2\frac{A}{2}}\right)^{\cos ec^2\frac{A}{2}} \geq 2^{\frac{18R}{2R-r}}$

5). $\prod\left(\frac{4R+r+2R\cos^2\frac{A}{2}}{4R+r-2R\cos^2\frac{A}{2}}\right)^{\sec^2\frac{A}{2}} \geq 2^{\frac{18R}{4R+r}}$

Mihály Bencze

PP. 16187. Prove that $\int\limits_x^{2x}\left(\frac{\sin t}{t}\right)^2 dt + \int\limits_x^{3x}\left(\frac{\sin t}{t}\right)^2 dt + \int\limits_x^{6x}\left(\frac{\sin t}{t}\right)^2 dt \leq \pi$ for all $x \geq 0$.

Mihály Bencze

PP. 16188. Prove that
$$\int\limits_0^1\left(\frac{1}{(1-x+x^2)^{\frac{3}{2}}} + \frac{1}{\left(1-\sqrt{2}x+x^2\right)^{\frac{3}{2}}} + \frac{1}{\left(1-\sqrt{3}x+x^2\right)^{\frac{3}{2}}}\right)dx < 12,37.$$

Mihály Bencze

PP. 16189. If $x \in R$, then $(\sin x)^{-2\sin^2 x}(\cos x)^{-2\cos^2 x} \leq e$.

Mihály Bencze

PP. 16190. If $x \in R$, then
$$\left(\frac{2}{e}\right)^2 \leq \left((\sin x)^{2tg^2 x} + (\sin x)^{2\sec^2 x}\right)\left((\cos x)^{2ctg^2 x} + (\cos x)^{2\cos ec^2 x}\right) \leq 1.$$

Mihály Bencze

PP. 16191. Let $0 < a_1 \leq a_2 \leq ... \leq a_n$ be an arithmetical progression with ratio $r > 0$. Prove that
$$\sum_{k=1}^{n}|cha_{k+1} - cha_k| \geq r\left(\sqrt{sha_1 sha_2} + \sqrt{sha_2 sha_3} + ... + \sqrt{sha_n sha_{n+1}}\right).$$

Mihály Bencze

PP. 16192. If $x \geq 0$, then $2x \leq tg\left(\frac{\pi}{2}thx\right) + \frac{\pi}{2}th\left(tgx\right).$

Mihály Bencze

PP. 16193. If $x, y, z > 0$ and $x^8 + y^8 + z^8 = x^3y^3z^3$, then $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \leq 1.$

Mihály Bencze

PP. 16194. If $0 < e \leq f, e + f = 1, 0 < c \leq d \leq 1$, $0 < a \leq b \leq 1$, then
$e\left(d - c\right)\ln\frac{b+1}{a+1} + f\left(b - a\right)\ln\frac{d+1}{c+1} \leq \int\limits_a^b\int\limits_c^d \frac{dxdy}{1+x^ey^f}.$

Mihály Bencze

PP. 16195. If $x \in \left(0, \frac{\pi}{4}\right]$, then $\arcsin\left(tg^2x\right) \geq \frac{6\left(1+\sqrt{\cos 2x}\right)}{1+4\cos x+\sqrt{\cos 2x}}.$

Mihály Bencze

PP. 16196. If $a_i > 0$ $(i = 1, 2, ..., n)$ and $k \in \{2, 3, ..., n\}$, then

$\sum\limits_{cyclic} \frac{a_1^m-(m-1)(a_1-1)}{a_2+a_3+...+a_k} \geq \frac{n}{k-1}$ for all $m \in N^*$.

Mihály Bencze

PP. 16197. 1). If $n \in N^*$, then $d^2\left(n\right) \leq 3n$

2). Determine all $k \in N^*$ such that $d^k\left(n\right) \leq \left(k + 1\right)n$ for all $n \in N^*$.

Mihály Bencze

PP. 16198. If $0 < a_1 < a_2 < ... < a_n$ are positive integers, then determine all $k \in N^*$ for which from the given sequence we can determine $n - k$ integers, such that for the remains $b_1, b_2, ..., b_k$ the numbers $\left|b_1 - b_2\right|, \left|b_2 - b_3\right|, ..., \left|b_k - b_1\right|$ are distinct.

Mihály Bencze

PP. 16199. Solve the following system: $\begin{cases} 1 + tgx + ctgy = \cos\left(z + \frac{\pi}{4}\right) \\ 1 + tgy + ctgz = \cos\left(x + \frac{\pi}{4}\right) \\ 1 + tgz + ctgx = \cos\left(y + \frac{\pi}{4}\right) \end{cases}.$

Mihály Bencze

PP. 16200. If $a, x_k > 0$ $(k = 1, 2, ..., n)$ and $\prod\limits_{k=1}^{n} x_k = a^n$, then determine $\min \sum\limits_{k=1}^{n} \frac{1}{\sqrt{x_k^2+1}}$.

Mihály Bencze

PP. 16201. If $x \in \left[0, \frac{\pi}{2}\right]$, then $\frac{1}{8}\sin^2 x \cos^2 x + \cos(\sin x) + \cos(\cos x) \leq \frac{25}{16}$.

Mihály Bencze

PP. 16202. Let consider the equation $x^2 + y^2 = z^2$

1). Prove that the equation have infinitely many solutions for which $x, \left(\frac{y}{2}\right)^2, z$ are consecutive integers.
2). Prove that the equation have infinitely many solutions for which $x, \left(\frac{y^2}{4}\right)^2, z$ are integers in arithmetical progression.

Mihály Bencze

PP. 16203. If $x \in (0,1) \cup (1,+\infty)$ and $\lambda \in [1,2]$, then $\sum\limits_{k=1}^{n} \left(x^k + \frac{1}{x^k}\right)^\lambda \leq \frac{\left(x^{n\lambda}-1\right)\left(x^{(n+1)\lambda}+1\right)}{x^{n\lambda}\left(x^\lambda-1\right)} - n \cdot 2^\lambda$.

Mihály Bencze

PP. 16204. If $x > 0$ and $\lambda \in [1,2]$, then $\sum\limits_{k=1}^{n} sh\lambda kx \geq n2^{\lambda-1}\left(\left(\frac{1}{n}\sum\limits_{k=1}^{n} shkx\right)^\lambda - 1\right)$.

Mihály Bencze

PP. 16205. Denote R_A the radius of the circle which is tangent to the sides AB, AC and interior to the circumscribed circle of triangle ABC etc. Prove that $R_A^a R_B^b R_C^c \geq \left(\frac{2Rr(4R+r)}{s^2}\right)^{2s}$.

Mihály Bencze

PP. 16206. Prove that $\prod\limits_{k=0}^{m}\left(\binom{m}{k}\binom{t}{k}\right)^{2^k} \le \left(\frac{\sum\limits_{k=0}^{m}\binom{m}{k}\binom{t+k}{m}}{2^{m+1}-1}\right)^{2^{m+1}-1}$.

Mihály Bencze

PP. 16207. If $x_n = \left[n\sqrt{k}\right]$, where $n \in N^*$ and $[\cdot]$ denote the integer part. Determine all $k \in N^*$ for which exist infinitely many $n \in N$ such that $x_n \equiv 0 \left(\text{mod } (k+1)^t\right)$, $t \in N$.

Mihály Bencze

PP. 16208. Let ABC be a triangle in which $a \ge b \ge c$. Prove that

$$\left(a\left(b-c\right)m_c + b\left(a-c\right)m_b\right)\left(m_b+m_c\right)^2 \ge c\left(a+b\right)m_a^3.$$

Mihály Bencze

PP. 16209. If F_k denote the k^{th} Fibonacci number, then

$$\sum_{k=1}^{n}\left(\frac{F_k}{1+F_kF_{k+1}}\right)^2 \le \frac{F_nF_{n+1}}{1+F_nF_{n+1}}.$$

Mihály Bencze

PP. 16210. Prove that:
1). $\sum\limits_{k=1}^{n}\frac{k+1}{k(2k+1)^2} \le \frac{n}{2n+1}$ 2). $\sum\limits_{k=1}^{n}\frac{(k+1)(k+2)}{k(5k^2+15k+8)^2} \le \frac{n(n+3)}{16(5n^2+15n+8)}$.

Mihály Bencze

PP. 16211. Determine all $a_k \in C$ $(k \in Z)$ such that

$$\sum_{k=-\infty}^{\infty} a_k\overline{a_{k-t}} = \begin{cases} \sin x & \text{if } \ t=0 \\ \cos x & \text{if } \ t\neq 0 \end{cases}.$$

Mihály Bencze

PP. 16212. In all triangle ABC
1). $\frac{1}{3}\left(s^2+r^2+4Rr\right) \le \frac{a^2b^2+b^2c^2+c^2a^2}{ab+bc+ca} \le \frac{2Rs^2}{9r}$
2). $3r\left(2R-r\right) \le s^2 \le 2R^2+r^2$

Mihály Bencze

PP. 16213. In all triangle ABC holds:

1). $\max\left\{\sum \sin\frac{A}{2}\cos ec\frac{B}{2}; \sum \sin\frac{A}{2}\cos ec\frac{C}{2}\right\} \leq \sqrt{\frac{(2R-r)(s^2+r^2-8Rr)}{2Rr^2}}$
2). $\max\left\{\sum \cos\frac{A}{2}\sec\frac{B}{2}; \sum \cos\frac{A}{2}\sec\frac{C}{2}\right\} \leq \sqrt{\frac{(4R+r)\left(s^2+(4R+r)^2\right)}{2Rs^2}}$
3). $\max\left\{\sum \cos\frac{A}{2}\sec\frac{B}{2}; \sum \cos\frac{A}{2}\sec\frac{C}{2}\right\} \leq \sqrt{\frac{(2R-r)\left(s^2+(4R+r)^2\right)}{2Rs^2}}$
4). $\max\left\{\sum \cos\frac{A}{2}\cos ec\frac{B}{2}; \sum \cos\frac{A}{2}\cos ec\frac{C}{2}\right\} \leq \sqrt{\frac{(4R+r)(s^2+r^2-8Rr)}{2Rr^2}}$

Mihály Bencze

PP. 16214. In all triangle ABC holds:

1). $\sum\left(2\left(\cos\frac{A}{4}+\sin\frac{A}{4}\right)\left(\cos\frac{B}{4}+\sin\frac{B}{4}\right)-4\cos\frac{A-B}{8}\cos\frac{B-C}{8}\cos\frac{C-A}{8}-\right.$
$\left.-\sum\sin\frac{\pi-A}{8}\right)\leq 1-\frac{r}{2R}$
2). $\sum\left(\cos\frac{A}{4}+\sin\frac{A}{4}\right)\left(\cos\frac{B}{4}+\sin\frac{B}{4}\right)\leq$
$\leq\frac{1}{2}\sum\sin\frac{A}{2}+6\cos\frac{A-B}{8}\cos\frac{B-C}{8}\cos\frac{C-A}{8}+\frac{3}{2}\sum\sin\frac{\pi-A}{8}$

Mihály Bencze

PP. 16215. In all triangle ABC holds:

1). $\frac{3R}{r}+\sum\frac{ab}{a^2+b^2}\geq\frac{3r}{R}+\sum\frac{b+c}{a}$
2). $\frac{\left(\sum a^2\right)\left(\sum a\right)}{\left(\sum ab\right)^2}\leq\frac{s}{18r^2}$
3). $\max\left\{\sum a^2b, \sum a^2c\right\}\leq 6sR^2$
4). $\left(s^2-r^2-4Rr\right)\left(\left(\frac{s^2+r^2+4Rr}{4sRr}\right)^2-\frac{1}{Rr}\right)\leq\frac{9R^2}{8r^2}$

Mihály Bencze

PP. 16216. In all triangle ABC holds: $2R^2+86Rr+9r^2-$

$$-2(R-2r)\sqrt{R^2-2Rr}\leq 6s^2\leq 31R^2+30Rr+7r^2+2(R-2r)\sqrt{R^2-2Rr}.$$

Mihály Bencze

PP. 16217. 1). In all triangle ABC holds: $\left(\sum\frac{a^3}{r_a}\right)\left(\sum\sqrt{\frac{a}{r_a}}\right)^2\leq\frac{9Rs^2}{r}$
2). Determine its minime.

Mihály Bencze

PP. 16218. Let ABC be a triangle, and $x, y > 0$. If $(x+y)\,a \leq xb + yc$, then $(x+y)\,A \leq xB + yC$.

Mihály Bencze

PP. 16219. Let ABC be a triangle in which $a \leq b \leq c$. Prove that

$$4s\,(a+h_a)\left(s^2+r^2+4Rr\right) \leq 3b\,(b+c+h_b+h_c)\left(a^2+ac+c^2\right).$$

Mihály Bencze

PP. 16220. In all triangle ABC holds: $\left(\sum a^t\right)\left(\sum (s-a)^{-t}\right) \geq 9 \cdot 2^t$ for all $t \in (-\infty, 0] \cup [1, +\infty)$.

Mihály Bencze

PP. 16221. In all triangle ABC holds:

1). $s^2\,(3R-8r) + 3Rr\,(4R+r) \geq 0$ 2). $\sum \left(\frac{a}{h_b+h_c}\right)^2 \geq 1$

Mihály Bencze

PP. 16222. In all triangle ABC holds:

1). $\sum \sqrt{\frac{r_a}{a}} \geq 6\sqrt{\frac{r}{s}}$ 2). $\sum \frac{r_a}{a^3} \geq \frac{9}{4sR}$ 3). $\sum \frac{1}{h_a} \geq \frac{3\sqrt{3}}{s}$

Mihály Bencze

PP. 16223. In all triangle ABC holds:

1). $8R^2 + 8Rr + 5r^2 \geq 2s^2$
2). $\sum \frac{1}{h_a} \geq \frac{9}{2R+5r}$
3). $\left(\sum r_a r_b\right)\left(\sum \frac{1}{h_a h_b}\right) \geq 9$

Mihály Bencze

PP. 16224. In all triangle ABC holds: $\frac{4\sqrt{3}sr}{R^2} \leq \left(\sum ab\right)\left(\sum \frac{1}{ab}\right) \leq \frac{\sqrt{3}sR}{r^2}$.

Mihály Bencze

PP. 16225. In all triangle ABC holds:

1). $\sum \left(\frac{a}{a+b}\right)^3 \geq \frac{s^2+r^2-8Rr}{s^2+r^2+2Rr}$
2). $\sum \left(\frac{-a+b+c}{c}\right)^3 \geq \frac{2(4R-5r)}{R}$
3). $\sum \left(\frac{r_a}{r_b+r_c}\right)^3 \geq \frac{4R-5r}{4R}$

Mihály Bencze

PP. 16226. If $a_k > 0$ $(k = 1, 2, ..., n)$, then
$\prod\limits_{cyclic} (a_1 + a_2 + ... + a_{n-1})^2 \geq \left(\frac{n-1}{n}\right)^2 \left(\sum\limits_{k=1}^{n} a_k\right)^n \prod\limits_{k=1}^{n} a_k.$

Mihály Bencze

PP. 16227. If $a_k > 0$ $(k = 1, 2, ..., n)$, then

$$\sum \frac{1}{\left(\sum \frac{a_1}{a_2}\right)^{n-1} - n^{n-1}\frac{a_2}{a_1}} \leq \frac{1}{\left(\sum \frac{a_1}{a_2}\right)^{n-1}}.$$

Mihály Bencze

PP. 16228. If $x \in \left(0, \frac{\pi}{2}\right)$, then $\frac{1}{\sin x} + \frac{1}{\cos x} \geq 2\sqrt{2} + \left(\sqrt{tgx} - \sqrt{ctgx}\right)^2$.

Mihály Bencze

PP. 16229. If $a_k, b_k, c_k > 0$, then
$$\frac{\left(\sum\limits_{k=1}^{n} a_k\right)^2 \left(\sum\limits_{k=1}^{n} b_k\right)^2 \left(\sum\limits_{k=1}^{n} c_k\right)^2}{\left(\sum\limits_{k=1}^{n} a_k + \sum\limits_{k=1}^{n} b_k\right)\left(\sum\limits_{k=1}^{n} b_k + \sum\limits_{k=1}^{n} c_k\right)\left(\sum\limits_{k=1}^{n} c_k + \sum\limits_{k=1}^{n} a_k\right)} \geq$$
$$\geq \left(\sum_{k=1}^{n} \frac{a_k b_k}{a_k+b_k}\right)\left(\sum_{k=1}^{n} \frac{b_k c_k}{b_k+c_k}\right)\left(\sum_{k=1}^{n} \frac{c_k a_k}{c_k+a_k}\right).$$

Mihály Bencze

PP. 16230. If $a, b, c > 0$, then $\sum ab^2 + \sum b^2c \leq abc + \sum \frac{a^2b^2}{c} + \sum a^3$.

Mihály Bencze

PP. 16231. If $\alpha, a_k > 0$ $(k = 1, 2, ..., n)$ and $\sum\limits_{k=1}^{n} a_k^{\alpha} = A^{\alpha}$, then $\sum\limits_{k=1}^{n} a_k^{\alpha+1} \leq A^{\alpha+1}$.

Mihály Bencze

PP. 16232. If $a_k > 0$ $(k = 1, 2, ..., n)$, then $\sum\limits_{k=1}^{n} a_k^{n} - n \prod\limits_{k=1}^{n} a_k \geq$

$$\geq (n-1) \max \left\{ \left(\frac{1}{n-1} (a_1 + ... + a_{i-1} + a_{i+1} + ... + a_n) - a_i \right)^n |1 \leq i \leq n \right\}.$$

Mihály Bencze

PP. 16233. If $a, b, c > 0$, then $\left(\sum a\right)\left(\sum \frac{1}{a}\right) \geq 9 + \frac{1}{3abc}\left(\sum |a-b|\sqrt{c}\right)^2$.

Mihály Bencze

PP. 16234. If $a, b, c > 0$, then

$$\sum \frac{a}{b+c} \geq \frac{3}{2} + \frac{1}{6(a+b)(b+c)(c+a)} \left(\sum \sqrt{b+c}\, |b-c| \right)^2.$$

Mihály Bencze

PP. 16235. If $a_i > 0$ $(i = 1, 2, ..., n)$ and $\prod\limits_{i=1}^{n} a_i = 1$, then

$$\sum_{i=1}^{n} \frac{1}{(1+a_i)^k} + \frac{k}{\prod\limits_{i=1}^{n} (1+a_i)} \geq \frac{n2^n + k \cdot 2^k}{2^{n+k}},$$ where $k \in N^*$.

Mihály Bencze

PP. 16236. If $a_k, b_k, \alpha > 0$ $(k = 1, 2, ..., n)$, then

$$\alpha \sum_{k=1}^{n} a_k b_k + \sqrt{\left(\sum_{k=1}^{n} a_k^2 \right) \left(\frac{1-\alpha^2}{n} \left(\sum_{k=1}^{n} b_k \right)^2 + \alpha^2 \sum_{k=1}^{n} b_k^2 \right)} \geq$$

$$\geq \frac{\alpha+1}{n} \left(\sum_{k=1}^{n} a_k \right) \left(\sum_{k=1}^{n} b_k \right).$$

Mihály Bencze

PP. 16237. If $a_k, b_k, c_k > 0$ $(k = 1, 2, ..., n)$, then

$$\frac{1}{2}\sum_{k=1}^{n} a_k b_k c_k + \sqrt[3]{\left(\sum_{k=1}^{n} a_k^3\right)\left(\sum_{k=1}^{n} b_k^3\right)\left(\frac{9}{8n}\left(\sum_{k=1}^{n} c_k\right)\left(\sum_{k=1}^{n} c_k^3\right) - \frac{1}{8}\sum_{k=1}^{n} c_k^3\right)} \geq$$
$$\geq \frac{3}{2n}\left(\sum_{k=1}^{n} a_k b_k\right)\left(\sum_{k=1}^{n} c_k\right).$$

Mihály Bencze

PP. 16238. If $a, b, c > 0$ and $a^2 + b^2 + c^2 = \frac{3}{4}$, then
$(1 - a)(1 - b)(1 - c) \geq abc.$

Mihály Bencze

PP. 16239. If $a, b, c \in R$, then

1). $\sum \sqrt[4]{1 + (b - c)^2 + (b - c)^4} \leq 3 - \sum ab + \sum a^2$
2). $\sum \sqrt[4]{1 + (b + c - 2a)^2 + (b + c - a)^4} \leq 3\left(1 - \sum ab + \sum a^2\right)$

Mihály Bencze

PP. 16240. If $a_k > 0$ $(k = 1, 2, ..., n)$ $(n \geq 3)$ and $\prod_{k=1}^{n} a_k = 1$,

$$\prod_{cyclic} (a_1 + a_2) \geq \prod_{k=1}^{n} (a_k + 1).$$

Mihály Bencze

PP. 16241. In all triangle ABC holds:
1). $\sum \left(\frac{a}{a+c}\right)^2 \geq \frac{s^2+r^2-2Rr}{s^2+r^2+2Rr}$
2). $\sum \left(\frac{-a+b+c}{b}\right)^2 \geq 4\left(1 - \frac{r}{2R}\right)$
3). $\sum \left(\frac{\sin\frac{A}{2}\cos\frac{C}{2}}{\cos\frac{B}{2}}\right)^2 \geq 1 - \frac{r}{2R}$
4). $\sum \left(\frac{\sin^2\frac{A}{2}}{\sin^2\frac{A}{2}+\sin^2\frac{C}{2}}\right)^2 \geq 1 - \frac{4Rr^2}{(2R-r)(s^2+r^2-8Rr)-2Rr}$
5). $\left(\frac{\cos^2\frac{A}{2}}{\cos^2\frac{A}{2}+\cos^2\frac{C}{2}}\right)^2 \geq 1 - \frac{4Rs^2}{(4R+r)^3+s^2(2R+r)}$

Mihály Bencze

PP. 16242. If $x, y, z, t > 0$, then $2\sqrt{(x^2+y^2)(y^2+z^2)(z^2+t^2)(t^2+x^2)}+$

$+\sqrt{(x^2+z^2)(y^2+t^2)(z^2+x^2)(t^2+y^2)} \geq \sum (x+y+z)\, xyz.$

Mihály Bencze

PP. 16243. If $a, b, c > 0$, then $\left(\sum \frac{a^2+b^2}{b}\right)\left(\sum \frac{a^2}{b+c}\right) \geq \frac{9\sum a^3}{\sum a}$.

Mihály Bencze

PP. 16244. In all triangle ABC holds:

1). $\prod \left(1+16\sin^4 \frac{A}{2}\right) \geq \frac{\left(R^3+8(2R-r)r^2\right)^2}{(R^4+16r^4)R^2}$
2). $\prod \left(1+\frac{16}{9}\cos^4 \frac{A}{2}\right) \geq \frac{\left(81R^3+8(4R+r)s^2\right)^2}{9(729R^4+16s^4)R^2}$.

Mihály Bencze

PP. 16245. If $x, y, z > 0$, then

$\sqrt{2(x^2+y^2)(y^2+z^2)(z^2+x^2)} \geq xyz + \sum x^2 z.$

Mihály Bencze

PP. 16246. If $a, b, c, d > 0$, then
1). $\sum \sqrt{(a^2+1)(b^2+1)(c^2+1)\left(1+\frac{1}{a^2b^2c^2}\right)} \geq \left(\sum a\right)\left(3+\frac{1}{abcd}\right)$
2). $\sum \left(a^2+1\right)\left(b^2+1\right)\left(c^2+1\right)\left(1+\frac{1}{a^2b^2c^2}\right) \geq \frac{1}{4}\left(\sum a\right)^2\left(3+\frac{1}{abcd}\right)^2$

Mihály Bencze

PP. 16247. In all triangle ABC holds:
1). $\prod \left(1+3tg^2\frac{A}{2}\right) \geq \frac{\left(s^2+9r(4R+r)\right)^2}{s^2+27r^2}$
2). $\prod \left(1+\frac{1}{3}ctg^2\frac{A}{2}\right) \geq \frac{\left(s^2+9r^2\right)^2}{s^2+27r^2}$

Mihály Bencze

PP. 16248. In all triangle ABC holds:
1). $16R^2 \geq s^2+r^2$
2). $16R^2\left(s^2+r^2\right) \geq \left(s^2+(4R+r)\,r\right)^2$

Mihály Bencze

PP. 16249. If $a_k > 0$ $(k = 1, 2, ..., n)$ and $\sum\limits_{k=1}^{n} a_k = 1$, then
$\sum\limits_{cyclic} \sqrt{a_1 + a_2^2 + a_3^3 + ... + a_n^n} \geq n\sqrt{\frac{3^n-1}{2\cdot 3^n}}$.

Mihály Bencze

PP. 16250. If $a, b, x, y \geq 0$, then
1). $2\left(a^2+1\right)\left(b^2+1\right)\left(a^2b^2+1\right) \geq \left(ab\left(a+b+1\right)+1\right)^2$
2). $\left(a^2+1\right)\left(b^2+1\right)\left(\left(a^xb^y\right)^{\frac{2}{x+y}}+1\right)\left(\left(a^yb^x\right)^{\frac{2}{x+y}}+1\right) \geq$
$\left(ab\left(a+b\right)+\left(a^xb^y\right)^{\frac{1}{x+y}}+\left(a^yb^x\right)^{\frac{1}{x+y}}\right)^2$.

Mihály Bencze

PP. 16251. If $a_k > 0$ $(k = 1, 2, ..., n)$ and $\sum\limits_{cyclic} a_1a_2 = n$, then
$\sum\limits_{cyclic} \frac{\left(a_1^2+1\right)\left(a_2^2+1\right)}{a_1+a_2} \geq 2n$.

Mihály Bencze

PP. 16252. If $a_k > 0$ $(k = 1, 2, ..., n)$, then $\sum\limits_{cyclic} \frac{\left(a_1^2+1\right)\left(a_2^2+1\right)}{a_1a_2+1} \geq 2\sum\limits_{k=1}^{n} a_k$.

Mihály Bencze

PP. 16253. If $1 \geq a_k > 0$ $(k = 1, 2, ..., n)$, then
$\sum\limits_{cyclic} \sqrt{\left(a_1^2+1\right)\left(a_2^2+1\right)} \geq n + \sum\limits_{k=1}^{n} a_k$.

Mihály Bencze

PP. 16254. If $a, b > 0$ and $ab \leq 1$, then $(a+b+2)^2 \geq 4\left(a+b\right)\left(ab+1\right)$.

Mihály Bencze

PP. 16255. Determine all $a, b > 0$ such that
$4a^2b^2 + 3\left(a^2+b^2\right) \geq 2ab + 4\left(a+b\right)$.

Mihály Bencze

PP. 16256. Prove that $\sum\limits_{k=1}^{n} k\sqrt{k} \geq (n-1)\left(k!\right)^{\frac{3}{2n}} + \frac{n(n+1)}{2}$.

Mihály Bencze

PP. 16257. If $a, b, c, x, y > 0$, then $\sum a^3 \geq \frac{3x-y}{2(x+y)} \sum a^2 b + \frac{3y-x}{2(x+y)} \sum ab^2$.

Mihály Bencze

PP. 16258. If $x_i > 0$ $(i = 1, 2, ..., n)$ and $k \in N^*$, then

$$\sum_{i=1}^{n} x_i \geq \frac{kn^{k+1}-(k+1)n^k+1}{(n-1)n^k} \sqrt[n]{\prod_{i=1}^{n} x_i} + \sum_{j=1}^{k} \frac{1}{n^j} \sqrt[j+1]{\frac{1}{n} \sum_{i=1}^{n} x_i^{j+1}}.$$

Mihály Bencze

PP. 16259. If $a_i > 0$ $(i = 1, 2, ..., n)$ and $k \in N$, then

$$\sum_{cyclic} \frac{a_1^{k+1}}{a_1^k + ... + a_{n-1}^k} \geq \frac{1}{n-1} \sum_{i=1}^{n} a_i.$$

Mihály Bencze

PP. 16260. Prove that $\frac{n(n+1)}{2} \geq (n-1)\sqrt[n]{n!} + \sqrt{\frac{n(n+1)(2n+1)}{6}}$ for all $n \in N^*$.

Mihály Bencze

PP. 16261. In all triangle ABC holds $\sum m_a^3 \left(m_b - m_c\right) m_b \geq 0$.

Mihály Bencze

PP. 16262. Solve the following equation:
$$\sum_{k=1}^{n} \sqrt{x_k} = \sqrt{x_1 + x_2 - x_3} + \sqrt{x_2 + x_3 - x_4} + ... + \sqrt{x_n + x_1 - x_2}.$$

Mihály Bencze

PP. 16263. Solve the following system: $\begin{cases} \sqrt{x} + \sqrt{y} - \sqrt{z} \geq \sqrt{x+y-z} \\ \sqrt{y} + \sqrt{z} - \sqrt{x} \geq \sqrt{y+z-x} \\ \sqrt{z} + \sqrt{x} - \sqrt{y} \geq \sqrt{z+x-y} \end{cases}$.

Mihály Bencze

PP. 16264. If $a, a_k \geq 1$ $(k = 1, 2, ..., n)$ and $\sum\limits_{k=1}^{n} a_k = na$, then

$$\sum_{k=1}^{n} \sqrt[m-1]{ma_k^{m-1} - m + 1} \leq n\left(ma - m + 1\right) \text{ for all } m \in N,\ m \geq 2.$$

Mihály Bencze

PP. 16265. In all triangle ABC holds

1). $\sum \left|\frac{a^3-b^3}{a+b}\right| \geq \left|\left(a-b\right)\left(b-c\right)\left(c-a\right)\right| \frac{s^2+r^2+4Rr}{2s(s^2+r^2+2Rr)}$

2). $4R\sum \left|\frac{r_a^3-r_b^3}{r_a+r_b}\right| \geq \left|\left(r_a-r_b\right)\left(r_b-r_c\right)\left(r_c-r_a\right)\right|$

3). $\sum \left|\frac{\sin^6\frac{A}{2}-\sin^6\frac{B}{2}}{\sin^2\frac{A}{2}+\sin^2\frac{B}{2}}\right| \geq$

$\left|\left(\sin^2\frac{A}{2}-\sin^2\frac{B}{2}\right)\left(\sin^2\frac{B}{2}-\sin^2\frac{C}{2}\right)\left(\sin^2\frac{C}{2}-\sin^2\frac{A}{2}\right)\right| \frac{2R\left(s^2+r^2-8Rr\right)}{(2R-r)(s^2+r^2-8Rr)-2Rr^2}$

4). $\left|\frac{\cos^6\frac{A}{2}-\cos^6\frac{B}{2}}{\cos^2\frac{A}{2}+\cos^2\frac{B}{2}}\right| \geq$

$\geq \left|\left(\cos^2\frac{A}{2}-\cos^2\frac{B}{2}\right)\left(\cos^2\frac{B}{2}-\cos^2\frac{C}{2}\right)\left(\cos^2\frac{C}{2}-\cos^2\frac{A}{2}\right)\right| \frac{2R\left(s^2+\left(4R+r\right)^2\right)}{\left(4R+r\right)^3+s^2(2R+r)}$

Mihály Bencze

PP. 16266. If $a_k > 0$ $(k = 1, 2, ..., n)$, then

$$\prod_{k=1}^{n}\left(3\sqrt{\left(a_k^4+1\right)^3} - 2\sqrt{2}a_k^3\right) \geq \left(2\sqrt{2}\right)^n \left(1 + \left(\prod_{k=1}^{n} a_k\right)^{\frac{6}{n}}\right)^n.$$

Mihály Bencze

PP. 16267. If $a_k \geq 1$ $(k = 1, 2, ..., n)$ and $\sum\limits_{k=1}^{n} a_k = na$, $a \geq 1$, then

$$n^2a + 4\sum_{k=1}^{n} \frac{1}{1+a_k^n} \geq n\left(n+2\right).$$

Mihály Bencze

PP. 16268. If $a_k > 0$ $(k = 1, 2, ..., n)$ and $\sum\limits_{k=1}^{n} a_k^3 = n$, then

$$\sum_{k=1}^{n}\left(a_k^2+1\right)\sqrt{a_k^2 - a_k + 1} \leq 2n.$$

Mihály Bencze

PP. 16269. If $a_k > 0$ $(k = 1, 2, ..., n)$, then
$$\prod_{k=1}^{n}\left(a_k + \frac{1}{a_k} - \sqrt{\frac{1}{2}\left(a_k^2 + \frac{1}{a_k^2}\right)}\right) \geq 1.$$

Mihály Bencze

PP. 16270. If $a_k > 0$ $(k = 1, 2, ..., n)$ and $\sum\limits_{k=1}^{n} a_k = n$, then $\sum\limits_{k=1}^{n} \frac{\left(a_k^2+1\right)^2}{a_k^3+1} \leq 2n.$

Mihály Bencze

PP. 16271. If $a_k > 0$ $(k = 1, 2, ..., n)$ and $\sum\limits_{k=1}^{n} a_k^4 = n$, then
$$\sum_{k=1}^{n} \sqrt[3]{\left(a_k^6 + a_k^3 + 1\right)^2} \leq n\sqrt[3]{9}.$$

Mihály Bencze

PP. 16272. If $a, b, c, d > 0$ and $abcd = 1$, then $\sum\limits_{cyclic} \frac{1}{2\sqrt{2}a^3+3\left(a^{\frac{4}{3}}+1\right)^{\frac{3}{2}}} \geq \frac{1}{2\sqrt{2}}.$

Mihály Bencze

PP. 16273. If $a_k > 0$ $(k = 1, 2, ..., n)$ and $\sum\limits_{k=1}^{n} a_k = n$, then
$$\sum_{k=1}^{n} \sqrt{a_k^2 - a_k + 1} + \sum_{k=1}^{n} \frac{a_k-1}{a_k^2+1} \leq n.$$

Mihály Bencze

PP. 16274. If $a_k > 0$ $(k = 1, 2, ..., n)$ then
$$\prod_{k=1}^{n}\left(a_k^4 + 1\right) \geq \left(\frac{8}{9}\right)^{\frac{n}{3}}\left(1 + \left(\prod_{k=1}^{n} a_k\right)^{\frac{3}{n}} + \left(\prod_{k=1}^{n} a_k\right)^{\frac{6}{n}}\right)^{\frac{2n}{3}}.$$

Mihály Bencze

PP. 16275. If $a_k > 0$ $(k = 1, 2, ..., n)$ then $\frac{\sum\limits_{k=1}^{n} a_k^3}{\sum\limits_{k=1}^{n} a_k^2} \geq \sqrt{\frac{\sum\limits_{k=1}^{n} a_k^3}{\sum\limits_{k=1}^{n} a_k}} \geq \sqrt[4]{\frac{1}{n}\sum\limits_{k=1}^{n} a_k^4}.$

Mihály Bencze

PP. 16276. If $a_k > 0$ $(k = 1, 2, ..., n)$ and $\sum\limits_{k=1}^{n} a_k^4 = n$, then
$n + 3\sum\limits_{k=1}^{n} a_k^2 \geq 2\left(\sum\limits_{k=1}^{n} a_k + \sum\limits_{k=1}^{n} a_k^3\right).$

Mihály Bencze

PP. 16277. If $a, x_k > 0$ $(k = 1, 2, ..., n)$ then

1). $\sum\limits_{cyclic} \frac{x_1}{\sqrt{x_2(ax_1+x_3)}} \geq \frac{n}{\sqrt{a+1}}$
2). $\frac{a+1}{n^2}\left(\sum\limits_{k=1}^{n} x_k\right)^3 \geq a\sum x_1^2x_2 + \sum x_1x_2x_3.$

Mihály Bencze

PP. 16278. If $x, y, a_k > 0$ $(k = 1, 2, ..., n)$, then $\sum\limits_{cyclic} \sqrt{\frac{a_1}{xa_1+ya_2}} \leq \frac{n}{\sqrt{x+y}}.$

Mihály Bencze

PP. 16279. In all triangle ABC holds $\sum \sqrt{ctg\frac{A}{2}\left(1 + \frac{r}{s}ctg\frac{A}{2}\right)} \geq \sqrt{\frac{2s}{r}}.$

Mihály Bencze

PP. 16280. In all triangle ABC holds

1). $\prod\left(s^2 + r^2ctg^2\frac{A}{2}\right) \geq \frac{15s^6}{4}$
2). $\sum \frac{tg^4\frac{A}{2}tg\frac{B}{2}}{tg^3\frac{A}{2}+tg^3\frac{C}{2}} \geq \frac{1}{2}$

Mihály Bencze

PP. 16281. If $a_k > 0$ $(k = 1, 2, ..., n)$ and $\prod\limits_{k=1}^{n} a_k = 1$, then
$\prod\limits_{cyclic}(a_1 + a_2) \geq \prod\limits_{k=1}^{n}(a_k + 1).$

Mihály Bencze

PP. 16282. If $a_k \in (-1, 1)$ $(k = 1, 2, ..., n)$, then $\sum\limits_{k=1}^{n} \frac{1}{1-a_k^n} \geq \frac{n}{1-\prod\limits_{k=1}^{n} a_k}.$

Mihály Bencze

PP. 16283. If $a, b, c \in R$ and $a+b+c=0$, then

$$\sqrt{3\sqrt{6}}\left|\sum ab\right| \geq \sum \left|a\left(b-c\right)\right|.$$

Mihály Bencze

PP. 16284. Find and draw the domain $D \subseteq R^2$, on which the following limit: $\lim\limits_{\substack{x\to 0\\ y\to 0}} \frac{x^3}{x+y}$, exists.

Laurenţiu Modan

PP. 16285. Let F_n, L_n denote the n^{th} Fibonacci respective Lucas numbers.

1). Determine all $k \in N$ for which $F_{kn} + F_n^k$ is divisible by k for all $n \in N$.
2). Determine all $p \in N$ for which $L_{pn} + L_n^p$ is divisible by p for all $n \in N$.

Mihály Bencze

PP. 16286. If $x_k \geq 1$ $(k = 1, 2, ..., n)$, $\alpha \geq 1$ and $\sum\limits_{k=1}^{n} x_k = 2n$, then

$$\prod_{k=1}^{n} x_k^{x_k^\alpha} \geq 2^{n2^\alpha}.$$

Mihály Bencze

PP. 16287. If $a_k > 0$ $(k = 1, 2, ..., n)$, then

$$\sqrt[n]{\prod_{k=1}^{n}\left(1+a_k^n\right)} \geq \frac{1}{n}\left(\sum_{k=1}^{n} a_k + \left(\sum_{k=1}^{n}\frac{1}{a_k}\right)\prod_{k=1}^{n} a_k\right).$$

Mihály Bencze

PP. 16288. If $a_i > 0$ $(i = 1, 2, ..., n)$ and $A = 2\sum\limits_{1\leq i<j\leq n} a_i a_j$, then

$$\sum_{i=1}^{n}\frac{a_i}{\sqrt[k]{a_i+\frac{A}{a_i}}} \geq \left(\sum_{i=1}^{n} a_i\right)^{\frac{k-1}{k}}.$$

Mihály Bencze

PP. 16289. If $a, b, c, \lambda > 0$, then

$$\left(a^{\lambda+1} + a^{\lambda}b^{\lambda} + b^{\lambda+1}\right)\left(b^{\lambda+1} + b^{\lambda}c^{\lambda} + c^{\lambda+1}\right)\left(c^{\lambda+1} + c^{\lambda}a^{\lambda} + a^{\lambda+1}\right) \geq$$
$$\geq \left(\sum (ab)^{\frac{2\lambda+1}{3}}\right)^3.$$

Mihály Bencze

PP. 16290. If $a, b, c, \lambda > 0$ then

$$\sum a^{\lambda+1}\left(b^{\lambda} + c^{\lambda}\right) \geq \sum (ab)^{\frac{2\lambda+1}{3}} \sqrt[3]{(a+b)\left(b^{\lambda}+c^{\lambda}\right)\left(c^{\lambda}+a^{\lambda}\right)}.$$

Mihály Bencze

PP. 16291. If $x_k > 0$ $(k = 1, 2, ..., n)$ then

$$\prod_{k=1}^{n}\left(2 + x_k + \frac{1}{x_k}\right) \geq \left(1 + \frac{n}{\sum\limits_{k=1}^{n} x_k}\right)^n \left(1 + \frac{n}{\sum\limits_{k=1}^{n} \frac{1}{x_k}}\right)^n.$$

Mihály Bencze

PP. 16292. If $a_i > 0\,(i = 1, 2, ..., n)$ and $k \in N^*$ $(k \geq 2)$, then

$$\sum_{cyclic} \frac{a_1}{k-1+a_2^k} \geq \frac{1}{k-1}\sum_{i=1}^{n} a_i - \frac{1}{k(k-1)}\sum_{cyclic} a_1 a_2^{k-1}.$$

Mihály Bencze

PP. 16293. If $a_k > 0\,(k = 1, 2, ..., n)$, then $\sum \frac{a_1^4}{a_1^2+a_2^2} \geq \sum\limits_{k=1}^{n} a_k^2 - \frac{1}{2}\sum a_1 a_2$.

Mihály Bencze

PP. 16294. If $a_i > 0\,(i = 1, 2, ..., n)$, $k \in N^*$ $(k \geq 2)$ then

$$\sum_{i=1}^{n} \frac{1}{a_i^k + k - 1} \geq \frac{n}{k-1} - \frac{1}{k(k-1)}\sum_{i=1}^{n} a_i^{k-1}.$$

Mihály Bencze

PP. 16295. If $a_i > 0\,(i = 1, 2, ..., n)$ and $k \in N$, then

$$\sum_{cyclic} \frac{a_1^k a_2^k}{a_1^{2k+1} + a_2^{2k+1} + a_1^k a_2^k (a_3 + a_4 + ... + a_n)} \leq \frac{n}{\sum\limits_{i=1}^{n} a_i}.$$

Mihály Bencze

PP. 16296. In all triangle ABC holds

$$\prod\left(\frac{a+b-c}{ab}\right)^{ab}\le\left(\frac{2s}{s^2+r^2+4Rr}\right)^{s^2+r^2+4Rr}.$$

Mihály Bencze

PP. 16297. If $a,b,c,\lambda>0$ and $k\in N$, $k\ge 2$, then

1). $\sum\frac{a}{\sqrt[k]{\lambda+b+c}}\ge\sqrt[k]{\frac{\left(\sum a\right)^{k+1}}{\lambda\sum a+2\sum ab}}$

2). $\sum\frac{a}{\sqrt[k]{a^2+\lambda bc}}\ge\sqrt[k]{\frac{\left(\sum a\right)^{k+1}}{\sum a^3+3\lambda abc}}$

3). $\sum\frac{a}{\sqrt[k]{a+2b}}\ge\left(\sum a\right)^{\frac{k-1}{k}}$

3). $\sum\frac{a}{\sqrt[k]{a^2+3ab+3b^2+2bc}}\ge\left(\sum a\right)^{\frac{k-2}{k}}$

Mihály Bencze

PP. 16298. If $x\ge 1$, then in all triangle ABC holds

$$\sum\frac{a}{xa-b+c}\ge\frac{3}{x+1}+\frac{2s^2}{(x^2-1)s^2-(x^2-2x-3)(4R+r)r}.$$

Mihály Bencze

PP. 16299. If $a_i>0$ $(i=1,2,...,n)$, $k\in\{2,3,...,n\}$, $\lambda>0$, then

$$\sum_{cyclic}\frac{a_1^2}{a_1^2+\lambda(a_1+a_2+...+a_k)^2}\ge\frac{n}{1+\lambda k^2}.$$

Mihály Bencze

PP. 16300. If $a,b,c>0$ and $n\in N$ then $\sum\limits_{cyclic}\frac{a^{n-1}b^{n-1}}{a^{2n+1}+b^{2n+1}+a^nb^nc}\le\frac{1}{abc}$.

Mihály Bencze

PP. 16301. If $a_i>0$ $(i=1,2,...,n)$ and $k\in N$ then

$$\sum_{cyclic}\frac{a_1^ka_2^ka_3}{a_1^{2k+1}+a_2^{2k+1}+a_1^ka_2^k(a_3+a_4+...+a_n)}\le 1.$$

Mihály Bencze

PP. 16302. If $x, y, z > 0$,then

1). $\left(\sum x\right)\left(\sum \frac{x^2y^2}{x^5+x^2y^2z+y^5}\right) \leq 3$
2). $\left(\sum x\right)\left(\sum \frac{x^ny^n}{x^{2n+1}+x^ny^nz+y^{2n+1}}\right) \leq 3$ for all $n \in N$.

Mihály Bencze

PP. 16303. If $\lambda, x_i > 0$ $(i = 1, 2, ..., n)$, $a \geq 1$, $k \in \{2, 3, ..., n\}$, then

$$\sum_{cyclic} \frac{x_1^k}{(\lambda+x_2)(\lambda+x_3)...(\lambda+x_k)} \geq \left(\frac{k}{a} - \frac{k-1}{a\sqrt[k-1]{a}}\right)\sum_{i=1}^{n} x_i - \frac{n(k-1)\lambda}{a\sqrt[k-1]{a}}.$$

Mihály Bencze

PP. 16304. In all triangle ABC holds $\sum \frac{1}{\sqrt{m_a+m_b-m_c}} \geq \frac{6}{s^2-r^2-4Rr}$.

Mihály Bencze

PP. 16305. If $x_k \geq \alpha > 0$ $(k = 1, 2, ..., n)$ and $\sum\limits_{k=1}^{n} \frac{1}{x_k} = \frac{n-1}{\alpha}$, then
$\sum\limits_{k=1}^{n} \sqrt{x_k - \alpha} \leq \sqrt{\sum\limits_{k=1}^{n} x_k}$.

Mihály Bencze

PP. 16306. If $a, b, c > 0$ and $\alpha \in [-1, 1]$, then

$$\sum \sqrt{\left(a^2 - 2\alpha ab + b^2\right)\left(b^2 - 2\alpha bc + c^2\right)} \geq \sum a^2 + (1 - 2\alpha)\sum ab.$$

Mihály Bencze

PP. 16307. If $a_k > 0$ $(k = 1, 2, ..., n)$, then $\sum\limits_{cyclic} \frac{a_1}{a_2^2+a_3^2+...+a_n^2} \geq \frac{4}{\sum\limits_{k=1}^{n} a_k}$.

Mihály Bencze

PP. 16308. If $a_k > 0$ $(k = 1, 2, ..., n)$, then $\sum\limits_{cyclic} \frac{a_1^3}{a_1^2+a_2^2} \geq \frac{1}{2}\sum\limits_{k=1}^{n} a_k$.

Mihály Bencze

PP. 16309. If $a_i > 0$ $(i = 1, 2, ..., n)$ and $k \in \{2, 3, ..., n-1\}$, then

$$\sum \frac{(a_1+a_2+...+a_k)^2}{a_1^2+a_2^2+...+a_k^2+ka_{k+1}^2} \le \frac{k(k-1)}{2}.$$

Mihály Bencze

PP. 16310. If $x_k > 0$ $(k = 1, 2, ..., n)$, then

$$\prod_{k=1}^{n} \left(x_k^2 + 4 + \frac{1}{x_k^2}\right) \le 6^n \cdot e^{\frac{2}{3}\left(\sum\limits_{k=1}^{n} x_k - 2n + \sum\limits_{k=1}^{n} \frac{1}{x_k}\right)}.$$

Mihály Bencze

PP. 16311. In all triangle ABC holds

1). $\sum a^5 b \le 2s^2\left(s^2 - 3r^2 - 6Rr\right)\left(s^2 - 3r^2 - 8Rr\right)$
2). $\sum (s-a)^5 (s-b) \le \frac{1}{2}s^2\left(s^2 - 12Rr\right)\left(s^2 - r^2 - 12Rr\right)$
3). $\sum r_a^5 r_b \le \frac{1}{2}\left((4R+r)^2 - 12s^2R\right)\left((4R+r)^3 - s^2r - 12s^2R\right)$

Mihály Bencze

PP. 16312. If $a, b, c > 0$, then
$$3\sum \frac{1}{\sqrt{4a^2+bc}} \ge 2\sqrt{2}\sum \frac{1}{\sqrt{(a+b)(2a+2b+c)}} + \sum \frac{1}{\sqrt{(a+b)^2+4c^2}}.$$

Mihály Bencze

PP. 16313. If $x > 0$ and $n, k \in N^*$, $n \ge 2$, then

$$((k+1)!)^{n-2}\left(x^n+1\right)^2\left(x^n+2\right)^2 ... \left(x^n+k\right)^2 \ge \left(x^2 + \sqrt[k]{k!}\right)^{nk}.$$

Mihály Bencze

PP. 16314. If $x_k \in [1, 2]$ $(k = 1, 2, ..., n)$, then

$$\prod_{k=1}^{n} \left(x_k^2 + 2\right) \ge 3^n \cdot e^{\frac{2}{3}\left(\sum\limits_{k=1}^{n} x_k - n\right)}.$$

Mihály Bencze

PP. 16315. If $a, b, c, x, y \in R$, then
$2\sum a^6 + x^2\sum a^4b^2 + y^2\sum a^2b^4 + 2xabc\sum a^2b + 2yabc\sum ab^2 \geq$

$\geq 2\sum a^3b^3 + 2x\sum a^5b + 2y\sum ab^5 + 6xya^2b^2c^2.$

Mihály Bencze

PP. 16316. If $x, y \in \left[0, \frac{\pi}{2}\right]$, then

$$\sqrt{\left(2+\sin^2 x\right)\left(2+\cos^2 y\right)} + \sqrt{\left(2+\cos^2 x\right)\left(2+\sin^2 y\right)} \geq \frac{9}{2} + \frac{1}{2}\sin\left(x+y\right).$$

Mihály Bencze

PP. 16317. If $a_k > 0$ $(k = 1, 2, ..., n)$, then
$$\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} \frac{1}{a_k}\right) \geq n^2 + \frac{8(n-1)}{n}\left(1 - \left(\frac{\sqrt[n]{\prod\limits_{k=1}^{n} a_k}}{\frac{1}{n}\sum\limits_{k=1}^{n} a_k}\right)^n\right).$$

Mihály Bencze

PP. 16318. If $x_i > 0$ $(i = 1, 2, ..., n)$ and $\sum\limits_{i=1}^{n} x_i = 1$, then
$\sum\limits_{cyclic} \sqrt{x_1 + x_2^2} \geq \sqrt{n+1}.$

Mihály Bencze

PP. 16319. If $a, b, c, x, y \in R$, then
$\sum a^4 + \left(\frac{x^2+y^2}{2} - 1\right)\sum a^2b^2 \geq (x+y)\sum a^3b + (xy - x - y)\, abc\sum a \geq 0.$

Mihály Bencze

PP. 16320. In all triangle ABC holds $\frac{2\left(s^2-r^2-Rr\right)}{s^2+r^2+2Rr} + \frac{s^2+r^2+4Rr}{2(s^2-r^2-4Rr)} \leq \frac{5}{2}$.

Mihály Bencze

PP. 16322. If $a, b, c > 0$ then $\sum \frac{1}{\sqrt{a^2+bc}} \geq \frac{9}{\sqrt{2}(a+b+c)}$.

Mihály Bencze

PP. 16322. Let M be a random point in the plane of the triangle ABC.

Prove that $\sum MA \cdot MB \left(MA^2 + MB^2\right) \geq 3AB \cdot BC \cdot CA \cdot MG$, where G is the centroid of the given triangle.

Mihály Bencze

PP. 16323. If $a_i > 0$ $(i = 1, 2, ..., n)$ and $\sum\limits_{cyclic} \frac{a_1+a_2}{a_1^2+a_2^2} = 1$, then $\sum\limits_{cyclic} \frac{a_1^3+a_2^3}{a_1^4+a_2^4} \leq 1$.

Mihály Bencze

PP. 16324. If ABC is a triangle and M is a random point in his plane, then $\sum \frac{MA}{MB} + \sum \frac{MB}{MA} \geq \frac{AB \cdot BC \cdot CA}{MA \cdot MB \cdot MC}$.

Mihály Bencze

PP. 16325. If $x, y, z > 0$ then $x^3 + y^3 + z^3 - 3xyz \geq \frac{1}{6} \left(\sum x\right) \left(\sum |x - y|\right)^2$.

Mihály Bencze

PP. 16326. If $a, b, c > 0$ then
1). $\left(\sum a^2\right)^2 \geq 3 \sum a^3 b + \frac{1}{6} \left(\sum \left|a^2 - 2ab + bc - c^2 + ca\right|\right)^2$.
2). $\left(\sum a^2\right)^2 \geq 3 \sum a^3 b + \frac{1}{18} \left(\sum \left|a^2 - 2b^2 + c^2 + 3bc - 3ca\right|\right)^2$

Mihály Bencze

PP. 16327. If $a, b, c > 0$, then $\frac{3abc}{\sum a} + \sum a^2 \geq \sum ab + \frac{1}{9} \left(\sum a\right)^2$.

Mihály Bencze

PP. 16328. If $a, b, c \in [0, 1]$ such that $\sum ab(a + b) = 2$, then $\sum a\sqrt{1 - b^3} + \sum a\sqrt{1 + c^3} \leq 1 + 2 \sum a$.

Mihály Bencze

PP. 16329. If $a, b, c > 0$, then
$(a + b)(b + c)(c + a) \geq 8abc + \frac{1}{3} \left(\sum |a - b| \sqrt{c}\right)^2$.

Mihály Bencze

PP. 16330. If $a, b, c > 0$ then $\sum \frac{a+1}{b+1} + \sum \frac{a+b^2}{a+bc} + \sum \frac{a+bc}{a+c^2} \leq 3 \sum \frac{a}{b}$.

Mihály Bencze

PP. 16331. If $a, b, c, d > 0$ and $a + b + c + d = 4$, then

$\sum a^2 \geq 4 + \frac{1}{4} \left(\sum |a-1|\right)^2$.

Mihály Bencze

PP. 16332. If $a_k > 0$ $(k = 1, 2, ..., n)$ and $\sum\limits_{cyclic} a_1 a_2 = 1$, then

$\sum\limits_{cyclic} \frac{1}{a_1+a_2} \geq \frac{n\sqrt{n}}{2}$.

Mihály Bencze

PP. 16333. In all triangle ABC

1). $\frac{2\left(s^2-r^2-Rr\right)}{s^2+r^2+2Rr} + \frac{3\sqrt[3]{4sRr}}{4s} \geq 2$

2). $\frac{s^2+r^2-8Rr}{4Rr} + \frac{3\sqrt[3]{sr^2}}{2s} \geq 2$

Mihály Bencze

PP. 16334. If $x, y, z > 0$ and $\sum \frac{1}{x+y} = 1$, then $\sum \frac{xy}{x^3+y^3} \leq 1$.

Mihály Bencze

PP. 16335. If $x_k > 0$ $(k = 1, 2, ..., n)$, then

$2^n \prod\limits_{k=1}^{n} \left(x_k - \sqrt{x_k} + 1\right)^2 \geq \left(1 + \sqrt[n]{\prod\limits_{k=1}^{n} x_k^2}\right)^n$.

Mihály Bencze

PP. 16336. If $a, b, c > 0$ and $f : R \to R$ is a convex function, then

1). $\sum af(a+2b) \geq \left(\sum a\right) f\left(\sum a\right)$

2). $\sum af\left(a^2+3ab+3b^2+2bc\right) \geq \left(\sum a\right) f\left(\left(\sum a\right)^2\right)$

Mihály Bencze

PP. 16337. In all acute triangle ABC holds $\frac{\sum(1+\sin A)^3}{\prod(1+\sin A)} + \frac{\sum(1+\cos A)^3}{\prod(1+\cos A)} \leq 10.$

Mihály Bencze

PP. 16338. If $x \in R$ and $a \geq b \geq 0$, then

$$(tgx)^{2a} + (ctgx)^{2a} \geq (tgx)^{2b} + (ctgx)^{2b}.$$

Mihály Bencze

PP. 16339. Let ABC be a triangle

1). If $A, B, C \in \left(\frac{\pi}{6}, \frac{\pi}{2}\right]$, then $(1-\sin A)(1-\sin B)(1-\sin C) \leq \frac{r(3R-s)^3}{2s^2R^2}$
2). If $A, B, C \in \left(0, \frac{\pi}{3}\right)$, then
$(1-\cos A)(1-\cos B)(1-\cos C) \leq \frac{\left(s^2-(2R+r)^2\right)(2R-r)^3}{4R^2(R+r)^3}.$

Mihály Bencze

PP. 16340. Let ABC be a triangle

1). If $A, B, C \in \left(\frac{\pi}{6}, \frac{\pi}{2}\right]$, then $\prod\left(\frac{\sin A+\sin B}{2-\sin A-\sin B}\right)^2 \leq \frac{2s^4r}{(3R-s)^3\left((2R+r)^2+s(s-2r-4R)\right)}$
2). If $A, B, C \in \left(0, \frac{\pi}{3}\right)$, then $\prod\left(\frac{\cos A+\cos B}{2-\cos A-\cos B}\right)^2 \leq \frac{(R+r)^3\left(s^2-(2R+r)^2\right)}{(2R-r)^3(3s^2+5r^2-12R^2)}$

Mihály Bencze

PP. 16341. If $a_k > 0$ $(k = 1, 2, ..., n)$, $\alpha > 0$, then

$$\sum_{cyclic} \frac{a_1^{\alpha+1}}{a_2+a_3+...+a_n} \geq \frac{n^{1-\alpha}}{n-1}\left(\sum_{k=1}^{n} a_k\right)^{\alpha}.$$

Mihály Bencze

PP. 16342. If $x_k > 0$ $(k = 1, 2, ..., n)$, $a_i, b_i, c_i, d_i > 0$ $(i = 1, 2)$, $a_1d_1 \geq b_1c_1$; $a_2d_2 \leq b_2c_2$ then
$n\sum_{k=1}^{n} \frac{a_1a_2x_k^2+(a_1b_2+a_2b_1)x_k+b_1b_2}{c_1c_2x_k^2+(c_1d_2+c_2d_1)x_k+d_1d_2} \leq \left(\sum_{k=1}^{n} \frac{a_1x_k+b_1}{c_1x_k+d_1}\right)\left(\sum_{k=1}^{n} \frac{a_2x_k+b_2}{c_2x_k+d_2}\right).$

Mihály Bencze

PP. 16343. If $a_i > 0$ $(i = 1, 2, ..., n)$, $f_i : R_+^n \to (0, +\infty)$ $(i = 1, 2, ..., n)$ such that $\sum\limits_{i=1}^{n} a_i f_i(a_1, a_2, ..., a_n) = \left(\sum\limits_{i=1}^{n} a_i\right)^p$, $p, k \in N^*$, then
$\sum\limits_{i=1}^{n} \frac{a_i}{\sqrt[k]{f_i(a_1, a_2, ..., a_n)}} \geq \left(\sum\limits_{i=1}^{n} a_i\right)^{\frac{k-p+1}{k}}$.

Mihály Bencze

PP. 16344. If $a, b, c, \lambda > 0$, then $\sum\limits_{cyclic} \frac{a}{\sqrt{b^2+\lambda ac}} \geq \frac{3}{2} - \frac{(\lambda+1)\sum ab^2}{2\left(\sum a\right)^3}$.

Mihály Bencze

PP. 16345. If $a_i > 0$ $(i = 1, 2, ..., n)$, $\lambda > 0$, $k \in N^*$, then
$\sum\limits_{cyclic} \frac{a_1}{\sqrt[k]{a_2^k+\lambda a_1 a_3 a_4 ... a_n}} \geq \frac{k+1}{k} - \frac{\sum a_1 a_2^k + \lambda \sum a_1^2 a_3 a_4 ... a_n}{k\left(\sum\limits_{i=1}^{n} a_i\right)^{k+1}}$.

Mihály Bencze

PP. 16346. Let a random variable $X \in Poisson\,(\lambda)$, $\lambda > 0$, be. Let also a random variable Y, be, having $m_Y = m_X$ and $\sigma_Y = \sigma_X^2$.
In a crop land, the production is the random variable Y. A selection of this production in tons, from 9 agriculture surfices of the land, is given by:
$y_1 = y_2 = y_3 = 5$; $y_4 = y_5 = 4,5$; $y_6 = y_7 = 6,5$; $y_8 = y_9 = 4$.
Find the parameter $\lambda > 0$, so that the confidence interval fro m_Y, at a confidence level $\alpha = 0,95$, will be $\left(\frac{1}{100}, 10\right)$.

Laurenţiu Modan

PP. 16347. Prove that $\sum\limits_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^n = n!$

Mihály Bencze

PP. 16348. Determine all $n, k \in N^*$ such that $\left[\frac{n}{k}\right] \equiv \left[\frac{n}{k}\right] \pmod{k}$, where $[\cdot]$ denote the integer part.

Mihály Bencze

PP. 16349. Prove that $\sum\limits_{k=0}^{n} \left(\binom{2n}{k}\right) = 2^{2n-1} + \frac{n+1}{2} \sum\limits_{k=0}^{\left[\frac{n}{2}\right]} \left(\binom{n}{k} - \binom{n}{k-1}\right)^2$.

Mihály Bencze

PP. 16350. If $x \geq 0$, then $\sum\limits_{k=1}^{n} \sqrt{[kx]} \leq \sqrt{\frac{2[nx]}{n(n+1)}}$, where $[\cdot]$ denote the integer part.

Mihály Bencze

PP. 16351. Prove that
$\sum\limits_{k=1}^{n} \left(\left[\sqrt{k^2 - \frac{3}{4}} + \frac{1}{2} \right] + \left[\sqrt{k^2 + 2k + \frac{1}{4}} + \frac{1}{2} \right] \right) = n\left(n+2\right)$, where $[\cdot]$ denote the integer part.

Mihály Bencze

PP. 16352. Prove that $\prod\limits_{k=1}^{p-1} \binom{p-1}{k} \equiv (-1)^{\frac{p(p-1)}{2}} \pmod{p}$, for all prime $p \geq 3$.

Mihály Bencze

PP. 16353. If $x, y, z \in R$, then $\sum \frac{e^{x+y}}{e^{2z}+2e^{x+y}} \leq 1 \leq \sum \frac{e^{2x}}{e^{2x}+2e^{y+z}}$.

Mihály Bencze

PP. 16354. In all triangle ABC holds:

1). $2\sum \sin\frac{A}{2}\sin\frac{B}{2} \leq \frac{5(2R-r)}{6R} + \sqrt[3]{\left(\frac{r}{4R}\right)^2}$

2). $2\sum \cos\frac{A}{2}\cos\frac{B}{2} \leq \frac{5(4R+r)}{6R} + \sqrt[3]{\left(\frac{s}{4R}\right)^2}$

Mihály Bencze

PP. 16355. In all triangle ABC holds:

1). $16r\left(4R+r\right) \leq 4s^2 + 3\sqrt[3]{\left(4sRr\right)^2}$

2). $16r\left(4R+r\right) \leq 5s^2 + 3r\sqrt[3]{rs^2}$

3). $16s^2 \leq 5\left(4R+r\right)^2 + 3s\sqrt[3]{sr^2}$

Mihály Bencze

PP. 16356. If $a, b, c, x, y, z > 0$, then
$\frac{xa^2+yb^2+zc^2}{bc} + \frac{xb^2+yc^2+za^2}{ca} + \frac{xc^2+ya^2+zb^2}{ab} \geq 3\left(x+y+z\right)$.

Mihály Bencze

PP. 16357. Prove that:

1). $\sum\limits_{1\le i<j\le n} ij\left(i^2+j^2\right) \le \frac{n^4(n+1)^4}{128}$

2). $\sum\limits_{1\le i<j\le n} i^2j^2\left(i^4+j^4\right) \le \frac{n^4(n+1)^4(2n+1)^4}{10368}$

3). $\sum\limits_{1\le i<j\le n} i^3j^3\left(i^6+j^6\right) \le \frac{n^8(n+1)^8}{1728}$

Mihály Bencze

PP. 16358. If $x_{n+3}=ax_{n+2}+bx_{n+1}+cx_n$ for all $n\in N^*$ and $x_1=x_2=1$, $x_3=4$, then determine all $a,b,c\in Z$ for which x_n is a perfect square for all $n\in N$.

Mihály Bencze

PP. 16359. If ABC is a triangle with positive integer sides, then the equation $x^2-\left(2\left(s^2-r^2-4Rr\right)+1\right)x+s^2+r^2+4Rr=0$ have integer roots if and only if ABC is equilateral.

Mihály Bencze

PP. 16360. If $a_i>0$ $(i=1,2,...,n)$ and $k\in N$, then $\sum\limits_{cyclic}\frac{a_1^{k+2}+a_2^{k+2}}{a_1a_2}\ge 2\sum\limits_{i=1}^n a_i^k$.

Mihály Bencze

PP. 16361. Determine all $x_k>0$ $(k=1,2,...,n)$ such that $\prod\limits_{cyclic}(x_1-x_2+x_3)\le\prod\limits_{k=1}^n x_k$.

Mihály Bencze

PP. 16362. Prove that $\sum\limits_{k=1}^n\frac{k}{2k+3}\ge\frac{(n+1)(n+6)}{12(2n+3)}$.

Mihály Bencze

PP. 16363. In all triangle ABC holds $\sum\sqrt{a}\ge\frac{2(4R+r)\sqrt{r}}{\sqrt{sR}}$.

Mihály Bencze

PP. 16364. Solve in N the equation $nx^n + x = (n-1)\,y^{n+1} + 2y$, where $n \in N$ is given.

Mihály Bencze

PP. 16365. Determine all $k \in N$ for which $n + k$ and $n^k + n + 1$ are perfect $k + 1$ powers, for all $n \in N$.

Mihály Bencze

PP. 16366. In all triangle ABC holds

1). $512s^2Rr \le 27\left(s^2 + r^2 + 2Rr\right)^2$
2). $4r\,(4R + r)^3 \le 27s^2R^2$
3). $512Rr^2\,(2R - r)^3 \le 27\left((2R - r)\left(s^2 + r^2 - 8Rr\right) - 2Rr^2\right)^2$
4). $512Rs^2\,(4R + r)^3 \le \Big((4R + r)^3 + s^2\,(2R + r)\Big)^2$

Mihály Bencze

PP. 16367. Prove that $\left[\sum\limits_{k=1}^{n} \frac{1}{n+k} + \frac{1}{n}\sum\limits_{k=1}^{2^n-1} \frac{1}{k}\right] = 1$, where $[\cdot]$ denote the integer part.

Mihály Bencze

PP. 16368. For all $n \in N$ at least of $n, n+1, n+2, ..., kn^{k-1}$ is perfect k power, where $k \in N$, $k \ge 2$.

Mihály Bencze

PP. 16369. 1). Prove that the equation $x^2 + 3x + 3 = y^3$ have no solution in Z.
2). Solve in N the equation $x^k + (k+1)\,n + k + 1 = y^{k+1}$, where $k \in N$ is given.

Mihály Bencze

PP. 16370. Solve in N the equation $(kx + 1)^n - (kx - 1)^n = y^k$, when $k, n \in N$ are given.

Mihály Bencze

PP. 16371. Determine all $a_k \in Z$ $(k = 1, 2, ..., n)$ for which

$$a_1a_2 + a_2a_3 + ... + a_na_n = 1 \text{ and } \prod_{k=1}^{n} \left(1 + a_k^2\right) \text{ is perfect square.}$$

Mihály Bencze

PP. 16372. Determine all $n, x, y \in N$ such that $x^{n+1} + y^n + x^{n-1} + y^{n-2}$ and $y^{n+1} + x^n + y^{n-1} + x^{n-2}$ are perfect squares.

Mihály Bencze

PP. 16373. If $x \in R$ and $\lambda \in [0, 1]$, then

$$\left(1 - \sin^{2\lambda} x\right)\left(1 + \sin^2 x\right)^{1-\lambda} + \left(1 - \cos^{2\lambda} x\right)\left(1 + \cos^2 x\right)^{1-\lambda} \le 1.$$

Mihály Bencze

PP. 16374. If $x_k \in R$ $(k = 1, 2, ..., n)$, then

$$\sum_{k=1}^{n} \frac{\sin^2 x_k}{\cos^2 x_k + \sum\limits_{i=1}^{n} \sin^2 x_i} + \sum_{k=1}^{n} \frac{\cos^2 x_k}{\sin^2 x_k + \sum\limits_{i=1}^{n} \cos^2 x_i} \le 2 - \prod_{k=1}^{n} \sin^2 x_k - \prod_{k=1}^{n} \cos^2 x_k.$$

Mihály Bencze

PP. 16375. Determine all $x, y \in N$ for which $x^y + y^x$ is perfect square.

Mihály Bencze

PP. 16376. Determine all $x, y, z, t \in N$ such that $\binom{x}{y}\binom{z}{t} = \binom{x}{t}\binom{z}{y}$.

Mihály Bencze

PP. 16377. In all acute triangle ABC holds $\sum \frac{1}{4m_a^2 - a^2} \ge \frac{1}{4Rr}$.

Mihály Bencze

PP. 16378. In all triangle ABC holds

1). $\sum \frac{\sin^2 \frac{A}{2}}{1 - \frac{r}{2R} + \cos^2 \frac{A}{2}} \le 1 - \frac{s^2}{16R^2}$

2). $\sum \frac{\cos^2 \frac{A}{2}}{2 + \frac{r}{2R} + \sin^2 \frac{A}{2}} \le 1 - \frac{r^2}{16R^2}$

Mihály Bencze

PP. 16379. Let $A_1A_2...A_n$ be a convex polygon inscribed in a circle, and $M \in Int\left(A_1A_2...A_n\right)$. The lines A_kM $(k=1,2,...,n)$, intersect the circle in points B_k $(k=1,2,...,n)$. Prove that $Area\left[A_1A_2...A_n\right] \le Area\left[B_1B_2...B_n\right]$.

Mihály Bencze

PP. 16380. Denote G the centroid of triangle ABC. Prove that
1). $\sum tg\left(\frac{1}{2}AGB\right) \ge \frac{3(4R+r)}{s}$
2). $\sum tg\left(\frac{1}{2}AGB\right) tg\left(\frac{1}{2}BGC\right) \ge 9$
3). $tg\left(\frac{1}{2}AGB\right) tg\left(\frac{1}{2}BGC\right) tg\left(\frac{1}{2}CGA\right) \ge \frac{27r}{s}$

Mihály Bencze

PP. 16381. If $a_k > 0$ $(k=1,2,...,n)$ and $x,y>0$, then

$$\frac{\left(\sum\limits_{k=1}^{n} a_k\right)^{x+y} - \sum\limits_{k=1}^{n} a_k^{x+y}}{\left(\sum\limits_{k=1}^{n} a_k\right)^{x} - \sum\limits_{k=1}^{n} a_k^{x}} \ge \frac{n^{x+y-1}}{n^{x+y-1}-n^y}\left(\sum_{k=1}^{n} a_k\right)^y.$$

Mihály Bencze

PP. 16382. Solve the following system: $\begin{cases} e^x - e^{-y} = 2\ln\left(z+\sqrt{1+z^2}\right) \\ e^y - e^{-z} = 2\ln\left(x+\sqrt{1+x^2}\right) \\ e^z - e^{-x} = 2\ln\left(y+\sqrt{1+y^2}\right) \end{cases}$.

Mihály Bencze

PP. 16383. If $x \in R$, then $2\left(-5sh^2x+\sqrt{5}shx+1\right) < \sqrt{5}\left(5sh^2x+1\right)chx$.

Mihály Bencze

PP. 16384. Determine all $p \in N$ such that $p\sum\limits_{k=1}^{n}\binom{pk-1}{k} = \sum\limits_{k=1}^{n}\binom{pk}{k}$ for all $n \in N^*$.

Mihály Bencze

PP. 16385. If $0 < a \le b$, then $\sum\limits_{k=1}^{n}\left((b-k)^3-(a-k)^3\right) \ge \frac{n\left(n^2-1\right)(b-a)}{4}$.

Mihály Bencze

PP. 16386. Determine all prime p for which $3^p + p^3$ is prime too.

Mihály Bencze

PP. 16387. If $x, y, z \in R$, then
$\max\left(\sin^2 x, \sin^2(x+1)\right) + \max\left(\sin^2 y, \sin^2(y+1)\right) +$

$+\max\left(\sin^2 z, \sin^2(z+1)\right) > \frac{1}{3}$.

Mihály Bencze

PP. 16388. In all triangle ABC holds $\left(\prod \sin \frac{\pi - A}{4}\right)\left(\sum \cos \frac{A}{2}\right) = \frac{s}{8R}$.

Mihály Bencze

PP. 16389. If $a, b, c > 0$ and $x \in [0,1]$ then

$\left|\sum a^{2x}\left(b^x c - bc^x\right)\right| \le (abc)^x \sum |a-b|$.

Mihály Bencze

PP. 16390. If $t \ge n$, then
$1 + \sum\limits_{k=1}^{n} (-1)^k \binom{n}{k} \frac{(a+t+1)(a+t+2)...(a+t+k)}{(a+1)(a+2)...(a+k)} = \frac{(-1)^n t!}{(t-n)!(a+1)(a+2)...(a+n)}$ for all
$a \in C \setminus \{-n, ..., -2, -1\}$.

Mihály Bencze

PP. 16391. If $k, n \in N, n \ge 3, k \ge 2$, then $\left[\prod\limits_{i=1}^{n}\left(1 + \frac{1}{k^i}\right)\right] = 1$, where $[\cdot]$ denote the integer part.

Mihály Bencze

PP. 16392. If the triangle ABC is inscribed in a circle with radius 1, then
$AB + BC + CA \ge AB \cdot BC \cdot CA$.

Mihály Bencze

PP. 16393. Prove that $\left[\frac{1}{n}\sum\limits_{k=1}^{n}\frac{\sigma(k)}{k}\right] = 1$ for all $n \in N^*$, when $[\cdot]$ denote the integer part.

Mihály Bencze

PP. 16394. If $x > 0$, $a_k > 0$ $(k = 1, 2, ..., n)$, then
$$\prod_{k=1}^{n} \left(x^{na_k} - 1\right) \geq \frac{n^n \prod\limits_{k=1}^{n} a_k}{\left(\sum\limits_{k=1}^{n} a_k\right)^n} \left(x^{\sum\limits_{k=1}^{n} a_k} - 1\right)^n .$$

Mihály Bencze

PP. 16395. In all triangle ABC holds $\sum \left(\frac{\cos \frac{A-B}{2}}{\sin \frac{C}{2}}\right)^{\alpha} \geq \frac{3}{4^{\alpha}}$ for all $\alpha \geq 1$.

Mihály Bencze

PP. 16396. If in triangle ABC we have $m_b \perp m_c$ then
1). $3a^2 + r^2 + 4Rr = s^2$
2). $b + c \leq a\sqrt{10}$
3). $\frac{4}{5} + \cos^2 A \geq \frac{s^2 - (2R+r)^2}{2R^2}$

Mihály Bencze

PP. 16397. If $0 < a_1 < a_2 < ... < a_{n+1}$ is an arithmetical progression with ratio r, then $\ln \frac{a_{n+1}}{a_1} > 2r \left(\frac{1}{a_1+a_2} + \frac{1}{a_2+a_3} + ... + \frac{1}{a_n+a_{n+1}}\right)$.

Mihály Bencze

PP. 16398. Determine all $p \in N^*$ for which
$\binom{pn}{n} + \binom{pn+1}{n} + ... + \binom{(p+1)n}{n} < \binom{(p+1)n+1}{n+1}$ for all $n \in N^*$.

Mihály Bencze

PP. 16399. Determine the general term of the sequence $(x_n)_{n \geq 0}$ defined in following way: $x_n + 4x_{n-2} = 2^n \sin nx$, $x_0 = 1$, $x_1 = 2$.

Mihály Bencze

PP. 16400. In all triangle ABC holds $\sum \frac{-a+b+c}{2a+b+c} \geq 1 + \frac{r}{R}$.

Mihály Bencze

PP. 16401. If $x \in \left(0, \frac{\pi}{2}\right)$, $a, b > 0$ and exist $\alpha \geq 1$ such that
$\frac{\sin^{2\alpha+2} x}{a^{\alpha}} + \frac{\cos^{2\alpha+2} x}{b^{\alpha}} = \frac{1}{(a+b)^{\alpha}}$, then the given relation is true for all $\alpha \in R$.

Mihály Bencze

PP. 16402. Solve in N the following equation
$\frac{x_1+1}{x_2+2}+\frac{x_2+2}{x_3+3}+...+\frac{x_n+n}{x_1+1}=\frac{3n-1}{2}-[\ln n]$, when $[\cdot]$ denote the integer part.

Mihály Bencze

PP. 16403. In all triangle ABC holds $\sum\left(\frac{\cos\frac{A-B}{8}}{\cos\frac{A+B}{8}}\right)^{\alpha}\geq 3\left(\sqrt{6}-\sqrt{2}\right)^{\alpha}$, for all $\alpha\geq 1$.

Mihály Bencze

PP. 16404. In all triangle ABC holds $3\left(s^2-(2R+r)^2\right)\leq 2r(R+r)$.

Mihály Bencze

PP. 16405. Solve the following system:
$a^{x_1-1}+a^{\frac{1}{\sqrt{x_2}}}=a^{x_2-1}+a^{\frac{1}{\sqrt{x_3}}}=...=a^{x_n-1}+a^{\frac{1}{\sqrt{x_1}}}=a+1$, where $a>1$.

Mihály Bencze

PP. 16406. Solve in Z the equation $\left(x^4+y^4\right)\left(y^4+z^4\right)=x^4t^4$.

Mihály Bencze

PP. 16407. In all triangle ABC we have $\sum\frac{m_a}{\sqrt{h_aw_a}}\leq\frac{s^2+r^2+10Rr}{8Rr}$.

Mihály Bencze

PP. 16408. In all triangle ABC holds $\prod\left(tg\frac{A}{2}\right)^a\leq\left(\frac{2R-r}{s}\right)^{2s}$.

Mihály Bencze

PP. 16409. If $a_k>0$ $(k=1,2,...,n)$, then $3\sum_{k=1}^{n}\frac{1}{a_k}\geq\sum_{cyclic}\frac{(a_1+a_2+a_3)^2}{a_1^3+a_2^3+a_3^3}$.

Mihály Bencze

PP. 16410. If $a,b,c>0$ and $\lambda\geq 0$, then
1). $\sum\frac{\left(a^2+2bc\right)^2}{b+\lambda c}\geq\frac{1}{\lambda+1}\left(\sum a\right)^3$
2). $\sum\frac{\left(a^3+3b^2c+3ca^2+2abc\right)^2}{b+\lambda c}\geq\frac{1}{\lambda+1}\left(\sum a\right)^5$

Mihály Bencze

PP. 16411. Determine all function $f : R \to R$ such that
$\sum\limits_{k=1}^{n} f\left(\frac{k}{2}\right) = f\left(\frac{n}{2}\right) f\left(\frac{n+1}{2}\right)$ for all $n \in N^*$.

Mihály Bencze

PP. 16412. In all triangle ABC holds
1). $\sum \sin\frac{A}{2} \le \frac{2\left(s^2-r^2-Rr\right)}{s^2+r^2+2Rr}$
2). $\sum \sin\frac{A}{2}\sin\frac{B}{2} \le \frac{s^2+r^2-2Rr}{s^2+r^2+2Rr}$
3). $s^2+r^2+2Rr \le 8R^2$

Mihály Bencze

PP. 16413. If $x, y \in \left[0, \ln\left(1+\sqrt{2}\right)\right]$, then $\frac{1}{chx} + \frac{1}{chy} \ge \frac{2}{\sqrt{1+shxshy}}$.

Mihály Bencze

PP. 16414. Determine all $n \in N$ such that $\sum\limits_{k=1}^{n} k \cdot 10^{-k} < 0,123456789$.

Mihály Bencze

PP. 16415. If $x \in R$, then $\left(\frac{\sin x}{\sin^4 x+\cos^2 x} + \frac{\cos x}{\cos^4 x+\sin^2 x}\right)\sin^2 2x \le 2$.

Mihály Bencze

PP. 16416. If $x \ge 0$, then $\frac{shx}{1} \le \frac{sh2x}{2} \le \frac{sh3x}{3} < ... < \frac{shnx}{n}$.

Mihály Bencze

PP. 16417. If $x, y, z, a > 0$ such that $\sum\left(x^{n-2}-1\right)x^3 = 2a$, $n \in N$ then
$\sum \frac{x^n}{y^2+z^2} \ge 3a$.

Mihály Bencze

PP. 16418. If $a, b, c > 0$, then $\sum(a+b)(a+c) \ge 6\sqrt{abc(a+b+c)}$

Mihály Bencze

PP. 16419. If $n \in N^*$, then $\left[\frac{5}{4} + \sum\limits_{k=1}^{n}\frac{1}{n+k}\right] = 1$, where $[\cdot]$ denote the integer part.

Mihály Bencze

PP. 16420. If $a, b, c, d > 0$, then
$2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)+9\left(\frac{1}{a+b+c}+\frac{1}{b+c+d}+\frac{1}{c+d+a}+\frac{1}{d+a+b}\right)\geq$
$8\left(\frac{1}{a+b}+\frac{1}{a+c}+\frac{1}{a+d}+\frac{1}{b+c}+\frac{1}{b+d}+\frac{1}{c+d}\right).$

Mihály Bencze

PP. 16421. If $x_k > 0$ $(k = 1, 2, ..., n)$ and $\sum\limits_{k=1}^{n} x_k^{\alpha} = 1$, where $\alpha \in R$, then
$\sum\limits_{k=1}^{n} \frac{x_k^{\alpha+1}}{x_k^2+x_k+1} \leq \frac{1}{3}.$

Mihály Bencze

PP. 16422. If $a \neq b \neq c$, then $\sum \frac{a^4+b^4+c^4-b^3-c^3-\left(a^2+b^2\right)c-a^2b-bc^2+3bc}{(a-b)(a-c)} = 3.$

Mihály Bencze

PP. 16423. In all triangle ABC holds $2\sum \frac{m_a}{m_b} \geq 3+\sum \frac{m_a}{m_c}$.

Mihály Bencze

PP. 16424. If $\alpha, \beta \geq 1$ and $S_\alpha = \sum\limits_{k=1}^{n} k^\alpha$, then
$\frac{S_{2\alpha}S_{2\beta}}{\left(S_{\alpha+\beta}\right)^2} \leq \frac{1}{4}\left(n^{\frac{\alpha+\beta}{2}}+n^{-\frac{\alpha+\beta}{2}}\right)^2.$

Mihály Bencze

PP. 16425. If $x \neq y \neq z > 0$ then $\sum\left(\frac{\sqrt[3]{x^2}}{\sqrt[3]{x^2}-\sqrt[3]{yz}}\right)^2+\sum\left(\frac{x}{x-y}\right)^2 \geq 2.$

Mihály Bencze

PP. 16426. 1). Prove that $\left[\sum\limits_{k=1}^{n}\frac{1}{k^3}\right] = 1$ for all $n \in N^*$, where $[\cdot]$ denote the integer part.
2). Determine all $p \in N$ such that $\left[\sum\limits_{k=1}^{n}\frac{1}{k^p}\right] = 1$

Mihály Bencze

PP. 16427. Prove that $\sum\limits_{k=1}^{\infty} \frac{1}{k} arctg \frac{k(e-1)}{k^2+e} = \frac{\pi-1}{2} + \frac{1}{2e} + \pi \int\limits_1^e \frac{dx}{x(e^{2\pi x}-1)}$.

Mihály Bencze

PP. 16428. Determine all $x_k \in R^*$ $(k = 1, 2, ..., n)$ such that $\sum\limits_{cyclic} arctg \frac{1-x_1}{x_2} = \frac{\pi}{4}$.

Mihály Bencze

PP. 16429. Solve in N the equation $\frac{x_1 x_2}{x_1^2+x_2^2} + \frac{x_2 x_3}{x_2^2+x_3^2} + ... + \frac{x_n x_1}{x_n^2+x_1^2} = \frac{n}{2}$.

Mihály Bencze

PP. 16430. Solve in N the equation $nx^2 + \sum\limits_{k=1}^{n} (x+k)^2 = ny^2$.

Mihály Bencze

PP. 16431. If $x_1 = 1, x_2 = x_3 = 2$ and $95800^{x_n} + 217519^{x_{n+1}} + 414560^{x_{n+2}} = 422481^{x_{n+3}}$ for all $n \geq 1$, then the sequence $(x_n)_{n\geq 1}$ is convergent and $\lim\limits_{n\to\infty} x_n = 4$. Compute $\lim\limits_{n\to\infty} n(x_n - 4)$.

Mihály Bencze

PP. 16432. Prove that

1). $\sum\limits_{k=1}^{n} \frac{k^2}{k+1} \leq \min\left\{\frac{n(n+1)(2n+1)}{2(n+2)}; \frac{n^2(n+1)^2(2n+1)}{4(3n^2+7n+2)}\right\}$

2). $\sum\limits_{k=1}^{n} \frac{k^3}{k^2+1} \leq \frac{n^2(n+1)^2}{2(n^2+n+2)}$

Mihály Bencze

PP. 16433. If $n \in N^*$ and $k \in \{1, 2, ..., n\}$, then $k! + \frac{(k+1)!}{2!} + ... + \frac{n!}{(n-k)!} \leq \left(\frac{n(n+1)}{2k}\right)^k$.

Mihály Bencze

PP. 16434. If F_k denote the k^{th} Fibonacci number, then $\sum\limits_{k=1}^{n} \frac{F_k F_{k+1}}{F_{k+2}} \leq \frac{(F_{n+2}-1)(F_{n+3}-2)}{F_{n+4}-3}$.

Mihály Bencze

PP. 16435. If $a, b, c, x, y, z > 0$, then
$abc \leq \frac{(xa+yb+zc)(xb+yc+za)(xc+ya+zb)}{(x+y+z)^3} \leq \left(\frac{a+b+c}{3}\right)^3$.

Mihály Bencze

PP. 16436. If $a_k > 0$ $(k = 1, 2, ..., n)$ and $A_n = \frac{1}{n}\sum\limits_{k=1}^{n} a_k$, $G_n = \sqrt[n]{\prod\limits_{k=1}^{n} a_k}$,
$H_n = \frac{n}{\sum\limits_{k=1}^{n}\frac{1}{a_k}}$, then $A_n^2 + G_n^2 + H_n^2 - A_nG_n - G_nH_n - H_nA_n \geq \frac{(A_n+G_n-2H_n)^3}{4(A_n+G_n+H_n)}$,
for all $n \in N^*$.

Mihály Bencze

PP. 16437. If $x_k \geq 1$ $(k = 1, 2, ..., n)$, then
$\sum\limits_{k=1}^{n} x_k - n\sqrt[n+1]{\prod\limits_{k=1}^{n} x_k} \geq n\left(\frac{1}{n}\sum\limits_{k=1}^{n}\sqrt[n+1]{x_k} - 1\right)^{n+1} - 1$.

Mihály Bencze

PP. 16438. Prove that
1). $\sum\limits_{k=1}^{\infty}\frac{1}{9k^2-1} = \frac{1}{2} - \frac{\pi\sqrt{3}}{18}$
2). $\sum\limits_{k=1}^{\infty}\frac{1}{144k^2-1} = \frac{1}{2} - \frac{3\left(2+\sqrt{3}\right)\pi}{72}$

Mihály Bencze

PP. 16439. 1). If $x_k \in [1, 2]$ $(k = 1, 2, ..., n)$, then
$\prod\limits_{k=1}^{n}\left(x_k^2 - x_k + 1\right) \geq e^{\sum\limits_{k=1}^{n} x_k - n}$

2). If $x_k \geq 2$ $(k = 1, 2, ..., n)$, then $\prod\limits_{k=1}^{n}\left(x_k^2 - x_k + 1\right) \leq e^{\sum\limits_{k=1}^{n} x_k - (2-\ln 2)n}$

Mihály Bencze

PP. 16440. If $\lambda_k, a_k > 0$ $(k = 1, 2, ..., n)$, then
$\sum\limits_{cyclic}\frac{a_1}{(\lambda_1 a_1+\lambda_2 a_2+...+\lambda_n a_n)(\lambda_1 a_2+\lambda_2 a_3+...+\lambda_n a_1)} \leq \frac{n}{\left(\sum\limits_{k=1}^{n} a_k\right)\left(\sum\limits_{k=1}^{n}\lambda_k\right)^2}$.

Mihály Bencze

PP. 16441. If $x_1 = 1, x_2 = 2, x_3 = 3$ and $2682440^{x_n} + 15365639^{x_{n+1}} + 18796760^{x_{n+2}} = 20615673^{x_{n+3}}$ for all $n \geq 1$, then the sequence $(x_n)_{n\geq 1}$ is convergent and $\lim\limits_{n\to\infty} x_n = 4$. Compute $\lim\limits_{n\to\infty} n\left(x_n - 4\right)$.

Mihály Bencze

PP. 16442. Solve in N the equation $x^2 + ny^2 = (n+1)\,z^2$, where $n \in N$ is given.

Mihály Bencze

PP. 16443. In all triangle ABC hold $\max\left\{\frac{2s^2+r^2}{4s^2}; \frac{r(4R-r)}{s^2}; \frac{(4R+r)^3-4s^2(2R+r)}{(4R+r)^3}; \frac{\left(s^2+r^2-8Rr\right)(2R-r)-8Rr^2}{4(2R-r)^3}; \frac{\left(s^2+(4R+r)^2\right)(4R+r)-8Rs^2}{4(4R+r)^3}\right\} \leq \frac{7}{27}$.

Mihály Bencze

PP. 16444. If $\alpha > 0$ then $\int\limits_1^e (\ln x)^{\frac{1}{\alpha}}\,dx \leq e - \frac{\alpha+2}{\alpha+1}$.

Mihály Bencze

PP. 16445. If $x_k \in \left[0, \frac{\pi}{2}\right]$ $(k = 1, 2, ..., n)$, then $2^{n+1} \geq \prod\limits_{k=1}^{n}(1+\sin x_k) + \prod\limits_{k=1}^{n}(1+\cos x_k) + \prod\limits_{k=1}^{n}\sin x_k + \prod\limits_{k=1}^{n}\cos x_k$.

Mihály Bencze

PP. 16446. In all triangle ABC holds $\sum\left(\frac{1}{\sqrt{a}}\cos\frac{A}{2}\right)^{\lambda} \leq 3^{1-\lambda}\left(\sqrt{\frac{s}{Rr}}\right)^{\lambda}$ for all $\lambda \geq 1$.

Mihály Bencze

PP. 16447. If $k, n \in N^*$, $k \leq n$, then $\int\limits_{2k}^{k+n}\Gamma(x)\,dx - \int\limits_{k+n}^{2n}\Gamma(x)\,dx \leq (n-k)\left((2n)! - (n+k)!\right)$, where Γ denote the Euler‘s Gamma function.

Mihály Bencze

PP. 16448. If $b \geq a > 0$ then $(b-a+4)\,e^{\frac{a+b}{2}} \leq (b-a+2)\,e^{a} + 2e^{b}$.

Mihály Bencze

PP. 16449. If $0 < x \leq y < \frac{\pi}{2}$, then

$$2tg\frac{x+y}{2} \leq tgx + tgy + \frac{y-x}{2}\left(\frac{1}{\cos^2 x} - \frac{1}{\cos^2\frac{x+y}{2}}\right).$$

Mihály Bencze

PP. 16450. If $0 < x_1 \leq x_2 \leq x_3 \leq ... \leq x_n$ is an arithmetical progression with ratio r, then $arctg\frac{r}{1+x_1x_2} + arctg\frac{r}{1+x_2x_3} + ... + arctg\frac{r}{1+x_{n-1}x_n} \leq \ln\sqrt{\frac{x_n}{x_1}}$.

Mihály Bencze

PP. 16451. If $ABCD$ is a concyclic quadrilateral then

$$\sum \sin^2\frac{A+B}{2}\sin^2\frac{B+C}{2} \geq \frac{ac+bd}{R^2}.$$

Mihály Bencze

PP. 16452. If $F_0 = 0, F_1 = 1, F_{n+2} = F_{n+1} + F_n$ for all $n \in N$, then determine all $k \in N$ such that $F_{p-1}^k + kF_p^k + F_{p+1}^k - (k+1)$ is divisible by p for all $n \in N$, where p is a prime number.

Mihály Bencze

PP. 16453. In all triangle ABC holds $\prod\left(\sqrt{a}\right)^{s-a} \leq \left(3\sqrt{\frac{Rr}{s}}\right)^{s}$.

Mihály Bencze

PP. 16454. If $x > y > 0$ and $a_k > 0\ (k = 1, 2, ..., n)$, then

$$\prod_{k=1}^{n}\left(x^{na_k} - y^{na_k}\right) \geq \left(\frac{n\sqrt[n]{\prod\limits_{k=1}^{n} a_k}}{\sum\limits_{k=1}^{n} a_k}\right)^{n}\left(x^{\sum\limits_{k=1}^{n} a_k} - y^{\sum\limits_{k=1}^{n} a_k}\right)^{n}.$$

Mihály Bencze

PP. 16455. If $t \geq 1, a_k, p_k > 0$ $(k = 1, 2, ..., n)$ and $\sum\limits_{k=1}^{n} \frac{1}{p_k} = 1$, then

$$\prod_{k=1}^{n} \left(t^{p_k a_k} - 1\right)^{\frac{1}{p_k}} \geq \frac{\prod\limits_{k=1}^{n} (p_k a_k)^{\frac{1}{p_k}}}{\sum\limits_{k=1}^{n} a_k} \left(t^{\sum\limits_{k=1}^{n} a_k} - 1\right).$$

Mihály Bencze

PP. 16456. In all triangle ABC holds

1). $\left|\prod (a-b)\right| \leq \frac{1}{3} s \left(s^2 - 7r^2 - 10Rr\right)\left(s^2 + r^2 + 2Rr\right)$
2). $\left|\prod (a-b)\right| \leq s \left(s^2 + r^2 - 14Rr\right)$

Mihály Bencze

PP. 16457. If $a, b, c > 0$ and $abc = 1$, then $\sum \frac{a^3+1}{a^2(b+c)} \geq 3 + 3\frac{\left|\prod (a-b)\right|}{\prod (a+b)}$.

Mihály Bencze

PP. 16458. In all triangle ABC holds $R - 2r \geq \frac{s^2 - 3r^2 - 12Rr}{2R}$.

Mihály Bencze

PP. 16459. Prove that $\frac{\left(\sum\limits_{k=1}^{n} a_k\right)!}{\left(\sum\limits_{k=1}^{n} a_k\right)^{\sum\limits_{k=1}^{n} a_k}} \leq \sum\limits_{k=1}^{n} \frac{a_k!}{a_k^{a_k}}$ for all $a_k \in N^*$ $(k = 1, 2, ..., n)$.

Mihály Bencze

PP. 16460. If $x_k > 0$ $(k = 1, 2, ..., n)$, then $\sum\limits_{cyclic} \frac{x_1}{x_2 + x_3} + \frac{2^{n-1} \prod\limits_{k=1}^{n} x_k}{\prod\limits_{cyclic} (x_1 + x_2)} \geq \frac{n+1}{2}$.

Mihály Bencze

PP. 16461. In all triangle ABC holds $\sum \left(\frac{\cos A}{\cos \frac{A}{2}}\right)^{2\lambda} \geq 3^{1-\lambda}$ for all $\lambda \geq 1$.

Mihály Bencze

PP. 16462. If $a_k > 0$ $(k = 1, 2, ..., n)$, then

$$\prod_{k=1}^{n} (1 + a_k)^{\frac{n(n+3)}{2}} \geq (n+1)^{n(n+1)} \prod_{k=1}^{n} a_k^n.$$

Mihály Bencze

PP. 16463. If $a_k > 0$ $(k = 1, 2, ..., n)$, then $\prod\limits_{cyclic} \frac{a_1}{(a_1+a_2+1)\sqrt{a_1+a_2}} \leq 4^{-n}$.

Mihály Bencze

PP. 16464. If $a_k \in (0, 1)$ $(k = 1, 2, ..., n)$ and $\sum\limits_{k=1}^{n} a_k^2 = 1$, then $\sum\limits_{k=1}^{n} \frac{1}{a_k\left(1-a_k^2\right)} \geq \frac{5\sqrt[4]{5}(n+1)}{4}$.

Mihály Bencze

PP. 16465. If $a_k \in (0, 1)$ $(k = 1, 2, ..., n)$ and $\sum\limits_{k=1}^{n} a_k = 1$, then $\sum\limits_{k=1}^{n} \frac{1}{a_k(1+a_k)\left(1+a_k^2\right)} \geq \frac{5\sqrt[4]{5}(n-1)}{4}$.

Mihály Bencze

PP. 16466. Let ABC be a triangle:

1). Prove that $\sqrt{a(s-a)}, \sqrt{b(s-b)}, \sqrt{c(s-c)}$ are the sides of a triangle.

2). Determine all $x, y, z \in R$ for which $\sqrt{xa^2 + yab + zac}$, $\sqrt{xb^2 + ybc + zba}$, $\sqrt{xc^2 + yca + zcb}$ are the sides of a triangle.

Mihály Bencze

PP. 16467. If $a_k \in [0, 1]$ $(k = 1, 2, ..., n)$, then $\prod\limits_{k=1}^{n} \left(2 - a_k + a_k^3\right) \geq 2^n \prod\limits_{k=1}^{n} a_k$.

Mihály Bencze

PP. 16468. If $a_i \in [0,1]$ $(i = 1, 2, ..., n)$ and $k \in \{1, 2, ..., n\}$, then
$(1 - a_1 a_2 ... a_k)(1 - a_2 a_3 ... a_{k+1}) ... (1 - a_n a_1 ... a_{k-1}) \geq \prod\limits_{i=1}^{n} \left(1 - a_i^k\right).$

Mihály Bencze

PP. 16469. If $x_k > 0$ $(k = 1, 2, ..., n)$, then
1). $\sum \frac{x_1^3(4x_1+x_2+x_3)}{x_1^2+x_2x_3} \geq \frac{7}{2} \sum\limits_{k=1}^{n} x_k^2 - \frac{1}{2} \sum x_1 x_2$
2). $\sum \frac{x_1^3\left(16x_1^2+x_2^2+x_3^2+4x_1x_2+2x_2x_3+4x_1x_3\right)}{x_1^2+x_2x_3} \geq \frac{31}{2} \sum\limits_{k=1}^{n} x_k^3 - \frac{3}{4} \sum x_1 x_2 (x_1 + x_2).$

Mihály Bencze

PP. 16470. If $a, b, c > 0$, then
1). $\sum \left(2a^2 + b^2 + c^2\right)(b + c) \geq 3(a + b)(b + c)(c + a)$
2). $\sum \frac{\left(3a^3+2b^3+c^3\right)(b+c)}{a+b} \geq \frac{9}{4}(a + b)(b + c)(c + a)$

Mihály Bencze

PP. 16471. If $x_k > 0$ $(k = 1, 2, ..., n)$, then
$\sum\limits_{cyclic} \frac{x_1}{\sqrt[n-1]{\left(x_1^2+x_2^2\right)...\left(x_1^2+x_n^2\right)}} \leq \frac{2}{n-1} \sum\limits_{1 \leq i < j \leq n} \frac{1}{x_i+x_j}.$

Mihály Bencze

PP. 16472. If $a_k > 0$ $(k = 1, 2, ..., n)$, then
1). $\sum\limits_{cyclic} \frac{a_1^3(2a_1+a_2)}{a_1^2+a_1a_2+a_2^2} \geq \sum\limits_{k=1}^{n} a_k^2$
2). $\sum\limits_{cyclic} \frac{a_1^3\left(4a_1^2+2a_1a_2+a_2^2\right)}{a_1^2+a_1a_2+a_2^2} \geq \frac{7}{3} \sum\limits_{k=1}^{n} a_k^3$

Mihály Bencze

PP. 16473. If $a_k > 0$ $(k = 1, 2, ..., n)$, then
1). $\sum \frac{a_1^3\left(5a_1^2+a_2^2\right)}{a_1+a_2} \geq 3 \sum\limits_{k=1}^{n} a_k^4$
2). $\sum \frac{a_1^3\left(25a_1^4+5a_1^2a_2^2+a_2^4\right)}{a_1+a_2} \geq \frac{31}{2} \sum\limits_{k=1}^{n} a_k^6$

Mihály Bencze

PP. 16474. If $x, y, z > 0$, then $\sum\limits_{cyclic} \frac{1}{xy\left(\sqrt{xy}+2z\right)} \ge \sum\limits_{cyclic} \frac{1}{\left(xz+x\sqrt{xy}+z\sqrt{yz}\right)y}$.

Mihály Bencze

PP. 16475. If $x, y, z > 0$, then $1 + \frac{3xyz}{\sum (xy)^{\frac{3}{2}}} \ge \frac{6\sqrt{xyz}}{\sum x^{\frac{3}{2}}}$.

Mihály Bencze

PP. 16476. If $x_i > 0$ $(i = 1, 2, ..., n)$ and $k \in N^*$, then

$$\frac{\sum\limits_{i=1}^{n} x_i^{4k-1}}{\left(\sum\limits_{i=1}^{n} x_i\right)^{2k-2}} \ge \frac{1}{(n-1)n^{2k-1}} \left(\sum\limits_{cyclic} x_1^k \sqrt{x_2 + x_3 + ... + x_n}\right)^2$$

Mihály Bencze

PP. 16477. If $a, b, c > 0$ and $a^2 + b^2 + c^2 = 3$, then
$\sum \frac{1}{(a^2+2)(b^2+2)} \le \frac{3}{(a+b+c)^2}$.

Mihály Bencze

PP. 16478. If $a_k > 0$ $(k = 1, 2, ..., n)$ and $\alpha \ge 1$, then
$\prod\limits_{k=1}^{n} \left(a_k^{\alpha} + n - 1\right) \ge n^{n-\alpha} \left(\sum\limits_{k=1}^{n} a_k\right)^{\alpha}$.

Mihály Bencze

PP. 16479. If $x_k > 0$ $(k = 1, 2, ..., n)$ then
$\sum\limits_{k=1}^{n} \frac{x_k}{(n-1)x_k + \sqrt[n-1]{x_1 \dots x_{k-1} x_{k+1} \dots x_n}} \le 1$.

Mihály Bencze

PP. 16480. If $\alpha \ge 1$ and $x, y, z > 0$, then $\sum \frac{\left(\sqrt{xy}\right)^{\alpha+3}}{\left(x\sqrt{x}+y\sqrt{y}\right)z^{\alpha}} \ge \frac{3\sqrt{xyz}}{2}$.

Mihály Bencze

PP. 16481. If $\alpha \ge 1$ and $a_k > 0$ $(k = 1, 2, ..., n)$, $S = \sum\limits_{p=1}^{n} a_p$, then

$\sum\limits_{k=1}^{n} \frac{a_k^{\alpha}}{S - a_k} \ge \frac{n}{n-1} \sqrt[n]{\prod\limits_{k=1}^{n} a_k^{\alpha-1}}$.

Mihály Bencze

PP. 16482. If $x_k \le 1$ $(k = 1, 2, ..., n)$ and $\sum\limits_{k=1}^{n} x_k^2 = 1$, then $\sum\limits_{k=1}^{n} \frac{2+x_k}{1+x_k^2} \le \frac{27(4n-1)}{50}$.

Mihály Bencze

PP. 16483. If $x_k \le 1$ $(k = 1, 2, ..., n)$ and $\sum\limits_{k=1}^{n} x_k^3 = 1$, then $\sum\limits_{k=1}^{n} \frac{4+2x_k+x_k^2}{1+x_k^2} \le \frac{27(8n-1)}{50}$.

Mihály Bencze

PP. 16484. If $x_k \in (0, 1)$ $(k = 1, 2, ..., n)$ then

$$\sum_{k=1}^{n} x_k + \sum_{cyclic} x_1^{-x_2} \le n + \sum_{cyclic} \frac{x_2}{x_1}.$$

Mihály Bencze

PP. 16485. If $x, y, z > 0$, then $\sum \frac{1+x^2+y^2}{3+y^2+z^2+4xz} \ge 1$.

Mihály Bencze

PP. 16486. If $x_k > 0$ $(k = 1, 2, ..., n)$ then $\sum\limits_{cyclic} \frac{x_1}{x_1 + \sqrt[n-1]{(x_1+x_2)(x_1+x_3)...(x_1+x_n)}} \le \frac{n}{3}$.

Mihály Bencze

PP. 16487. If $x_k \le 1$ $(k = 1, 2, ..., n)$ and $\sum\limits_{k=1}^{n} x_k = 1$, then $\sum\limits_{k=1}^{n} \frac{1}{1+x_k^2} \le \frac{27(2n-1)}{50}$.

Mihály Bencze

PP. 16488. If $a_k \in [0, 1]$ $(k = 1, 2, ..., n)$, then $\sqrt{\prod\limits_{k=1}^{n} a_k} + \sqrt{\prod\limits_{k=1}^{n} (1 - a_k)} \le 1$.

Mihály Bencze

PP. 16489. If $a_k > 0$ $(k = 1, 2, ..., n)$ then $\sum\limits_{k=1}^{n} \frac{a_k}{(S-a_k)^2} \geq \frac{n^2}{(n-1)^2 \sum\limits_{k=1}^{n} a_k}$, where $S = \sum\limits_{p=1}^{n} a_p$.

Mihály Bencze

PP. 16490. If $a_k > 0$ $(k = 1, 2, ..., n)$ and $p \in N$ then $\sum\limits_{cyclic} \frac{a_1^{2p+1}}{a_2^{2p}} \geq \sum\limits_{cyclic} \frac{a_1^{2p}}{a_2^{2p-1}}$.

Mihály Bencze

PP. 16491. If $a_k > 0$ $(k = 1, 2, ..., n)$ then $\min \left\{ \sum\limits_{cyclic} \frac{2a_1^2+a_2^2+a_3^2}{a_1+a_2}; \sum\limits_{cyclic} \frac{2a_1^2+a_2^2+a_3^2}{a_1+a_3} \right\} \geq 2 \sum\limits_{k=1}^{n} a_k$.

Mihály Bencze

PP. 16492. If $a_k > 0$ $(k = 1, 2, ..., n)$ then $\min \left\{ \sum\limits_{cyclic} \frac{3a_1^3+2a_2^3+a_3^3}{(a_1+a_2)^2}; \sum\limits_{cyclic} \frac{3a_1^3+2a_2^3+a_3^3}{(a_1+a_2)(a_1+a_3)} \right\} \geq \frac{3}{2} \sum\limits_{k=1}^{n} a_k$.

Mihály Bencze

PP. 16493. If $\alpha, a_k > 0$ $(k = 1, 2, ..., n)$ then $\sum\limits_{cyclic} \sqrt{a_1^2 + (\alpha - a_2)^2} \geq \frac{n\alpha\sqrt{2}}{2}$.

Mihály Bencze

PP. 16494. If $x, y, z > 0$, then $\prod (2x + z) \left(3x^2 + 2y^2 + z^2\right) \geq 5832x^3y^3z^3$.

Mihály Bencze

PP. 16495. If $a, b, c > 0$, then $\sum \frac{a+b}{c} \geq 4 \sum \frac{a^2}{a^2+bc}$.

Mihály Bencze

PP. 16496. If $0 \leq x_k \leq 1$ $(k = 1, 2, ..., n)$, then $\prod\limits_{k=1}^{n} x_k \left(1 - x_k^2\right) \leq \left(\frac{2\sqrt{3}}{9}\right)^n$.

Mihály Bencze

PP. 16497. If $x, y, z > 0$, then $2\sum \frac{x^3\left(x^2+y^2\right)}{x+y} \geq \sum x^4 + \sum x^2y^2$.

Mihály Bencze

PP. 16498. If $a_k > 0$ $(k = 1, 2, ..., n)$ then
$\sum\limits_{cyclic} \frac{a_1^6}{\left(a_1^4+a_1^2a_2^2+a_2^4\right)\left(\sqrt{2}a_1+a_2\right)} \geq \frac{\sqrt{2}-1}{3} \sum\limits_{k=1}^{n} a_k$.

Mihály Bencze

PP. 16499. If $x, y, z > 0$, then $\left(\sum x\right)\left(\sum \frac{x^3}{x^2+yz}\right) \geq \frac{1}{2}\sum x^2 + \sum xy$.

Mihály Bencze

PP. 16500. If $S_k\left(n\right) = 1^k + 2^k + ... + n^k$, then
$5\sum\limits_{k=1}^{n} k^6\left(n-k+1\right)^6 \leq 3S_{10}\left(n\right) + 2S_{15}\left(n\right)$.

Mihály Bencze

PP. 16501. If $a_k > 0$ $\left(k = 1, 2, ..., n\right)$, then
$\frac{n}{3} + \sum\limits_{cyclic} \frac{a_1^3}{a_2\left(a_1^2+a_1a_2+a_2^2\right)} \geq \frac{2}{3}\sum\limits_{cyclic} \frac{a_1}{a_2}$.

Mihály Bencze

PP. 16502. If $x_k > 0$ $\left(k = 1, 2, ..., n\right)$, then
$\sum\limits_{cyclic} \frac{x_1^3}{\left(x_1+x_2\right)\left(\sqrt{5}x_1+x_2\right)} \geq \frac{\sqrt{5}-1}{8}\sum\limits_{k=1}^{n} x_k$.

Mihály Bencze

PP. 16503. If $x, y, z > 0$, then $\sum \frac{1}{x\left(xz+2y^2+6z\right)} \leq \frac{1}{xyz}$.

Mihály Bencze

PP. 16504. If $x \in \left[0, \frac{\pi}{2}\right]$, then
1). $2\left(1+\sin^3 x+\cos^3 x\right) \leq$
$\leq \left(1+\sin 2x\right)\left(2-\sin 2x\right) + \left(2-\sin^2 2x\right)\sqrt{1+\sin 2x}$
2). $\frac{1-\cos 2x}{1+\cos 2x+\sin^2 2x} + \frac{1+\cos 2x}{1-\cos 2x+\sin^2 x} \geq \frac{3}{2} - \sin^2 x\cos^2 x$

Mihály Bencze

PP. 16505. If $a_k, b_k, c_k > 0$ $(k = 1, 2)$ and $a_1^2 + b_1^2 + c_1^2 = a_2 + b_2 + c_2 = 1$, then $\left(\sum \frac{1-a_1^2}{b_1+c_1}\right)\left(\sum \frac{a_2^2+b_2^2}{1-c_2}\right) \geq \frac{9\sum a_2^2}{\sum a_1}$.

Mihály Bencze

PP. 16506. Let ABC be a triangle, and AE, BF, CD the interior bisectors of angles BAC, CBA, ACB, where $E \in BC$, $F \in AC$, $D \in AB$. Prove that $(a+b)(b+c)(AD+EC-b)+(b+c)(c+a)(BE+FA-c)+$ $(c+a)(a+b)(CF+DB-a) = 4sRr\left(\frac{2r}{R}-1\right)$.

Mihály Bencze

PP. 16507. Solve in N the following equation $\sum\limits_{k=1}^{m} x_k! = (m+1)^n \cdot n!$.

Mihály Bencze

PP. 16508. In all triangle ABC holds $\sum \frac{tg\frac{A}{2}}{\cos\frac{A}{2}\cos\frac{B-C}{2}} \geq 2$.

Mihály Bencze

PP. 16509. If $x \in R$, then

1). $3\sqrt{2} + |\sin x| + |\cos x| \leq 2\left(\sqrt{1+2\sin^2 x} + \sqrt{1+2\cos^2 x}\right)$

2). $1 + 3\sqrt{2}chx + |shx| \leq 2\left(\sqrt{2+ch^2x} + \sqrt{1+3sh^2x}\right)$

Mihály Bencze

PP. 16510. Let ABC be rectangle triangle with catetes a and b and ipotenuse c. Prove that $a + b + 3\sqrt{2}c \leq 2\left(\sqrt{c^2+2a^2} + \sqrt{c^2+2b^2}\right)$.

Mihály Bencze

PP. 16511. If $a, b > 0$, then
$\sqrt{|3a^2-b^2|} + \sqrt{|3b^2-a^2|} + 6\sqrt{a^2+b^2} \leq 4\sqrt{2}(a+b)$.

Mihály Bencze

PP. 16512. If $a \geq b \geq c \geq d \geq 0$ and $a^\alpha + b^\alpha + c^\alpha + d^\alpha = 1$, then determine all $\alpha \in R$ which implies $a + b \geq 1 \geq c + d$.

Mihály Bencze

PP. 16513. If $x_k \in R$ $(k = 1, 2, ..., n)$, then
$-n \le \sum\limits_{k=1}^{n} \sin x_k + \sum\limits_{1 \le i < j \le n} \sin(x_i - x_j) \le n.$

Mihály Bencze

PP. 16514. Determine all $a \in N^*$ for which $\sum\limits_{p=1}^{(a^k-1)n} \frac{1}{n+p} > k$ for all $n, k \in N^*$.

Mihály Bencze

PP. 16515. If $x_k \in R$ $(k = 1, 2, ..., n)$, then
$10 \sum\limits_{k=1}^{n} ch^2 x_k \le 4 \sum\limits_{cyclic} ch^4 x_1 ch^4 x_2 + \sum\limits_{cyclic} (chx_1 + chx_2)^4.$

Mihály Bencze

PP. 16516. In all triangle ABC holds

$$12\left(s^2 - r^2 - Rr\right)\left(s^2 - r^2 - 4Rr\right) \ge \left(s^2 + r^2 + 2Rr\right)\left(11s^2 - 15r^2 - 60Rr\right).$$

Mihály Bencze

PP. 16517. Let $ABCD$ be a convex quadrilateral with sides a, b, c, d. Prove that $(s+a)(s+b)(s+c)(s+d) \ge$
$\ge 81(Area(ABCD))^2 + \frac{81}{2}abcd\left(\cos^2\frac{A+C}{2} + \cos^2\frac{B+D}{2}\right).$

Mihály Bencze

PP. 16518. If $x_0 > 0$ and $x_n x_{n+1} \ge x_{n+1} + 2 \ge x_{n+1} x_{n+2}$ for all $n \in N$, then compute $\lim\limits_{n \to \infty} n(x_n - 2)$.

Mihály Bencze

PP. 16519. If $n, k \in N$, then for all $x_i > 0$ $(i = 1, 2, ..., n)$, then we have
$\sum\limits_{cyclic} x_1^{k+1}\left(\frac{1}{x_2} + \frac{1}{x_n}\right) \ge 2\sum\limits_{i=1}^{n} x_i^k.$

Mihály Bencze

PP. 16520. If $b > 1$, $a_1 \in \left(0, \frac{1}{b}\right)$ and $a_{n+1} = a_n - ba_n^2$ for all $n \in N^*$, then compute $\lim\limits_{n\to\infty} n\left(na_n - \frac{1}{b}\right)$.

Mihály Bencze

PP. 16521. Determine all functions $f : R \to R$ such that
$f(xf(y)) + f(f(x) - f(y)) = yf(x) + f(f(y)) - f(f(x))$ for all $x, y \in R$.

Mihály Bencze

PP. 16522. Determine all $m, n \in N$ such that $m^{n-1} - n^{m-1}$ is divisible by $m + n$.

Mihály Bencze

PP. 16523. Determine all $x, y \in Z$ such that $(x+y)^4 = x^3y + x^2y^2 + xy^3$.

Mihály Bencze

PP. 16524. If from the equations $x^2 + a_kx + b_k = 0$ $(k = 1, 2, ..., n)$, all $n-1$ equations have a common root, then determine the min and the max of the expression $\sum\limits_{k=1}^{n} (a_k^2 + b_k^2)$, where $a_k, b_k \in R$ $(k = 1, 2, ..., n)$.

Mihály Bencze

PP. 16525. In all triangle ABC holds
$s\left(\sum \frac{1}{m_a+m_b} - \frac{1}{\sqrt{2(s^2-r^2-4Rr)}}\right) \geq \sum \frac{c}{m_a+m_b}$.

Mihály Bencze

PP. 16526. If $P_0(x) = 1$, $Q_0(x) = -1$, $P_{n+1}(x) + (n+1)P_n(x) = x^{n+1}$, $Q_{n+1}(x) + (n+1)Q_n(x) = -x^{n+1}$ for all $n \in N$ and $x \in R$, then compute min and max of $\frac{P_n(x)}{Q_n(x)}$.

Mihály Bencze

PP. 16527. Let ABC be a triangle, M a random point in the plane of the given triangle. Denote D and E the midpoints of AC and BM. The line DE cut AB and CM in P and Q. Deterine all M such that $APQ\measuredangle = MQP\measuredangle$.

Mihály Bencze

PP. 16528. If $0 < m \le a_k \le M\ (k = 1, 2, ..., n)$, then compute
$\int\limits_m^M \left(\sum\limits_{k=1}^n \frac{1}{x+a_k} \right) \ln x dx$.

Mihály Bencze

PP. 16529. If $a, b \in [-1, 1]$, then compute $\int \frac{(a-b)\cos x - (a+b)\sin x}{e^x + a\sin x + b\cos x} dx$.

Mihály Bencze

PP. 16530. If $x_k > 0\ (k = 1, 2, ..., n)$, then
$n \sum\limits_{k=1}^n x_k^2 \ge \left(\sum\limits_{k=1}^n x_k \right)^2 + \frac{1}{n} \left(\sum\limits_{\substack{i,j=1 \\ i \ne j}}^n |x_i - x_j| \right)^2$.

Mihály Bencze

PP. 16531. In all triangle ABC holds
$2R(4R+r)^2 \le (2R-r)\left(s^2 + (4R+r)^2\right)$.

Mihály Bencze

PP. 16532. If $a, b, c, d > 1$, then $\prod \log_a \frac{b+c}{2} \log_a \frac{c+d}{2} \log_a \frac{d+b}{2} \ge 1$.

Mihály Bencze

PP. 16533. If $z \in C$, then
$2(|1+z| + |1-z|) + \left|1 + z + z^2 + z^3 + z^4\right| + \left|1 - z + z^2 - z^3 + z^4\right| \ge 2$.

Mihály Bencze

PP. 16534. If $a_k = \frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{kn}$, then compute $\sum\limits_{k=2}^n [a_k]$, where $[\cdot]$ denote the integer part.

Mihály Bencze

PP. 16535. In all triangle ABC holds $AB + BC + CA \ge \frac{r^2 s}{2} \left(\frac{4\pi\sqrt{3}}{9} \right)^3$.

Mihály Bencze

PP. 16536. In all triangle ABC holds $\sum \frac{(a+b)c}{a+b-c} \ge \frac{8sr}{R}$.

Mihály Bencze

PP. 16537. Prove that $\left[\frac{1}{2}\sum\limits_{k=0}^{n}\frac{(2n-3k+1)2^k\binom{n+1}{k}}{k+1}\right]=2^n-1$, where $[\cdot]$ denote the integer part.

Mihály Bencze

PP. 16538. If $x\geq 1$, then $\exp\left(\frac{n+1}{\sqrt{x}+\sqrt{x+n+1}}\right)\geq\prod\limits_{k=0}^{n}\left(1+\frac{1}{2\sqrt{x+k}}\right)$.

Mihály Bencze

PP. 16539. If $x\in\left[0,\frac{\pi}{2}\right]$, then
$\frac{\sin x}{1+\sqrt{1+\sin^2 x}}+\frac{\cos x}{1+\sqrt{1+\cos^2 x}}\geq\ln\left(1+\sqrt{2}\cos\left(x-\frac{\pi}{4}\right)+\frac{1}{8}\sin 2x\right)$.

Mihály Bencze

PP. 16540. If $x,y,z>0$, then
$(x+y)^{xy^2}(y+z)^{yz^2}(z+x)^{zx^2}\geq 8^{xyz}x^{zx^2}y^{xy^2}z^{yz^2}$.

Mihály Bencze

PP. 16541. Prove that $\int\limits_0^1\left(1+x^2+x^4+...+x^{2n}\right)e^{nx}dx\geq\frac{(n+1)(e^n-1)}{n(2n+1)}$.

Mihály Bencze

PP. 16542. If $x,y,z>0$ and $xyz=1$, then
$\sum\frac{1}{x+y}\left(\frac{1}{x^3}+\frac{1}{z^3}\right)+\frac{1}{2}\sum\frac{1}{x^3y}\geq 3\sum\frac{1}{x+1}$.

Mihály Bencze

PP. 16543. If $x\in[0,1]$, then $\frac{\sqrt{x}}{1+\sqrt{x+1}}\geq\ln\left(1+\frac{\sqrt{x}}{2}\right)$.

Mihály Bencze

PP. 16544. If $a,b,x_k>0$ $(k=1,2,...,n)$, then
$\min\left\{\sum\limits_{cyclic}\frac{x_1^2+x_1x_2+x_2^2}{ax_1+bx_2};\sum\limits_{cyclic}\frac{x_1^2+x_1x_2+x_2^2}{ax_2+bx_1}\right\}\geq\frac{3}{a+b}\sum\limits_{k=1}^{n}x_k$.

Mihály Bencze

PP. 16545. Solve in N the equation $\frac{x_1+2009}{x_2+2009}+...+\frac{x_n+2009}{x_1+2009}=n$.

Mihály Bencze

PP. 16546. Denote T the Toricelli's point in triangle ABC. Prove that $\left(\sum AT\right)\left(\sum AT\cdot BT\right)\geq 4\sqrt{3}sRr$.

Mihály Bencze

PP. 16547. If $a, b, x_k > 0$ $(k = 1, 2, ..., n)$, then $\sum\limits_{cyclic}\frac{(ax_1+bx_2)(ax_2+bx_1)}{x_1^2+x_1x_2+x_2^2}\leq\frac{n(a+b)^2}{3}$.

Mihály Bencze

PP. 16548. If $x_k > 0$ $(k = 1, 2, ..., n)$ and $\sum\limits_{k=1}^{n}x_k = 1$, then $\sum\limits_{cyclic}\frac{1}{x_1+x_2}+n\sum x_1x_2\geq 1+\frac{n^2}{2}$.

Mihály Bencze

PP. 16549. Let ABC be a triangle. An exterior tangent circle to the circumcircle and to the sides AB and AC have the tangent points M and N with sides AB and AC. In same way we define the points P, Q and L, K. Prove that $\frac{MN}{AB\cdot AC}+\frac{PQ}{BC\cdot BA}+\frac{LK}{CA\cdot CB}\geq\frac{2}{s}\left(\sum\sqrt{\sin\frac{A}{2}}\right)^2$.

Mihály Bencze

PP. 16550. If $x_k > 0$ $(k = 1, 2, ..., n)$ and $m\in N^*$, then $\sum\limits_{cyclic}\frac{1}{(x_1+x_2)^m+m-1}\leq\frac{1}{2m}\sum\limits_{k=1}^{n}\frac{1}{x_k}$ (A generalization of problem 26123, Gazeta Matematica, Bucuresti).

Mihály Bencze

PP. 16551. If $a, b, c > 0$, then $3\prod\left(a^2c^2+ab^2c+b^4\right)\geq\left(\sum a^2b\right)^2\left(\sum a^2c\right)^2$.

Mihály Bencze

PP. 16552. In all convex quadrilateral $ABCD$ holds

$$1+\frac{5R}{r}\geq\left(\sqrt{\frac{s-a}{s-b}}+\sqrt{\frac{s-b}{s-c}}+\sqrt{\frac{s-c}{s-d}}+\sqrt{\frac{s-d}{s-a}}\right)^2\geq 16.$$

Mihály Bencze

PP. 16553. If $n \in N^*$, then
$\frac{1}{3}+\frac{1}{5}+...+\frac{1}{2n+1} < \ln\frac{(2n+1)!}{4^n(n!)^2} < \frac{1}{2}+\frac{1}{4}+...+\frac{1}{2n} < \ln\frac{4^n(n!)^2}{(2n)!} < 1+\frac{1}{3}+\frac{1}{5}+...+\frac{1}{2n-1}.$

Mihály Bencze

PP. 16554. If $a, b, c > 0$, then $\left(\sum a^2 + \sum ab\right)\sum\left(\frac{a}{b}+\frac{b}{a}\right) \geq 4\left(\sum a\right)^2.$

Mihály Bencze

PP. 16555. If $a, b, c \in [0,1]$, then $abc + \frac{\left(\sum a\right)^2}{\sum a + 3abc} \leq \frac{5}{2}$.

Mihály Bencze

PP. 16556. In all triangle ABC holds:
1). $2\sum\frac{m_a^2}{1+\cos A} \geq \left(\frac{\sum\sqrt{2bc(b^2+c^2)-a^2bc}}{\sum a}\right)^2$
2). $\sum\frac{m_a^2}{1+\cos A} \geq \frac{1}{2}\left(s^2+r^2+4Rr\right)$

Mihály Bencze

PP. 16557. Compute $\int\limits_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{\left(\sin^2 nx-\sin^2(n-1)x\right)\left(e^x+e^{-x}\right)}{\sin x}dx$, where $n \in N^*$.

Mihály Bencze

PP. 16558. If $x, y, z > 0$, then $\left(\sum x\right)^2\left(\sum xy\right)^2 \leq \left(2\sum x^2 + \sum xy\right)^3.$

Mihály Bencze

PP. 16559. Solve in R the following system: $\begin{cases} [x]\{y\} = z\,|t| \\ [y]\{z\} = t\,|x| \\ [z]\{t\} = x\,|y| \\ [t]\{x\} = y\,|z| \end{cases}$, where $[\cdot]$ and $\{\cdot\}$ denote the integer part, respective the fractional part.

Mihály Bencze

PP. 16560. If $x \in R\backslash\left\{k\pi;(2k+1)\frac{\pi}{2}|k \in Z\right\}$, then
$\log_{\sin^2 x} 2\cos^2 x + \log_{\cos^2 x} 2\sin^2 x \geq 0.$

Mihály Bencze

PP. 16561. If $x_k \in R$ $(k = 1, 2, ..., n)$ and $\sum\limits_{k=1}^{n} x_k = n$, then $\sum\limits_{k=1}^{n} \frac{x_k^{12}+x_k^4+1}{x_k^8+1} \geq n$.

Mihály Bencze

PP. 16562. If $x_k \in R$ $(k = 1, 2, ..., n)$ then
$3 \sum\limits_{k=1}^{n} x_k^2 + n \geq 2 \sum\limits_{k=1}^{n} x_k + 2 \sum\limits_{cyclic} x_1 x_2$.

Mihály Bencze

PP. 16563. If $e(x) = \left(1 + \frac{1}{x}\right)^x$, then $\sum\limits_{cyclic} \frac{1}{\log_{e(x)}(e(y)+e(z))} \geq \frac{3}{2}$ for all $x, y, z > 0$.

Mihály Bencze

PP. 16564. Prove that $b_1, b_2, ..., b_n, b_{n+1}, b_{n+2} > 0$ is a geometrical progression if and only if $\left(\sum\limits_{k=1}^{n} b_k^3\right)\left(\sum\limits_{k=2}^{n+1} b_k^3\right)\left(\sum\limits_{k=3}^{n+2} b_k^3\right) = \left(\sum\limits_{k=1}^{n} b_k b_{k+1} b_{k+2}\right)^3$.

Mihály Bencze

PP. 16565. If $x \in R$, then
1). $\frac{1+\sin^8 x+\sin^{24} x}{1+\sin^{16} x} + \frac{1+\cos^8 x+\cos^{24} x}{1+\cos^{16} x} \geq 1$
2). $4\left(\sin^4 x + \cos^4 x\right) + 3 \geq 3\left(\sin x + \cos x\right)$

Mihály Bencze

PP. 16566. If $n \in N^*$, then determine the integer part of the following expressions:
1). $\left(1 - \frac{1}{n}\right)^{1-\frac{1}{n}} + \left(1 + \frac{1}{n}\right)^{1+\frac{1}{n}}$
2). $\left(1 - \frac{1}{n}\right)^{1+\frac{1}{n}} + \left(1 + \frac{1}{n}\right)^{1-\frac{1}{n}}$

Mihály Bencze

PP. 16567. If $x, y, z > 0$, then $1 + \frac{\sum x^2}{\sqrt[3]{x^2y^2z^2}} \geq \frac{2}{3}\left(\frac{\sum x}{\sqrt[3]{xyz}} + \sqrt[3]{xyz}\left(\sum \frac{1}{x}\right)\right)$.

Mihály Bencze

PP. 16568. If $a, b, c > 0$, then $\sum \frac{1}{a^2+bc} \leq \frac{\sum a}{2abc}$.

Mihály Bencze

PP. 16569. In all acute triangle ABC holds:

1). $\sum \frac{m_a^4}{1+\cos A} \geq \frac{9R\left(s^2-r^2-4Rr\right)^2}{4(4R+r)}$
2). $\sum \frac{m_a^2}{1+\cos A} \geq \frac{81Rr^2}{4R+r}$

Mihály Bencze

PP. 16570. If $x, y, z > 0$ and $x+y+z = xyz$, then
$\sum \frac{1}{1+x^2} + \frac{2}{\left(1+\sqrt[3]{x^2y^2z^2}\right)^{\frac{3}{2}}} \leq 1.$

Mihály Bencze

PP. 16571. If $x, y, z > 0$ and $\sum xy = 1$, then
$2\left(1+\frac{xyz}{\sqrt{(1+x^2)(1+y^2)(1+z^2)}}\right) \geq \frac{9}{3+\sum x^2}.$

Mihály Bencze

PP. 16572. If $a, b \in N^*$, then determine all $x, y \in N$ for which

1). $a^{[x,y]} + b^{(x,y)}$ is a perfect square
2). $a^{\left[\frac{x}{y}\right]} + b^{\left[\frac{y}{x}\right]}$ is a perfect square, where $[\cdot]$ denote the integer part
3). $a^{xb} + b^{ya}$ is a perfect square

Mihály Bencze

PP. 16573. If $a, b > 0$, then $\frac{(1+a+b)^2}{(1+a+a^2)(1+b+b^2)} \leq \frac{3\left(a^2+ab+b^2\right)}{(a+ab+b)^2}.$

Mihály Bencze

PP. 16574. If $x, y, z \geq 0$, then in all triangle ABC holds
$x^n \cos\frac{A}{2} + y^n \cos\frac{B}{2} + z^n \cos\frac{C}{2} \geq \left(\frac{y+z}{2}\right)^n \sin A + \left(\frac{z+x}{2}\right)^n \sin B + \left(\frac{x+y}{2}\right)^n \sin C.$

Mihály Bencze and Nicusor Minculete

PP. 16575. If $x \in \left(0, \frac{\pi}{2}\right)$, then $\sin x + \cos x + \frac{1}{\sin x} + \frac{1}{\cos x} \geq$
$18 + \frac{12}{\sin^2 2x}\left(1-\sin 2x\right)\left(\sin 2x + \min\left(\sin^2 x, \cos^2 x\right)\right).$

Mihály Bencze

PP. 16576. If $x \in \left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$, then
$\sin 2x < 2\left(\left|\frac{\cos 2x}{n\left(\sqrt[n]{\cos^2 x} - \sqrt[n]{\sin^2 x}\right)}\right|\right)^{\frac{n}{n-1}} < 1$, for all $n \in N$, $n \geq 2$.

Mihály Bencze

PP. 16577. If $x \geq 0$, then $8 \leq \frac{2\sqrt{2}shx}{1+chx} \leq \frac{\sqrt{2}shx}{\sqrt{chx}} \leq \frac{shx(1+chx)}{\sqrt{2}chx} \leq chx + \frac{1}{chx}$.

Mihály Bencze

PP. 16578. In all triangle ABC holds:

1). $s^2 + r^2 + 4Rr \leq \frac{1}{4}\sum\sqrt{a^4 + 14a^2b^2 + b^4} \leq 2\left(s^2 - r^2 - 4Rr\right)$
2). $s^2 \leq \frac{1}{4}\sum\sqrt{r_a^4 + 14r_a^2r_b^2 + r_b^4} \leq (4R+r)^2 - s^2$
3). $r(4R+r) \leq \frac{1}{4}\sum\sqrt{(s-a)^4 + 14(s-a)^2(s-b)^2 + (s-b)^4} \leq$
$\leq s^2 - 2r^2 - 8Rr$
4). $\frac{2s^2r}{R} \leq \frac{1}{4}\sum\sqrt{h_a^4 + 14h_a^2h_b^2 + h_b^4} \leq \frac{\left(s^2+r^2+4Rr\right)^2 - 16s^2Rr}{4R^2}$
5). $\frac{s^2+r^2-8Rr}{16R^2} \leq \frac{1}{4}\sum\sqrt{\sin^8\frac{A}{2} + 14\sin^4\frac{A}{2}\sin^4\frac{B}{2} + \sin^8\frac{B}{2}} \leq \frac{8R^2+r^2-s^2}{8R^2}$
6). $\frac{s^2+(4R+r)^2}{16R^2} \leq \frac{1}{4}\sum\sqrt{\cos^8\frac{A}{2} + 14\cos^4\frac{A}{2}\cos^4\frac{B}{2} + \cos^8\frac{B}{2}} \leq \frac{(4R+r)^2-s^2}{8R^2}$

Mihály Bencze

PP. 16579. If $x \in \left[0, \frac{\pi}{2}\right]$, then $\left(\frac{(\sin x+\cos x)\left(\sqrt[3]{\sin^2 x}+\sqrt[3]{\cos^2 x}\right)}{4}\right)^3 \leq$
$\frac{(\sin x+\cos x)\left(\sqrt[3]{\sin^2 x}+\sqrt[3]{\cos^2 x}\right)\left(\sqrt[3]{\sin^4 x}+\sqrt[3]{\cos^4 x}\right)}{16} \leq \frac{\sin^3 x+\cos^3 x}{4}$.

Mihály Bencze

PP. 16580. Let ABC be a triangle, and $M \in Int(ABC)$,
$AA_1 \cap BB_1 \cap CC_1 = \{M\}$, where $A_1 \in BC$; $B_1 \in CA$; $C_1 \in AB$. Determine all points M such that
$\min\left\{\frac{MA}{MA_1}, \frac{MB}{MB_1}, \frac{MC}{MC_1}\right\} \leq 2 \leq \max\left\{\frac{MA}{MA_1}, \frac{MB}{MB_1}, \frac{MC}{MC_1}\right\}$.

Mihály Bencze

PP. 16581. In all triangle ABC hold $\frac{s^2-(2R+r)^2}{2R(R+r)} \leq \frac{r}{3R}$.

Mihály Bencze

PP. 16582. Let ABC be a triangle. Determine all $x > 1$ such that
$\sum \left(\frac{\cos \frac{A-B}{x}}{\sin \frac{C}{x}} \right)^x \geq 3x^x$.

Mihály Bencze

PP. 16583. 1). If $x \in R$, then $\max \left\{ \sin \left(sh^2 x \right); \sin \left(ch^2 x \right) \right\} > \frac{1}{3}$
2). Deterine $\max \left\{ \cos \left(sh^2 x \right); \cos \left(ch^2 x \right) \right\}$

Mihály Bencze

PP. 16584. If $x \in R$ then $a^{\sin x} b^{\cos x} + a^{\cos x} b^{\sin x} \leq 2a^{\frac{\ln a}{\sqrt{\ln^2 a + \ln^2 b}}} b^{\frac{\ln b}{\sqrt{\ln^2 a + \ln^2 b}}}$.

Mihály Bencze

PP. 16585. If $x \in [0,1]$, then $\prod\limits_{k=1}^{n} x^k \left(1 - x^k\right)^k \leq \frac{1}{(n+1)^{n+1}}$

Mihály Bencze

PP. 16586. If $a > 2$, then $\prod\limits_{k=1}^{n} k! \leq a^{\frac{n(n+1)(2n+1)}{12}}$.

Mihály Bencze

PP. 16587. Determine all $x, y \in [0,1]$ such that
$(\arcsin x)^y + (\arccos y)^x = (arctgx)^y + (arcctgy)^x$.

Mihály Bencze

PP. 16588. If $n \in N^*$, then $\prod\limits_{k=1}^{n} (k!)^{\frac{n-k+1}{k}} \leq 2^{\frac{-n(n+1)}{2}} \prod\limits_{k=1}^{n} (k+1)!$.

Mihály Bencze

PP. 16589. If $a, b, c > 0$ and $a + b + c = 1$, then $\sum \frac{a}{a+bc} \geq \frac{9}{4}$.

Mihály Bencze

PP. 16590. If $a, b, c \in (0,1)$ and $a+b+c \geq 2$, then $\sum \frac{a}{1-a} \geq 6$.

Mihály Bencze

PP. 16591. Solve in Z the equation $x^2+y^2+z^2=2\left(12t^2+1\right)$.

Mihály Bencze

PP. 16592. If $x, y, z \in R$, then $\prod\left(\sin^2 x \cos^2 y+\cos^2 x \sin^2 y\right) \leq \frac{1}{8}$.

Mihály Bencze

PP. 16593. If $a_k \in N^*$ $(k=1,2,...,n)$ and $d=(a_1, a_2, ..., a_n)$, then $\sum\limits_{k=1}^{n} \varphi^n(a_k) \geq n\varphi(d)\varphi\left(\prod\limits_{k=1}^{n} a_k\right)$.

Mihály Bencze

PP. 16594. Solve in N the following equation $\sum\limits_{k=1}^{n}(k+1)^x=y^2$.

Mihály Bencze

PP. 16595. If $a, b, c>0$, then $\sum \frac{a(2a+b+c)}{ab+bc+ca+c^2} \geq 3$.

Mihály Bencze

PP. 16596. Prove that:
1). $\sum\limits_{k=1}^{n}(k!)^{-\frac{2}{k}} \geq \frac{2(2n-1)}{n+1}$
2). $\sum\limits_{k=1}^{n}(k+1)(k!)^{-\frac{3}{k}} \geq \frac{4n}{n+1}$

Mihály Bencze

PP. 16597. Prove that:
1). $\sum\limits_{k=1}^{n}\left(3k^2+3k-1\right)(k!)^{-\frac{4}{k}} \geq \frac{10(2n-1)}{n+1}$
2). $\sum\limits_{k=1}^{n}(k+1)\left(2k^2+2k-1\right)(k!)^{-\frac{5}{k}} \geq \frac{12n}{n+1}$

Mihály Bencze

PP. 16598. If $a, b > 0$, then $\frac{b}{a(a^2+4ab+3b^2)} + \frac{a}{b(3a^2+4ab+b^2)}] \geq \frac{4ab}{(a+b)^4}$.

Mihály Bencze

PP. 16599. If $x \in \left[0, \frac{\pi}{2}\right]$, then

1). $\frac{\sin x}{\sin x+\cos ecx} + \frac{\cos x}{\cos x+\sec x} \geq \frac{1}{3}\left(1+\sin 2x\right)$
2). $\sqrt[3]{\sin^2 x} + \sqrt[3]{\cos^2 x} \leq \sqrt[3]{4}$

Mihály Bencze

PP. 16600. If $a, b, c > 0$ and $\lambda \geq 0$, then $\sum \frac{a+\lambda}{a+2b} \geq 1 + \lambda \left(\frac{\sum \sqrt{a}}{\sum a}\right)^2$.

Mihály Bencze

PP. 16601. If $a \in R$, $b \in \left(-\frac{1}{4}, +\infty\right)$ and $f : [0,1] \to (0, +\infty)$ is continuous,

then $\int\limits_0^1 f^2(x)\,dx \left(\int\limits_0^1 x^{2a} f(x)\,dx\right)^2 \geq (4b+1)\left(\int\limits_0^1 x^{a+b} f(x)\,dx\right)^4$.

Mihály Bencze

PP. 16602. If $b > a > 0$, then
$\left(\int\limits_a^b \frac{dx}{x\left(x+\sin^2 t\cos^2 t\right)+\sin^4 t}\right)^{-1} + \left(\int\limits_a^b \frac{dx}{x\left(x+\sin^2 t\cos^2 t\right)+\cos^4 t}\right)^{-1} \geq \frac{1}{\ln b-\ln a}$ for all $t \in R$.

Mihály Bencze

PP. 16603. If $a, b, c > 0$, then $\sum \frac{a}{a+b+\frac{bc}{a}} \geq 1$.

Mihály Bencze

PP. 16604. If $a, b > 0$, then
$\frac{b}{a\left(a^{2n-2}+C_{2n}^1 a^{2n-3}b+...+C_{2n}^{n-1}a^{n-1}b^{n-1}+\frac{1}{2}C_{2n}^n a^{n-2}b^n\right)} +$
$\frac{a}{b\left(\frac{1}{2}C_{2n}^n a^n b^{n-2}+C_{2n}^{n+1}a^{n-1}b^{n-1}+...+C_{2n}^{2n-1}ab^{2n-3}+b^{2n-2}\right)} \geq \frac{4ab}{(a+b)^{2n}}$ for all $n \in N^*$, $n \geq 2$.

Mihály Bencze

PP. 16605. If $a, b > 0$, then

$\frac{b}{a\left(a^{2n-1}+C_{2n+1}^{1}a^{2n-2}b+...+C_{2n+1}^{n}a^{n-1}b^{n}\right)}+\frac{a}{b\left(C_{2n+1}^{n+1}a^{n}b^{n-1}+...+C_{2n+1}^{2n}ab^{2n-2}+b^{2n-1}\right)}\geq$
$\frac{4ab}{(a+b)^{2n+1}}$, for all $n \in N^*$.

Mihály Bencze

PP. 16606. If $x_0 \in (-1, 1)$ and $x_{n+1} = x_n^k - x_n^p + 1$ for all $n \in N$, then determine all $k, p \in N$ for which $\lim\limits_{n\to\infty} x_1 x_2 ... x_n = 0$.

Mihály Bencze

PP. 16607. In all triangle ABC holds:

1). $\sum \frac{1}{a(b+1)} \geq \frac{3}{1+4sRr}$
2). $\sum \frac{1}{r_a(r_b+1)} \geq \frac{3}{1+s^2r}$
3). $\sum \frac{1}{h_a(h_b+1)} \geq \frac{3R}{R+2s^2r^2}$
4). $\sum \frac{1}{\sin^2\frac{A}{2}\left(1+\sin^2\frac{B}{2}\right)} \geq \frac{48R^2}{16R^2+r^2}$
5). $\sum \frac{1}{\cos^2\frac{A}{2}\left(1+\cos^2\frac{B}{2}\right)} \geq \frac{48R^2}{16R^2+s^2}$

Mihály Bencze

PP. 16608. In all triangle ABC holds:

1). $\sqrt[3]{256s^2} \geq 24\sqrt[3]{2Rr} + \sqrt[3]{9\left(s^2 - 3r^2 - 6Rr\right)}$
2). $9\sqrt[3]{s^2} \geq 24\sqrt[3]{r^2} + \sqrt[3]{9\left(s^2 - 12Rr\right)}$
3). $9\left(4R + r\right) \geq 24\sqrt[3]{s^2r} + \sqrt[3]{9\left(\left(4R + r\right)^3 - 12s^2R\right)}$
4). $9\left(1 - \frac{r}{2R}\right) \geq 12\sqrt[3]{\frac{r^2}{2R^2}} + \sqrt[3]{\frac{9\left((2R-r)\left((4R+r)^2-3s^2\right)+6Rr^2\right)}{32R^3}}$
5). $9\left(2 + \frac{r}{2R}\right) \geq 12\sqrt[3]{\frac{s^2}{2R^2}} + \sqrt[3]{\frac{9\left((4R+r)^3-3s^2(2R+r)\right)}{32R^3}}$

Mihály Bencze

PP. 16609. In all triangle ABC holds:
1). $9\left(s^2 + r^2 + 2Rr\right) \geq 16s\sqrt{6Rr}$
2). $3\sqrt{3}R \geq 2s$
3). $9s\left(s^2 + r^2 + 2Rr\right) \geq 4\sqrt{3}\left(s^2 + r^2 + 4Rr\right)^{\frac{3}{2}}$

4). $9sR \geq 2\left(4R+r\right)^{\frac{3}{2}}\sqrt{3r}$
5). $9\left(\left(2R-r\right)\left(s^2+r^2-8Rr\right)-2Rr^2\right) \geq 64R^2r\left(1-\frac{r}{2R}\right)^{\frac{3}{2}}\sqrt{3}$
6). $9\left(\left(4R+r\right)^3+s^2\left(2R+r\right)\right) \geq 64R^2s\left(2+\frac{r}{2R}\right)^{\frac{3}{2}}\sqrt{3}$

Mihály Bencze

PP. 16610. If $x, y > 0$ and $a, b \in [0,1]$, then $\frac{a}{\sqrt{xb^2+y}}+\frac{b}{\sqrt{xa^2+y}} \leq \frac{2}{\sqrt{x+y}}$.

Mihály Bencze

PP. 16611. If $x \in [0,1]$, $n, k, p \in N$, $k \geq 1$, $n \geq 2$, then
$\max\left\{x\left(1-x^n\right); x\left(\frac{1}{k}-\frac{x^n}{\binom{n+k}{n+1}}\right); x\left(\frac{1}{p+2}-\frac{x^n}{n+p+2}\right)\left(p+1\right)\right\} \leq \frac{n}{\sqrt[n]{(n+1)^{n+1}}}$.

Mihály Bencze

Solutions

PP. 12741. If $x = \frac{r}{h_a}, x = \frac{r}{h_b}, z = \frac{r}{h_c}, t = \frac{r}{h_d}$, then

$$\sum \frac{h_a h_b + h_b h_c + h_c h_a}{h_a^3 + h_b^3 + h_c^3} = \frac{1}{r} \sum \frac{(xyz)^2 (x+y+z)}{(xy)^3 + (yz)^3 + (zx)^3} \leq \frac{1}{3r} \sum (x+y+z) = \frac{1}{r}$$

If $x = \frac{r}{r_a}, x = \frac{r}{r_b}, z = \frac{r}{r_c}, t = \frac{r}{r_d}$ then

$$\sum \frac{r_a r_b + r_b r_c + r_c r_a}{r_a^3 + r_b^3 + r_c^3} = \frac{1}{r} \sum \frac{(xyz)^2 (x+y+z)}{(xy)^3 + (yz)^3 + (zx)^3} \leq \frac{1}{3r} \sum (x+y+z) = \frac{2}{r}$$

Traian Ianculescu

PP. 12824. In Inegalitati from C. Panaitopol, V. Bandila, M. Lascu is proved

$$(k+1) \sqrt[k+1]{(k+1)!} - k \sqrt[k]{k!} < k+1$$

for all $k \geq 2$. For $k = 1$ we get $2\sqrt{2} < 3$, then
$\prod\limits_{k=1}^{n} \left((k+1) \sqrt[k+1]{(k+1)!} - k \sqrt[k]{k!} \right) < \prod\limits_{k=1}^{n} (k+1) = (n+1)!$.

Traian Ianculescu

PP. 12932. From the previous problems we get

$$\sum \left(\frac{h_a h_b + h_b h_c + h_c h_a}{h_a^3 + h_b^3 + h_c^3} \right)^\alpha + \sum \left(\frac{r_a r_b + r_b r_c + r_c r_a}{r_a^3 + r_b^3 + r_c^3} \right)^\beta \leq \left(\frac{1}{r} \right)^\alpha + \left(\frac{2}{r} \right)^\beta$$

for all $\alpha, \beta > 0$.
Remark. *If* $\alpha, \beta \in [0,1]$ *then*

$$\sum \left(\frac{h_a h_b + h_b h_c + h_c h_a}{h_a^3 + h_b^3 + h_c^3} \right)^\alpha + \sum \left(\frac{r_a r_b + r_b r_c + r_c r_a}{r_a^3 + r_b^3 + r_c^3} \right)^\beta \leq$$

$$\leq 3 \left(\left(\frac{1}{3r} \right)^\alpha + \left(\frac{2}{3r} \right)^\beta \right)$$

Traian Ianculescu

PP. 12974. We have

$$\Gamma\left(k+\frac{1}{2}\right)=\frac{(2k)!\sqrt{\pi}}{4^k\cdot k!}$$

therefore

$$\prod_{k=1}^{n}\Gamma\left(k+\frac{1}{2}\right)\le\left(\frac{1}{n}\sum_{k=1}^{n}\Gamma\left(k+\frac{1}{2}\right)\right)^n=\left(\frac{\sqrt{\pi}}{n}\sum_{k=1}^{n}\frac{(2k)!}{4^kk!}\right)$$

Traian Ianculescu

PP. 12977. Because $1,77<\sqrt{\pi}<1,78$ we get $1,77<\frac{4^n\Gamma\left(n+\frac{1}{2}\right)}{\binom{2n}{n}n!}<1,78$ because

$$\Gamma\left(n+\frac{1}{2}\right)=\left(\frac{n!}{4^n}\binom{2n}{n}\right)\sqrt{\pi}$$

Traian Ianculescu

PP. 12997. In inequalities $\ln\left(x+\sqrt{1+x^2}\right)\le x\le\ln\left(\frac{2+x}{2-x}\right)$ we take

$x=\frac{1}{k(k+1)}$ $(k=1,2,...,n)$ therefore

$$\sum_{k=1}^{n}\ln\left(\frac{1+\sqrt{k^2\left(k+1\right)^2+1}}{k\left(k+1\right)}\right)\le\sum_{k=1}^{n}\frac{1}{k\left(k+1\right)}=\frac{n}{n+1}\le$$

$$\le\sum_{k=1}^{n}\ln\left(\frac{2k\left(k+1\right)+1}{2k\left(k+1\right)-1}\right)$$

Traian Ianculescu

PP. 13014. We use the Corollary 1.2. from the paper: "About Exponencial Functions", author M. Bencze and Gy. Szöllõsy namely:

If $a_k\in(0,1)\cup(1,+\infty)$ $(k=1,2,...,n)$, then

$$\frac{\sum\limits_{k=1}^{n}a_k-1}{\ln\left(\sum\limits_{k=1}^{n}a_k\right)}\le\sum_{k=1}^{n}\frac{a_k-1}{\ln a_k}$$

(see Octogon Mathematical Magazine, Vol. 15, Nr. 2A, October 2007)
If $n=2$, $a_1=F_k, a_2=F_{k+1}$, then we get

$$(F_k-1)\ln F_{k+1}\ln F_{k+2}+(F_{k+1}-1)\ln F_k\ln F_{k+2}\geq(F_{k+2}-1)\ln F_k\ln F_{k+1}$$

Traian Ianculescu

PP. 13015. If in previous problem we take $n=2$ and $a_1=L_k, a_2=L_{k+1}$ then

$$\frac{L_k-1}{\ln L_k}+\frac{L_{k+1}-1}{\ln L_{k+1}}\geq\frac{L_{k+2}-1}{\ln L_{k+2}}$$

Traian Ianculescu

PP. 13017. We have

$$\ln P=\ln\prod_{k=1}^{n}\frac{k^2+k+1}{k\,(k+1)}=\sum_{k=1}^{n}\ln\left(1+\frac{1}{k\,(k+1)}\right)<\sum_{k=1}^{n}\frac{1}{k\,(k+1)}=\frac{n}{n+1}.$$

Traian Ianculescu

PP. 13018. In the paper "New Inequalities in Triangle" authors M. Bencze and S. Arslanagic (Octogon Mathematical Magazine, Vol. 14, Nr. 1, April 2006) is proved that if $x,y,z\in[0,1]$, then

$$\frac{x}{yz+1}+\frac{y}{zx+1}+\frac{z}{xy+1}\leq 2$$

In this we take $x=\frac{F_n}{F_{n+1}}, y=\frac{L_k}{L_{k+1}}, z=\frac{P_n}{P_{m+1}}\in[0,1]$ so we obtain

$$\frac{F_nL_{k+1}P_{m+1}}{F_{n+1}\left(L_kP_m+L_{k+1}P_{m+1}\right)}+\frac{L_kF_{n+1}F_{m+1}}{L_{k+1}\left(F_nP_m+F_{n+1}P_{m+1}\right)}+$$

$$+\frac{P_mF_{n+1}L_{k+1}}{P_{m+1}\left(F_nL_k+F_{n+1}L_{k+1}\right)}\leq 2$$

Traian Ianculescu

PP. 13022. In $\frac{\sin b}{\sin a} \leq \frac{b}{a}$ for all $a, b \in \left(0, \frac{\pi}{2}\right]$, $a \leq b$ we take $a = \frac{\pi}{k+1}$, $b = \frac{\pi}{k}$ $(k = 1, 2, ..., n)$ and we get

$$\sin\frac{\pi}{k}\cos ec\frac{\pi}{k+1} \leq \frac{k+1}{k} \text{ or}$$

$$\sum_{k=1}^{n} k\sin\frac{\pi}{k}\cos ec\frac{\pi}{k+1} \leq \sum_{k=1}^{n}(k-1) = \frac{n(n+3)}{2}$$

Traian Ianculescu

PP. 13023. If in $\frac{\sin b}{\sin a} \leq \frac{b}{a}$ we take $b = \frac{\pi}{k(k+1)}$, $a = \frac{\pi}{k^2+k+1}$ $(k = 1, 2, ..., n)$ then

$$\sum_{k=1}^{n}\sin\frac{\pi}{k(k+1)}\cos ec\frac{\pi}{k^2+k+1} \leq \sum_{k=1}^{n}\left(1+\frac{1}{k(k+1)}\right) = \frac{n(n+2)}{n+1}$$

Traian Ianculescu

PP. 13043. From $\frac{x}{\sqrt{1+x^2}} \leq arctgx \leq x$ we have for $x = \frac{1}{\sqrt{k}}$ the following $\frac{1}{\sqrt{k+1}} \leq arctg\frac{1}{\sqrt{k}} \leq \frac{1}{\sqrt{k}}$ or $k \leq \frac{1}{\left(arctg\frac{1}{\sqrt{k}}\right)^2} \leq k+1$ $(k = 1, 2, ..., n)$, therefore

$$\frac{n(n+1)}{2} = \sum_{k=1}^{n} k \leq \sum_{k=1}^{n}\frac{1}{\left(arctg\frac{1}{\sqrt{k}}\right)^2} \leq \sum_{k=1}^{n}(k+1) = \frac{n(n+3)}{2}$$

Traian Ianculescu

PP. 13088. We have

$$\sum\frac{x}{x+\sqrt{3(x^2+y^2+z^2)}} \leq \sum\frac{x}{2x+y+z} = 3 - s\sum\frac{1}{x+s} \leq 3 - \frac{9}{4} = \frac{3}{4}$$

Traian Ianculescu

PP. 13106. In $(1+x)^k \geq 1+kx$ we take $x = t-1$ so $t^k \geq 1+k(t-1)$, after then $t = u^{\frac{1}{k}}$ so $u^{\frac{1}{k}} \leq 1+\frac{u-1}{k}$ and finally setting $t = \frac{a}{b}$ we get $\left(ab^{k-1}\right)^{\frac{1}{k}} \leq b+\frac{a-b}{k}$ and $\left(ba^{k-1}\right)^{\frac{1}{k}} \leq a+\frac{b-a}{k}$, therefore

$$\sum_{k=1}^{n}\left(ab^{k-1}\right)^{\frac{1}{k}} \leq nb+a-b \text{ and } \sum_{k=1}^{n}\left(ba^{k-1}\right)^{\frac{1}{k}} \leq na+b-a$$

after addition we get:

$$\sum_{k=1}^{n}\left(\left(ab^{k-1}\right)^{\frac{1}{k}}+\left(ba^{k-1}\right)^{\frac{1}{k}}\right)\leq n\left(a+b\right)$$

Traian Ianculescu

PP. 13166. If $x_i \in R$ $(i=1,2,3,4,5)$ and $\sum\limits_{i=1}^{5} x_i = 0$, then $\sum\limits_{i=1}^{n} |\cos x_i| \geq 1$.

In this we take $x_1 = F_n, x_2 = F_{n+1}, x_3 = F_{n+3}, x_4 = F_{n+5}, x_5 = -F_{n+6}$ and

after then $x_1 = L_n, x_2 = L_{n+1}, x_3 = L_{n+3}, x_4 = L_{n+5}, x_5 = -L_{n+6}$ so we obtain:

$$|\cos F_n| + |\cos F_{n+1}| + |\cos F_{n+3}| + |\cos F_{n+5}| + |\cos F_{n+6}| \geq 1$$

and

$$|\cos L_n| + |\cos L_{n+1}| + |\cos L_{n+3}| + |\cos L_{n+5}| + |\cos L_{n+6}| \geq 1$$

for all $n \in N$.

Traian Ianculescu

PP. 13200. We have $\sum \frac{m_a}{h_a} \leq \frac{3R}{2r}$ (see P.S.L 7/Revista de Matematica din Timisoara), therefore

$$\sum\left(\frac{m_a}{h_a}\right)^{\alpha}\leq 3\left(\frac{R}{2r}\right)^{\alpha}$$

Traian Ianculescu

PP. 13201. Because $\sum a \cdot AG \geq 4sr$ (P.S.G 7/Revista de Matematica din Timisoara), then we have

$$\sum\left(a\cdot AG\right)^{\alpha}\geq 3\left(\frac{1}{3}\sum a\cdot AG\right)^{\alpha}\geq 3^{1-\alpha}\left(4sr\right)^{\alpha}$$

Traian Ianculescu

PP. 13202. Because $R_1 + R_2 + R_3 \geq 3R$ (see P.S.L 9, Revista de Matematica din Timisoara), then we have

$$\sum R_1^\alpha \geq 3\left(\frac{1}{3}\sum R_1\right)^\alpha \geq 3R^\alpha$$

Traian Ianculescu

PP. 13215. If $P \in [OH]$, then $6r \leq PA + PB + PC \leq 3R$ (see P.S.L.I - Revista de Matematica din Timisoara), therefore

$$\sum (MA)^x \leq 3\left(\frac{1}{3}\sum MA\right)^x \leq 3R^x \text{ and}$$

$$\sum (MA)^y \geq 3\left(\frac{1}{3}\sum MA\right)^y \geq 3\,(2r)^x$$

after addition we get

$$3\,(2r)^y + \sum (MA)^x \leq 3R^x + \sum (MA)^y$$

for all $0 \leq x \leq 1 \leq y$.

Traian Ianculescu

PP. 13224. We have

$$\sum \left(\frac{1}{a}\right)^x \leq 3\left(\frac{1}{3}\sum \frac{1}{a}\right)^x \leq 3\left(\frac{\sqrt{3}}{6r}\right)^x \text{ and}$$

$$\sum \left(\frac{1}{a}\right)^y \geq 3\left(\frac{1}{3}\sum \frac{1}{a}\right)^y \geq 3\left(\frac{\sqrt{3}}{3R}\right)^y$$

After addition we obtain:

$$3\left(\frac{\sqrt{3}}{3R}\right)^y + \sum \left(\frac{1}{a}\right)^x \leq 3\left(\frac{\sqrt{3}}{6r}\right)^x + \sum \left(\frac{1}{a}\right)^y$$

for all $0 \leq x \leq 1 \leq y$.

Traian Ianculescu

PP. 13225. We have

$$\sum \left(a^2\right)^x \leq 3\left(\frac{1}{3}\sum a^2\right)^x \leq 3\left(\frac{2s^2}{3}\right)^x \text{ and}$$

$$\sum \left(a^2\right)^y \geq 3\left(\frac{1}{3}\sum a^2\right)^y \geq 3\left(\frac{2s}{3}\right)^{2y}$$

and after addition we get

$$3\left(\frac{2s}{3}\right)^{2y} + \sum a^{2x} \leq 3\left(\frac{2s^2}{3}\right)^x + \sum a^{2y}$$

for all $0 \leq x \leq 1 \leq y$.

Traian Ianculescu

PP. 13226. We multiply the inequalities $\sum w_a \leq \sum_a \leq \sum r_a$ and $\prod w_a \leq \prod r_a \leq \prod m_a$ and we obtain:

$$\left(\sum w_a\right) w_a w_b w_c \leq \left(\sum m_a\right) r_a r_b r_c \leq \left(\sum r_a\right) m_a m_b m_c$$

Traian Ianculescu

PP. 13227. We starting from

$$\sum w_a^2 \leq s^2 \leq \sum r_a^2, \text{ therefore}$$

$$\sum \left(w_a^2\right)^x \leq 3\left(\frac{1}{3}\sum w_a^2\right)^x \leq 3\left(\frac{s^2}{3}\right)^x \text{ and}$$

$$\sum \left(r_a^2\right)^y \geq 3\left(\frac{1}{3}\sum r_a^2\right)^y \geq 3\left(\frac{s^2}{3}\right)^y$$

and after addition we get

$$3\left(\frac{s^2}{3}\right)^y + \sum w_a^{2x} \leq 3\left(\frac{s^2}{3}\right)^x \leq \sum r_a^{2y}$$

for all $0 \leq x \leq 1 \leq y$.

Traian Ianculescu

PP. 13236. Using the problem O.X. 57-Revista de Matematica din Timisoara we have

$$\frac{1}{2}<\sum\frac{OA}{AB+AC}\leq\frac{3\sqrt{2}}{4} \text{ so}$$

$$\sum\left(\frac{OA}{AB+AC}\right)^x\leq 3\left(\frac{1}{3}\sum\frac{OA}{AB+AC}\right)^x\leq 3\left(\frac{\sqrt{2}}{4}\right)^x=3\cdot 2^{-\frac{3x}{2}} \text{ and}$$

$$\sum\left(\frac{OA}{AB+AC}\right)^y\geq 3\left(\frac{1}{3}\sum\frac{OA}{AB+AC}\right)^y\geq 3\cdot 6^{-y}$$

and after addition we get:

$$3\cdot 6^{-y}+\sum\left(\frac{OA}{AB+AC}\right)^x\leq 3\cdot 2^{-\frac{3x}{2}}+\sum\left(\frac{OA}{AB+AC}\right)^y$$

for all $0\leq x\leq 1\leq y$.

Traian Ianculescu

PP. 13238. We have

$$\left(\sum x\right)\left(\sum\frac{1}{x}\right)=1+\frac{(x+y)(y+z)(z+x)}{xyz} \text{ and}$$

$$\left(\sum x\right)\left(\sum\frac{1}{x}\right)=1+\underbrace{\frac{(x+y)(y+z)(z+x)}{nxyz}+...+\frac{(x+y)(y+z)(z+x)}{nxyz}}_{n-time}\geq$$

$$\geq(n+1)\sqrt[n+1]{\left(\frac{(x+y)(y+z)(z+x)}{nxyz}\right)^n}$$

this is a generalization of problem O.IX. 43-Revista de Matematica din Timisoara of author Robert Csetnek (for $n=8$ we obtain this problem).

Traian Ianculescu

PP. 13239. We have

$$x_k+\underbrace{(1+...+1)}_{(m-1)-\text{ time}}\geq m\sqrt[m]{x_k}\text{ or}$$

$$\frac{1}{\sqrt[m]{x_k}}\geq\frac{m}{x_k+m-1}\ \ (k=1,2,...,n)\text{, therefore}$$

$$\sum_{k=1}^{n}\frac{1}{\sqrt[m]{x_k}}\geq m\sum_{k=1}^{n}\frac{1}{x_k+m-1}\geq\frac{mn^2}{m(m-1)+\sum\limits_{k=1}^{n}x_k}$$

If $n=m=3$, then

$$\frac{1}{\sqrt[3]{x}}+\frac{1}{\sqrt[3]{y}}+\frac{1}{\sqrt[3]{z}}\geq\frac{27}{x+y+z+6}$$

for all $x,y,z>0$, which is generalized by

$$\sum_{k=1}^{n}\frac{1}{\sqrt[m]{x_k}}\geq\frac{mn^2}{n(m-1)+\sum\limits_{k=1}^{n}x_k}$$

Traian Ianculescu

PP. 13240. We have

$$\sum\left(\frac{\log_k^2 x}{1+2\log_k y}\right)^{\alpha}\geq 3\left(\frac{1}{3}\sum\frac{\log_k^2 x}{1+2\log_k y}\right)^{\alpha}\text{ but}$$

$$\sum\frac{\log_k^2 x}{1+2\log_k y}\geq\frac{\left(\sum\log_k x\right)^2}{3+2\sum\log_k y}=\frac{\log_k^2(xyz)}{3+2\log_k(xyz)}=\frac{a^2}{3+2a}\geq 1$$

because $a=\log_k(xyz)\geq\log_k k^3=3$, therefore

$$\sum\left(\frac{\log_k^2 x}{1+2\log_k y}\right)^{\alpha}\geq 3^{1-\alpha}$$

for all $\alpha\geq 1$.

Traian Ianculescu

PP. 13247. In IX. 126-Revista de Matematica din Timisoara is proved that

$$\sum \frac{a}{b+1} \geq \frac{3\sum a}{\sum a+3},$$

when $a, b, c > -1$, therefore

$$\sum \left(\frac{a}{b+1}\right)^{\alpha} \geq 3\left(\frac{1}{3}\sum \frac{a}{b+1}\right)^{\alpha} \geq 3\left(\frac{a+b+c}{a+b+c+3}\right)^{\alpha}$$

for all $\alpha \geq 1$.

Traian Ianculescu

PP. 13248. We have

$$\sum x^{-\frac{\alpha}{3}} = \sum \left(\frac{1}{\sqrt[3]{x}}\right)^{\alpha} \geq 3\left(\frac{1}{3}\sum \frac{1}{\sqrt[3]{x}}\right)^{\alpha} \geq 3\left(\frac{9}{6+\sum x}\right)^{\alpha}$$

because from PP. 13239 we have $\sum \frac{1}{\sqrt[3]{x}} \geq \frac{27}{x+y+z+6}$, therefore

$$\sum x^{-\frac{\alpha}{3}} \geq 3^{2\alpha+1}\left(\sum x\right)^{-\alpha}$$

for all $\alpha \geq 1$.

Traian Ianculescu

PP. 13260. We have $[x] + \left[x+\frac{1}{2}\right] = [2x]$ and

$$\left[\frac{x+1}{3}\right] = \left[2\left(\frac{x+1}{6}\right)\right] = \left[\frac{x+1}{6}\right] + \left[\frac{x+4}{6}\right]$$

from $[x] + \left[x+\frac{1}{3}\right] + \left[x+\frac{2}{3}\right] = [3x]$ so

$$\left[\frac{x+1}{2}\right] = \left[3\left(\frac{x+1}{6}\right)\right] = \left[\frac{x+1}{6}\right] + \left[\frac{x+1}{6}+\frac{1}{3}\right] + \left[\frac{x+1}{6}+\frac{2}{3}\right] =$$

$$= \left[\frac{x+1}{6}\right] + \left[\frac{x+3}{6}\right] + \left[\frac{x+5}{6}\right]$$

and finally we get

$$\left[\frac{x+1}{6}\right] + \left[\frac{x+3}{6}\right] + \left[\frac{x+5}{6}\right] = \left[\frac{x+1}{6}\right] + \left[\frac{x+4}{6}\right]$$

and

$$\left(\left[\frac{x+1}{6}\right]\right)^{\alpha}+\left(\left[\frac{x+3}{6}\right]\right)^{\alpha}+\left(\left[\frac{x+5}{6}\right]\right)^{\alpha}\leq$$

$$\leq 3\left(\frac{1}{3}\left(\left[\frac{x+1}{6}\right]+\left[\frac{x+3}{6}\right]+\left[\frac{x+5}{6}\right]\right)\right)^{\alpha}=3^{1-\alpha}\left(\left[\frac{x+1}{6}\right]+\left[\frac{x+4}{6}\right]\right)^{\alpha}$$

for all $\alpha\in[0,1]$.

Traian Ianculescu

PP. 13293. If $H_k=\frac{k}{\frac{1}{a_1}+\ldots+\frac{1}{a_k}}=\frac{k}{x+y}$, $H_{k-1}=\frac{k-1}{x}$, then $\frac{k^2}{x+y}=\frac{(k-1)^2}{x}\leq\frac{1}{y}\Leftrightarrow((k-1)y-x)^2\geq 0$, therefore

$$kH_k-(k-1)H_{k-1}\leq a_k \text{ and}$$

$$\sum_{k=1}^{n}\frac{1}{kH_k-(k-1)H_{k-1}}\geq\sum_{k=1}^{n}\frac{1}{a_k}=\frac{n}{H_n}$$

Traian Ianculescu

PP. 13300. If $f:[0,1]\to R$, where

$$f(x)=\sqrt[n]{1+x}+\sqrt[n]{1-x}+\frac{(n-1)x^2}{n^2}$$

then

$$f'(x)=\frac{1}{n}(1+x)^{\frac{1}{n}-1}-\frac{1}{n}(1-x)^{\frac{1}{n}-1}+\frac{2(n-1)x}{n^2},\ f'(0)=0$$

$$f''(x)=\frac{1}{n}\left(\frac{1}{n}-1\right)(1+x)^{\frac{1}{n}-2}-\frac{1}{n}\left(\frac{1}{n}-1\right)(1-x)^{\frac{1}{n}-2}+\frac{2(n-1)}{n^2},\ f''(0)=0$$

$$f'''(x)=\frac{1}{n}\left(\frac{1}{n}-1\right)\left(\frac{1}{n}-2\right)\left((1+x)^{\frac{1}{n}-3}-(1-x)^{\frac{1}{n}-3}\right)<0,$$

therefore f'' is decreasing and $f''(x)\leq f''(0)=0$, f' is decreasing and $f'(x)\leq f'(0)=0$, f is decreasing and $f(x)\leq f(0)=2$, therefore

$$\sqrt[n]{1+x}+\sqrt[n]{1-x}\le 2-\frac{(n-1)\,x^2}{n^2} \text{ or}$$

$$S_1=\sqrt[n]{1+\sin x}+\sqrt[n]{1-\sin x}\le 2-\frac{(n-1)\sin^2 x}{n^2} \text{ and}$$

$$S_2=\sqrt[n]{1+\cos x}+\sqrt[n]{1-\cos x}\le 2-\frac{(n-1)\cos^2 x}{n^2} \text{ and}$$

$$\sqrt[n]{1+\sin x}+\sqrt[n]{1-\cos x}\le S_1+S_2\le 4-\frac{n-1}{n^2}$$

for all $x\in\left[0,\frac{\pi}{2}\right]$.

Traian Ianculescu

PP. 13305. Using the inequality

$$\sqrt[n]{1+x}+\sqrt[n]{1-x}\le 2-\frac{(n-1)\,x^2}{n^2}$$

for all $x\in[0,1]$ we get

$$S_1=\sum\left(\sqrt[n]{1+\sin^2\frac{A}{2}}+\sqrt[n]{\cos^2\frac{A}{2}}\right)\le 6-\frac{n-1}{n^2}\sum\sin^4\frac{A}{2}=$$

$$=6-\frac{(n-1)\left(8R^2+r^2-s^2\right)}{8n^2R^2}$$

$$S_2=\sum\left(\sqrt[n]{1+\cos^2\frac{A}{2}}+\sqrt[n]{\sin^2\frac{A}{2}}\right)\le 6-\frac{n-1}{n^2}\cos^4\frac{A}{2}=$$

$$=6-\frac{(n-1)\left((4R+r)^2-s^2\right)}{8n^2R^2}$$

Traian Ianculescu

PP. 13306. We have

$$\sqrt[n]{1+\frac{1}{k}}+\sqrt[n]{1-\frac{1}{k}}\le 2-\frac{n-1}{n^2k^2} \text{ so}$$

$$2-\sqrt[n]{\frac{k+1}{k}}-\sqrt[n]{\frac{k-1}{k}}\ge\frac{n-1}{n^2k^2} \text{ and}$$

$$\sum_{k=1}^{\infty}\left(2-\sqrt[n]{\frac{k+1}{k}}-\sqrt[n]{\frac{k-1}{k}}\right)\geq\frac{n-1}{n^2}\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{(n-1)\,\pi^2}{6n^2}$$

Traian Ianculescu

PP. 13330. In inequality

$$\sqrt[n]{1+x}+\sqrt[n]{1-x}\leq 2-\frac{(n-1)\,x^2}{n^2}$$

we take $x=\frac{1}{\sqrt{k(k+1)}}$ and we get

$$\sum_{k=1}^{n}\left(\sqrt[n]{1+\frac{1}{\sqrt{k\,(k+1)}}}+\sqrt[n]{1-\frac{1}{\sqrt{k\,(k+1)}}}\right)\leq$$

$$\leq\sum_{k=1}^{n}\left(2-\frac{n-1}{n^2k\,(k+1)}\right)=\frac{2n^3+2n^2-n+1}{n\,(n+1)}$$

Traian Ianculescu

PP. 13347. In Revista de Matematica din Timisoara, G. Buth and L. Vlaicu proved that:

1). if $a\geq b$ then $s_1=\sum\frac{x}{ax+by+bz}\leq\frac{3}{a+2b}$

2). if $a\leq b$ then $s_2=\sum\frac{x}{ax+by+bz}\geq\frac{3}{a+2b}$
after addition we get

$$\frac{3}{c+2b}+s_1\leq\frac{3}{a+2b}+s_2$$

Traian Ianculescu

PP. 13467. From $(a-b)\left(a^3-b^3\right)\geq 0$ we get $\frac{a^4+b^4}{ab}\geq a^2+b^2$, therefore

$$\sum\frac{a^4+b^4}{ab}\geq 2\left(\sum a^2\right)\geq\frac{2}{3}\sum a$$

for all $a,b,c>0$, therefore

$$\frac{F_n^4+F_{n+1}^4}{F_nF_{n+1}}+\frac{F_{n+1}^4+F_{n+3}^4}{F_{n+1}F_{n+3}}+\frac{F_{n+3}^4+F_n^4}{F_{n+3}F_n}\geq\frac{2}{3}\left(F_n+F_{n+1}+F_{n+3}\right)^2=$$

$$=\frac{2}{3}\left(F_{n+2}+F_{n+3}\right)^2=\frac{2}{3}F_{n+4}^2$$

Traian Ianculescu

PP. 13474. From

$$\sum_{k=0}^{n}\binom{p}{k}\binom{q}{m-k}=\binom{p+q}{m}$$

we get

$$\sum_{k=0}^{2n}\binom{2n-1}{k}\binom{2n+1}{2n-k}=\binom{4n}{2n}\text{ and}$$

$$S\left(n\right)=\sum_{k=1}^{2n-1}\binom{2n-1}{k}\binom{2n+1}{2n-k}=\binom{4n}{2n}-2n-1$$

therefore

$$\lim_{n\to\infty}\frac{S\left(n\right)}{n}=-2+\lim_{n\to\infty}\frac{(4n)!!}{\left((2n)!\right)^2n}>-2+\lim_{n\to\infty}\frac{2^n}{n}>-2+\lim_{n\to\infty}n=+\infty$$

Laurentiu Modan

PP. 13497. We have

$$\sum_{k=1}^{n}\frac{a_k^2}{a_k+b_k+c_k}\geq\frac{\left(\sum\limits_{k=1}^{n}a_k\right)^2}{\sum\limits_{k=1}^{n}\left(a_k+b_k+c_k\right)}=\frac{\left(\sum\limits_{k=1}^{n}a_k\right)^2}{3\sum\limits_{k=1}^{n}a_k}=3$$

Traian Ianculescu

PP. 13514. We have $k!>\left(\frac{k+1}{e}\right)^k>\frac{k!}{k+1}$ or $(k!)^{\frac{1}{k}}>\frac{k+1}{e}>\left(\frac{k!}{k+1}\right)^{\frac{1}{k}}$ $(k=1,2,...,n)$ and

$$\sum_{k=1}^{n}(k!)^{\frac{1}{k}}>\frac{n\left(n+3\right)}{2e}>\sum_{k=1}^{n}\left(\frac{k!}{k+1}\right)^{\frac{1}{k}}$$

Traian Ianculescu

PP. 13717. Using the Chebyshev's inequality we get

$$\sum a\cos^2\frac{A}{2}\leq\frac{1}{3}\left(\sum a\right)\left(\sum\cos^2\frac{A}{2}\right)\text{ but}$$

$$\sum\cos^2\frac{A}{2}=2\left(1+\prod\sin\frac{A}{2}\right)=2\left(1+\frac{r}{4R}\right)\leq\frac{9}{4}\Rightarrow\sum a\cos^2\frac{A}{2}\leq\frac{2s}{3}\cdot\frac{9}{4}=\frac{3s}{2}$$

Traian Ianculescu

PP. 13718. In all triangle ABC holds

$2<\sum\frac{a+b}{c}-\frac{\sum a^3}{abc}\leq 3\Leftrightarrow 8abc<\left(\sum a\right)\left(2\sum ab-\sum a^2\right)\leq 9abc$ but

$\sum ab=s^2+r^2+4Rr$, $\sum a^2=2s^2-2r^2-8Rr$, $abc=4sRr$ so we obtain

$32sRr<4sr\left(r+4R\right)\leq 36sRr$ or $8R<8R+2r\leq 9R$ which is true.
If $a\to m_a, b\to m_b, c\to m_c$, then we obtain

$$2<\sum\frac{m_a+m_b}{m_c}-\frac{\sum m_a^3}{m_am_bm_c}\leq 3$$

Traian Ianculescu

PP. 13731. We have

$$\sum\left(a\cos A\right)^{\alpha}\leq 3^{1-\alpha}\left(\sum a\cos A\right)^{\alpha}=3^{1-\alpha}\left(\frac{2rs}{R}\right)^{\alpha}\leq 3^{1-\alpha}s^{\alpha}$$

for all $\alpha\in\left[0,1\right]$.

Traian Ianculescu

PP. 13732. Using the P. Vasic's inequality we get

$$\sum\left(x\sin A\right)^{\alpha}\leq 3^{1-\alpha}\left(\sum x\sin A\right)^{\alpha}\leq 3^{1-\alpha}\left(\frac{\sqrt{3}}{2}\sum\frac{xy}{z}\right)^{\alpha}=2^{-\alpha}\cdot 3^{1-\frac{\alpha}{2}}\left(\sum\frac{xy}{z}\right)^{\alpha}$$

for all $\alpha\in\left[0,1\right]$.

Traian Ianculescu

PP. 13733. Using the D.F. Barrow‘s inequality we get

$$\sum (x\cos A)^{\alpha} \leq 3^{1-\alpha}\left(\sum x\cos A\right)^{\alpha} \leq 3^{1-\alpha}\left(\frac{1}{2}\sum \frac{xy}{z}\right)^{\alpha} = 3^{1-\alpha}2^{-\alpha}\left(\sum \frac{xy}{z}\right)^{\alpha}$$

for all $\alpha \in [0,1]$.

Traian Ianculescu

PP. 13736. We have

$$w_a^2 = \frac{4bc}{(b+c)^2}s(s-a) \leq s(s-a) \text{ so}$$

$$\left(\sum w_a\right)^2 \leq 3\sum w_a^2 \leq 3s^2 \text{ and } \sum w_a \leq s\sqrt{3}$$

therefore

$$\sum w_a^{\alpha} \leq 3^{1-\alpha}\left(\sum w_a\right)^{\alpha} \leq 3^{1-\frac{\alpha}{2}}s^{\alpha}$$

for all $\alpha \in [0,1]$.

Traian Ianculescu

PP. 13740. Starting from

$$\sum r_a w_a \leq s^2 \leq \sum m_a w_a$$

we get

$$\left(\sum r_a w_a\right)^x \leq 3^{1-x}\left(\sum r_a w_a\right)^x \leq 3^{1-x}s^{2x} \text{ and}$$

$$\left(\sum m_a w_a\right)^y \geq 3^{1-y}\left(\sum m_a w_a\right)^y \geq 3^{1-y}s^{2y}$$

and after addition we get

$$3^{1-y}s^{2y} + \left(\sum r_a w_a\right)^x \leq 3^{1-x}s^{2x} + \left(\sum m_a w_a\right)^y$$

for all $0 \leq x \leq 1 \leq y$.

Traian Ianculescu

PP. 13746. From L. Bankhoff‘s inequality we have

$$\sum h_a \leq 2R + 5r$$

therefore

$$\sum h_a^\alpha \leq 3^{1-\alpha} \left(\sum h_a\right)^\alpha \leq 3^{1-\alpha} (2R + 5r)^\alpha$$

for all $\alpha \in [0,1]$.

Remark. *Using the L. Panaitopol‘s inequality* $\sum a^2 \leq 8R^2 + 4r^2$ *and* $\sum a^2 = 2s^2 - 2r^2 - 8Rr$ *we get* $s^2 \leq 4R^2 + 4Rr + 3r^2$ *but* $4R^2 + 4Rr + 3r^2 \leq 4R^2 + 6Rr - r^2$ *therefore* $s^2 + r^2 + 4Rr \leq 4R^2 + 10Rr$ *or*

$$\sum h_a = \frac{2sr}{abc} \sum ab = \frac{s^2 + r^2 + 4Rr}{2R} \leq 2R + 5r$$

Traian Ianculescu

PP. 13750. Starting from L. Bankhoff’s inequality

$$\left(\sum \sin \frac{A}{2}\right)^2 \geq \left(\sum \sin A\right)^2$$

and using that

$$\sum a^2 = 2\left(s^2 - r^2 - 4Rr\right), \ a = 2R \sin A \text{ etc.}$$

we get

$$\sum \sin \frac{A}{2} \geq \sqrt{\frac{s^2 - 4Rr - r^2}{2R^2}}$$

Traian Ianculescu

PP. 13753. J.F. Darling proved that

$$\sum a^2 \geq \frac{36}{35} \left(s^2 + \frac{abc}{s}\right)$$

therefore

$$\sum a^{2\alpha} \geq 3 \left(\frac{1}{3} \sum a^2\right)^\alpha \geq 3 \left(\frac{12}{35} \left(s^2 + 4Rr\right)\right)^\alpha$$

for all $\alpha \geq 1$.

Traian Ianculescu

PP. 13754. C.T. Nedelcu proved that

$$\sum \cos A < \sum \frac{aC}{cA}$$

(see problem 18899/GM), therefore

$$\sum (\cos A)^{\alpha} \leq 3^{1-\alpha} \left(\sum \cos A\right)^{\alpha} \leq 3^{1-\alpha} \left(\sum \frac{aC}{cA}\right)^{\alpha}$$

for all $\alpha \in [0,1]$.

Traian Ianculescu

PP. 13757. C.T. Nedelcu proved that

$$\sum \frac{a-b+c}{aA} > \frac{\pi}{4}$$

(see Gazeta Matematica, 4/1980), therefore

$$\sum \left(\frac{a-b+c}{aA}\right)^{\alpha} \geq 3^{1-\alpha} \left(\sum \frac{a-b+c}{aA}\right)^{\alpha} \geq 3 \left(\frac{\pi}{12}\right)^{\alpha}$$

Traian Ianculescu

PP. 13758. We have

$$\sum \frac{1}{h_a} \geq 3 \sqrt[3]{\prod \frac{1}{h_a}} = 3 \sqrt[3]{\frac{R}{2s^2r^2}} \text{ so}$$

$$\sum \left(\frac{1}{h_a}\right)^{\alpha} \geq 3^{1-\alpha} \left(\sum \frac{1}{h_a}\right)^{\alpha} \geq 3 \left(\frac{R}{2s^2r^2}\right)^{\frac{\alpha}{3}}$$

for all $\alpha \geq 1$.

Traian Ianculescu

PP. 13775. We have

$$\sum \left(\frac{r_a}{h_a}\right)^{\alpha} \geq 3 \left(\prod \frac{r_a}{h_a}\right)^{\frac{\alpha}{3}} = 3 \left(\frac{R}{2r}\right)^{\alpha} \geq 3$$

Traian Ianculescu

PP. 13777. We have

$$\sum\left(\frac{h_ar_a}{r_br_c}\right)^{\alpha}\geq 3\left(\frac{1}{3}\sum\frac{h_ar_a}{r_br_c}\right)^{\alpha}=3\left(\frac{2}{3}\sum\frac{r_a}{r_b+r_c}\right)^{\alpha}\geq 3$$

because $\sum\frac{r_a}{r_b+r_c}\geq\frac{3}{2}$.

Traian Ianculescu

PP. 13779. We have

$$\sum h_a^{\alpha}\leq 3\left(\frac{1}{3}\sum h_a\right)^{\alpha}\leq 3\left(\frac{2R+5R}{3}\right)^{\alpha}\leq 3\left(R+r\right)^{\alpha}$$

Traian Ianculescu

PP. 13829. We have

$$\sum\frac{r_a}{a}=sr\sum\frac{1}{a\left(s-a\right)}\geq\frac{9sr}{\sum a\left(s-a\right)}=\frac{9s}{8R+2r}\geq\frac{s}{R}\geq\frac{3\sqrt{3}r}{R}$$

If $a\leq b\leq c$ then $r_a\leq r_b\leq r_c$ and $\frac{1}{a}\geq\frac{1}{b}\geq\frac{1}{c}$ and from Chebyshev‘s inequality we get

$$\sum\frac{r_a}{a}\leq\frac{1}{3}\left(\sum r_a\right)\left(\sum\frac{1}{a}\right)\leq\frac{\left(4R+r\right)\sqrt{3}}{6r}\leq\frac{3\sqrt{3}R}{4r}$$

so $\frac{3\sqrt{3}}{R}\leq\sum\frac{r_a}{a}\leq\frac{3\sqrt{3}R}{4r}$ and

$\sum\left(\frac{r_a}{a}\right)^x\leq 3\left(\frac{1}{3}\sum\frac{r_a}{a}\right)^x\leq 3\left(\frac{R\sqrt{3}}{4r}\right)^x$ and $\sum\left(\frac{r_a}{a}\right)^y\geq 3\left(\frac{1}{3}\sum\frac{r_a}{a}\right)^y\geq 3\left(\frac{\sqrt{3}r}{R}\right)^y$ and

$$3\left(\frac{r\sqrt{3}}{R}\right)^y+\sum\left(\frac{r_a}{a}\right)^x\leq 3\left(\frac{R\sqrt{3}}{4r}\right)^x+\sum\left(\frac{r_a}{a}\right)^y$$

for all $0\leq x\leq 1\leq y$.

Traian Ianculescu

PP. 13840. We have

$$\sum \frac{r_a}{a} = r\left(\sum \frac{1}{a} + \sum \frac{1}{s-a}\right) = \frac{s^2+(4R+r)^2}{4sR} \geq \frac{9}{\sum \frac{a}{r_a}} = \frac{9}{2\sum tg\frac{A}{2}} =$$

$$= \frac{9s}{2(4R+r)}$$

so

$$\frac{s^2+(4R+r)^2}{4sR} \geq \frac{9s}{2(4R+r)} \text{ or } s^2+(4R+r)^2 \geq \frac{18s^2R}{4R+r}$$

Traian Ianculescu

PP. 13918. We have

$$\prod_{k=0}^{n} (\cos kx)^{\binom{n}{k}} \leq \left(\frac{\sum\limits_{k=0}^{n}\binom{n}{k}\cos kx}{\sum\limits_{k=0}^{n}\binom{n}{k}}\right)^{\sum\limits_{k=0}^{n}\binom{n}{k}} = \left(\cos\frac{nx}{2}\left(\cos\frac{x}{2}\right)^n\right)^{2^n}$$

$$\prod_{k=1}^{n} (\sin kx)^{\binom{n}{k}} \leq \left(\frac{\sum\limits_{k=1}^{n}\binom{n}{k}\sin kx}{\sum\limits_{k=1}^{n}\binom{n}{k}}\right)^{\sum\limits_{k=1}^{n}\binom{n}{k}} = \left(\frac{2^n\sin\frac{nx}{2}\left(\cos\frac{x}{2}\right)^n}{2^n-1}\right)^{2^n-1}$$

for all $x \in \left(0, \frac{\pi}{2n}\right)$.

Traian Ianculescu

PP. 13923. We have

$$\prod_{k=0}^{n} \binom{2n}{k}^{\binom{n}{k}} \leq \left(\frac{\sum\limits_{k=0}^{n}\binom{2n}{k}\binom{n}{k}}{\sum\limits_{k=0}^{n}\binom{n}{k}}\right)^{\sum\limits_{k=0}^{n}\binom{n}{k}} = \left(2^{-n}\binom{3n}{n}\right)^{2^n}$$

Traian Ianculescu

PP. 14099. We have

$$\sum_{cyclic}\frac{a_1}{a_2+...+a_n-a_1}=\sum\frac{a_1^2}{a_1\left(a_2+...+a_n\right)-a_i^2}\geq$$

$$\geq\frac{\left(\sum\limits_{k=1}^{n}a_k\right)^2}{\sum a_1\left(a_2+...+a_n\right)-\sum\limits_{k=1}^{n}a_k^2}\geq\frac{n}{n-2}$$

because $\left(\sum\limits_{k=1}^{n}a_k\right)^2\leq n\sum\limits_{k=1}^{n}a_k^2$.

Traian Ianculescu

PP. 14103. We have

$$\sum\frac{a_1^2}{a_2+...+a_n}\geq\frac{\left(\sum\limits_{k=1}^{n}a_k\right)^2}{\sum\left(a_2+...+a_n\right)}=\frac{\left(\sum\limits_{k=1}^{n}a_k\right)^2}{(n-1)\sum\limits_{k=1}^{n}a_k}=\frac{1}{n-1}\sum_{k=1}^{n}a_k$$

Traian Ianculescu

PP. 14196. Because $\sqrt[3]{\cos x}\leq\frac{\sin x}{x}$, then we obtain $\frac{\sin^3 x}{\cos x}\geq x$ in which we take $x=\frac{1}{\sqrt[3]{k(k+1)}}$ $(k=1,2,...,n)$ therefore

$$\sum_{k=1}^{n}\left(\sin\frac{1}{\sqrt[3]{k\left(k+1\right)}}\right)^3\sec\frac{1}{\sqrt[3]{k\left(k+1\right)}}\geq\sum_{k=1}^{n}\frac{1}{k\left(k+1\right)}=\frac{n}{n+1}$$

Traian Ianculescu

PP. 14269.

1). We have $ctg\frac{A}{2}+\frac{(n+1)(2n+1)}{3}\left(ctgB+ctgC\right)=$

$=ctg\frac{A}{2}+\frac{n(n+1)(2n+1)}{6}\cdot\frac{2(ctgB+ctgC)}{n}=ctg\frac{A}{2}+n\sum\limits_{k=1}^{n}k^2\frac{2(ctgB+ctgC)}{n^2}\geq$

$\geq ctg\frac{A}{2}+\frac{n^2(n+1)^2}{4}\cdot\frac{2(ctgB+ctgC)}{n^2}$ but

$ctg\frac{A}{2}\left(ctgB+ctgC\right)=\frac{a^2}{2(s-a)(s-c)}=\frac{(s-b+s-c)^2}{2(s-b)(s-c)}\geq 2$, therefore

$$ctg\frac{A}{2}+\frac{(n+1)\,(2n+1)}{3}\,(ctgB+ctgC)\geq 2\,(n+1)$$

2). $\frac{s}{r}+\frac{n(n+1)(2n+1)}{6}\cdot\frac{2s^2-8Rr-2r^2}{nsr}=\frac{s}{r}+\frac{\left(\sum\limits_{k=1}^{n}k^2\right)\left(\sum a^2\right)}{nsr}=$
$=\frac{s}{r}+\left(n\sum\limits_{k=1}^{n}k^2\right)\frac{\sum a^2}{n^2sr}\geq\frac{s}{r}+\frac{(n+1)^2}{4}\cdot\frac{36r^2}{sr}$, therefore

$$\frac{s}{r}+\frac{(n+1)\,(2n+1)\left(s^2-4Rr-r^2\right)}{3sr}\geq 6\,(n+1)$$

3). We have $\sum a^2\geq 36r^2$ so $s^2+\lambda^2\sum a^2\geq s^2+36\lambda^2r^2\geq 12\lambda sr$ but $\sum a^2=2s^2-8Rr-2r^2$, therefore $s^2+\lambda^2\left(2s^2-8Rr-2r^2\right)\geq 12\lambda sr$ and finally

$$\left(1+2\lambda^2\right)s^2\geq 2\lambda r\,(\lambda r+4\lambda R+6s)$$

for all $n\in N$.

Traian Ianculescu

PP. 14566. We have

$$\prod_{k=0}^{n}\binom{n}{k}^{\frac{1}{(k+1)(k+2)}}\leq\left(\frac{\sum\limits_{k=0}^{n}\frac{\binom{n}{k}}{(k+1)(k+2)}}{\sum\limits_{k=1}^{n}\frac{1}{(k+1)(k+2)}}\right)^{\sum\limits_{k=1}^{n}\frac{1}{(k+1)(k+2)}}=\left(\frac{2^{n+2}-n-3}{(n+1)^2}\right)^{\frac{n+1}{n+2}}$$

Traian Ianculescu

PP. 14571. We have

$$\prod_{k=0}^{n}\binom{n}{k}^{k}\leq\left(\frac{\sum\limits_{k=1}^{n}k\binom{n}{k}}{\sum\limits_{k=1}^{n}k}\right)^{\sum\limits_{k=1}^{n}k}=\left(\frac{2^n}{n+1}\right)^{\frac{n(n+1)}{2}}$$

Traian Ianculescu

PP. 14572. We have

$$\prod_{k=0}^{n}\binom{n}{k}^{\frac{1}{k+1}} \le \left(\frac{\sum\limits_{k=0}^{n}\frac{\binom{n}{k}}{k+1}}{\sum\limits_{k=0}^{n}\frac{1}{k+1}}\right)^{\sum\limits_{k=0}^{n}\frac{1}{k+1}} = \left(\frac{2^{n+1}-1}{(n+1)\sum\limits_{k=1}^{n+1}\frac{1}{k}}\right)^{\sum\limits_{k=1}^{n+1}\frac{1}{k}}$$

Traian Ianculescu

PP. 14592. We have

$$\prod_{k=0}^{n}\left(tg2^{k}x\right)^{2^k} \le \left(\frac{\sum\limits_{k=0}^{n}2^k tg2^k x}{\sum\limits_{k=0}^{n}2^k}\right)^{\sum\limits_{k=0}^{n}2^k} = \left(\frac{ctgx-2^{n+1}ctg2^{n+1}x}{2^{n+1}-1}\right)^{2^{n+1}-1}$$

for all $x \in \left(0, \frac{\pi}{2^{n+1}}\right)$.

Traian Ianculescu

PP. 17788. We have

$$\sum\left(\frac{h_a}{w_a}\right)^{\alpha} \ge 3\left(\prod\frac{h_a}{w_a}\right)^{\frac{\alpha}{3}} \ge 3\left(\frac{2r}{R}\right)^{\frac{\alpha}{3}}$$

because $\prod h_a = \frac{2s^2r^2}{R}$ and $\prod w_a \le s^2 r$ (because $w_a \le \sqrt{s\,(s-a)}$).

Traian Ianculescu

Open questions

OQ. 3356. 1). Determine all functions $f, g : R \to R$ for which $f(x) + g(x) = c > 0$, and $\int\limits_a^b \left(\frac{\ln f(x)}{x} + \frac{\ln g(x)}{x} \right) dx = d\pi^2$, when $d \in R$ is a constant.

A solution is $f(x) = 1 + x, g(x) = 1 - x, c = 2, d = -\frac{1}{12}, a = 0, b = 1$

2). Determine all functions $f, g : R \to R$ for which $f(x) + g(x) = c > 0$, and $\int\limits_a^b \left(\frac{\ln x}{f(x)} + \frac{\ln x}{g(x)} \right) dx = d\pi^2$.

A solution is $f(x) = x + 1, g(x) = 1 - x, c = 2, d = \frac{1}{12}, a = 0, b = 1$.

Mihály Bencze

OQ. 3357. 1). Determine all functions $f, g : R \to R$ for which $\int\limits_0^1 \ln f(x)\, dx = \int\limits_0^\infty \ln g(x)\, dx = \frac{\pi^2}{4}$. A solution is $f(x) = \frac{1+x}{1-x}$ and $g(x) = \frac{e^x+1}{e^x-1}$

2). Solve the equation $f(x) = g(x)$
3). Compute $\int f(\ln x)\, g(\ln x)\, dx$.

Mihály Bencze

OQ. 3358. Let be the equation $\sum\limits_{i=1}^n a_i^3 - 3 \sum\limits_{1 \le i < j < k \le n} a_i a_j a_k = b^m$

1). Solve in N
2). Solve in Z
3). Solve in Q

4). If p is a given prime, then the equation $\sum\limits_{i=1}^n a_i^3 - 3 \sum\limits_{1 \le i < j < k \le n} a_i a_j a_k = p$ have infinitely many solutions in Z.
5). Solve in Z the equation $\sum\limits_{i=1}^n \frac{1}{a_i^3} - 3 \sum\limits_{1 \le i < j < k \le n} \frac{1}{a_i a_j a_k} = 1$.

Mihály Bencze

OQ. 3359. 1). Solve in Z the equation
$\left(x^3+y^3\right)\left(x^4+y^4\right)=\left(u^3+v^3\right)\left(u^4+v^4\right)$

2). Let be $\left(\sum\limits_{k=1}^{n} x_k^a\right)\left(\sum\limits_{k=1}^{m} y_k^b\right)=\left(\sum\limits_{k=1}^{m} x_k^b\right)\left(\sum\limits_{k=1}^{n} y_k^a\right)$, where $a,b\in Z$. Solve

a). in Z b). in N c). in Q

Mihály Bencze

OQ. 3360. Let be $A\in M_n(C)$ such that $A^2=A+I_n$ $(n\geq 2)$. Determine all $k,p\in Z$ such that $\det A=\left(\frac{1-\sqrt{5}}{2}\right)^k\left(\frac{1+\sqrt{5}}{2}\right)^p$.

Mihály Bencze and Gheorghe Stoica

OQ. 3361. Solve in Z the equation

$\left(x^2+y^3+z^4\right)\left(y^2+z^3+x^4\right)\left(z^2+x^3+y^4\right)=(x+y+z)^9$.

Mihály Bencze

OQ. 3362. Let ABC be a triangle. Determine all $x,y\in R$ such that

$\frac{1}{R^2}\leq\sum\frac{\left(\sin\frac{A}{2}\right)^x}{(bc)^y}\leq\frac{1}{4r^2}$. For $x=2$ and $y=1$ we have a solution.

Mihály Bencze

OQ. 3363. Let ABC be a triangle. Determine all $x,y\in R$ such that

$\frac{\sqrt{3}}{R}\leq\sum\frac{\left(\sin\frac{A}{2}\right)^x}{a^y}\leq\frac{9}{2s}$. For $x=y=1$ we obtain a solution.

Mihály Bencze

OQ. 3364. Solve the following system: $\begin{cases}\sigma(n)=2m+3\\ \sigma(m)=2n+3\end{cases}$.

Mihály Bencze

OQ. 3365. Solve in Z the equation $\frac{1}{x^n+(n-2)y}+\frac{1}{y^n+(n-2)z}=\frac{2}{z^n+(n-2)x}$, when $n\in N$.

Mihály Bencze

OQ. 3366. Solve in Z the equation $x_1^{2k+1}+x_2^{2k+1}+...+x_n^{2k+1}=1$, where $k\in Z$.
The equation have infinitely many solutions in Z, because for $n=2p$ we have $x_1=1, x_2=0, x_4=-x_3\in Z, ..., x_{2p}=-x_{2p-1}\in Z$, and for $n=2p+1, x_1=1, x_3=-x_2\in Z, ..., x_{2p+1}=-x_{2p}\in Z$ are solutions.

Mihály Bencze

OQ. 3367. Solve in Z the equation
$\left(x_1^n-x_2\right)\left(x_2^n-x_3\right)...\left(x_n^n-x_1\right)=\left(x_1+x_2+...+x_n\right)^{n+1}$.

Mihály Bencze

OQ. 3368. Solve in Z the equation
$\left(x_1^n+x_2^{n-1}+...+x_{n-1}^2+x_n\right)\left(x_2^n+x_3^{n-1}+...+x_n^2+x_1\right)\cdot...$
$\cdot\left(x_n^n+x_1^{n-1}+...+x_{n-2}^2+x_{n-1}\right)=\left(x_1+x_2...+x_n\right)^n$.

Mihály Bencze

OQ. 3369. 1). Solve in Z the equation $3^x+3^y+3^z+3^t=u^3$. In N this equation have infinitely many solutions
$x=3a, y=2a+b+1, z=a+2b+1, t=3b, u=3^a+3^b$, where $a,b\in N$
2). Solve in Z the equation $n^{x_1}+n^{x_2}+...+n^{x_n}+n^{x_{n+1}}=y^n$.

Mihály Bencze

OQ. 3370. Solve in Z the equation $\sum\limits_{k=1}^{n}x_k^{y_k}=\sum\limits_{k=1}^{n}x_k+(n+1)\,y^z$.

Mihály Bencze

OQ. 3371. Solve in Z the equation $x^3+y^3+z^3-3xyz=t^k$, where $k\in Z$.

Mihály Bencze

OQ. 3372. Solve in Z the equation $x^3+y^3+z^3=xyzt^k$, where $k\in Z$.

Mihály Bencze

OQ. 3373. 1). Solve in Z the equation $x^2+y^2+z^2=xyz+t^2$

2). Solve this equation in Q.

3). Solve in Z the equation $\sum\limits_{k=1}^{n} x_k^2=\prod\limits_{k=1}^{n} x_k+y^2$

4). Solve this equation in Q.

Mihály Bencze

OQ. 3374. 1). Solve in Z the equation $\sum\limits_{k=1}^{n} x_k^2+y\prod\limits_{k=1}^{n} x_k=1$

2). Solve in Q the given equation.

Mihály Bencze

OQ. 3375. Solve in N the equations

1). $\sigma(n)=3n-1$ 2). $\Psi(n)=2n+5$

Mihály Bencze

OQ. 3376. Solve in N the equations

1). $\frac{\sigma(n)}{\Psi(m)}+\frac{\Psi(m)}{\sigma(n)}=2$ 2). $\frac{\sigma(n)+d(m)}{\Psi(m)+d(n)}+\frac{\Psi(k)+d(p)}{\sigma(p)+d(k)}=2$

Mihály Bencze

OQ. 3377. 1). Solve in Z the equation $\frac{x_1^3}{x_2+k}+\frac{x_2^3}{x_3+k}+...+\frac{x_n^3}{x_1+k}=n$, when $k\in Z$

2). Solve in Z the equation $\frac{x_1^p}{x_2+k}+\frac{x_2^p}{x_3+k}+...+\frac{x_n^p}{x_1+k}=n$, when $k,p\in Z$

Mihály Bencze

OQ. 3378. Let be $A\in M_{m\times n}(C)$ and $\|A\|_F^2=tr\left(A^TA\right)$, the Frobenius norm.

Determine the min and the max of the expression $\|A\|_2+\|A\|_3+...+\|A\|_k$.

Mihály Bencze

OQ. 3379. Let be $A \in M_{n\times n}(R)$ and $e^{At} = I_n + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + ...$

1). Determine all $A_k \in M_{n\times n}(R)$ $(k = 1, 2, ..., m)$ such that $\prod\limits_{k=1}^{m} e^{A_k t} = I_n$

2). Determine all $A, B, C \in M_{n\times n}(R)$ such that $\begin{cases} \det\left(e^{At}\right) = e^{tr(B)} \\ \det\left(e^{Bt}\right) = e^{tr(C)} \\ \det\left(e^{Ct}\right) = e^{tr(A)} \end{cases}$.

Mihály Bencze

OQ. 3380. The prime p is called $k-$ pannonical if and only if $k^{p-1} \equiv 1 \left(\text{mod } p^k\right)$

1). Prove that are infinitely many $k-$ pannonical primes. Determine all.
2). If $k = 2$, we obtain the Wieferich's prime, by example 1093, 3511 etc.

3). Let $S_k^\alpha = \frac{1}{p_1^\alpha} + \frac{1}{p_2^\alpha} + ... + \frac{1}{p_n^\alpha} + ...$, where $p_1, p_2, ..., p_n, ...$ are $k-$ pannonical primes, and $\alpha > 1$. Compute S_k^α and prove that S_k^α is irrational and transcendential.

Mihály Bencze

OQ. 3381. 1). Determine all prime p_k $(k = 1, 2, ..., n+1)$ such that $2^{p_{n+1}} - 1 = p_1 p_2 ... p_n$

2). Determine all prime p_k $(k = 1, 2, ..., n+1)$ such that $2^{p_1 p_2 \cdots p_n} - 1 = p_{n+1}$
3). Determine all prime q_k $(k = 1, 2, ..., n+1)$ such that $2^{q_{n+1}} + 1 = q_1 q_2 ... q_n$
4). Determine all prime q_k $(k = 1, 2, ..., n+1)$ such that $2^{q_1 q_2 \cdots q_n} + 1 = q_{n+1}$.

Mihály Bencze

OQ. 3382. Let $z = \prod\limits_{k=1}^{n} \left(k^p + i\right)$

1). Determine all $p \in Z$ for which $\text{Re}(z) > 0$ and $\text{Im}(z) > 0$
2). Determine all $p \in R$ for which $\text{Re}(z) > 0$ and $\text{Im}(z) < 0$
3). Determine all $p \in Q$ for which $\text{Re}(z) < 0$ and $\text{Im}(z) < 0$

Mihály Bencze

OQ. 3383. Determine all $A_k \in M_4(R)$ $(k = 1, 2, ..., n)$ such that $\sum\limits_{cyclic} (\det(A_1 + A_2) + \det(A_1 - A_2)) = 8 \sum\limits_{k=1}^{n} \det A_k$.

Mihály Bencze

OQ. 3384. Solve in N the equation $\sum\limits_{k=1}^{n} k^p = m^r$.

Mihály Bencze

OQ. 3385. Determine all $z_k \in C$ $(k = 1, 2, ..., n)$ such that $\left(\operatorname{Re}\left(\sum\limits_{k=1}^{n} z_k\right)\right)^2 + \left(\operatorname{Im}\left(\sum\limits_{k=1}^{n} z_k\right)\right)^2 \leq n^2$. If $z_k = z^{2k}$ $(k = 1, 2, ..., n)$, where $|z| = 1$, then we obtain a solution.

Mihály Bencze

OQ. 3386. Let be $\sum\limits_{k=1}^{n} x_k^2 = n \prod\limits_{k=1}^{n} x_k$.

1). Solve in N 2). Solve in Z 3). Solve in Q

Mihály Bencze

OQ. 3387. Let $a_k \in Q$ $(k = 1, 2, ..., n)$ such that $\prod\limits_{k=1}^{n} a_k = 1$. Determine the set $M = \left\{ x \in R \backslash Q \middle| \sum\limits_{k=1}^{n} x^{a_k} \in Z \right\}$.

Mihály Bencze

OQ. 3388. Let $a_k \in Q$ $(k = 1, 2, ..., n)$ such that $\sum\limits_{k=1}^{n} a_k = 0$. Determine the set $K = \left\{ x \in R \backslash Q \middle| \sum\limits_{k=1}^{n} x^{a_k} \in Z \right\}$.

Mihály Bencze

OQ. 3389. Solve in Z the equation $\prod\limits_{k=1}^{n} \left(x_k^8 + 2x_k^6 - x_k^4 + 2x_k^2 + 1\right) = y^2 + z^2$.

Mihály Bencze

OQ. 3390. Let $x^2 - 4xy + y^2 = z^2$. Solve the given equation

1). in N 2). in Z 3). in Q

Mihály Bencze

OQ. 3391. Determine all $a, b \in N$ for which if $\left[\frac{a^b}{b}\right]$ is a power of a, then b is a power of a, when $[\cdot]$ denote the integer part. If $a = 2$ and $b \geq 4, b \in N$, then we have a solution.

Mihály Bencze

OQ. 3392. Determine all $n, k \in N$ for which exist primes p and q such that

$1 + \frac{1}{2^k} + \frac{1}{3^k} + ... + \frac{1}{n^k} = \frac{1}{p} + \frac{1}{q}$.

Mihály Bencze

OQ. 3393. Let be $k, n \in N^*$ given. Determine all $m \in N$ such that

$\left[\sum\limits_{i=1}^{(m-1)n} \frac{1}{n+i}\right] = k$, when $[\cdot]$ denote the integer part.

Mihály Bencze

OQ. 3394. 1). If $x_k > 0$ $(k = 1, 2, ..., n)$, then

$$\left(\sum_{k=1}^{n} x_k\right)\left(\sum_{k=1}^{n} \frac{1}{x_k}\right) \geq n + (n-1)\sqrt{\frac{\left(\sum\limits_{k=1}^{n} x_k\right)\left(\sum\limits_{k=1}^{n} x_k^{n-1}\right)}{\prod\limits_{k=1}^{n} x_k}} \geq n^2.$$

2). Determine the best constant $c > 0$ such that

$$\left(\sum_{k=1}^{n} x_k\right)\left(\sum_{k=1}^{n} \frac{1}{x_k}\right) \geq n + (n-1)\sqrt{\frac{\left(\sum\limits_{k=1}^{n} x_k\right)\left(\sum\limits_{k=1}^{n} x_k^{n-1}\right)}{\prod\limits_{k=1}^{n} x_k}} +$$

$$+c\left((x_1 - x_2)^2 + (x_2 - x_3)^2 + ... + (x_n - x_1)^2\right) \geq n^2.$$

Mihály Bencze

OQ. 3395. Determine all $a_k \in C$ $(k = 1, 2, ..., n)$ for which $\sum\limits_{cyclic} \frac{a_1 - a_2}{a_1 + a_2} + \prod\limits_{cyclic} \frac{a_1 - a_2}{a_1 + a_2} = 0$.

Mihály Bencze

OQ. 3396. Determine the best $c \geq 0$ such that

$$\prod_{cyclic} (a_1 + a_2) \geq 2^n \prod_{k=1}^{n} a_k + c \sum_{cyclic} (a_1 - a_2)^2 \text{ for all } a_k > 0 \ (k = 1, 2, ..., n).$$

Mihály Bencze

OQ. 3397. If $a_i, p_i > 0 \ (i = 1, 2, ..., n)$ and $F_k (a_1, a_2, ..., a_n) = \frac{\sum\limits_{i=1}^{n} p_i a_i^{k+1}}{\sum\limits_{i=1}^{n} p_i a_i^{k}}$,

then $F_k (a_1, a_2, ..., a_n) \geq F_{k-1} (a_1, a_2, ..., a_n) \geq ... \geq F_1 (a_1, a_2, ..., a_n) \geq$
$\geq F_0 (a_1, a_2, ..., a_n)$.

Mihály Bencze

OQ. 3398. Determine all $m, n, k, p \in N$ such that
$(x_1^m + x_2^m + ... + x_k^m) \left(x_2^m + x_3^m + ... + x_{k+1}^m\right) ... \left(x_n^m + x_1^m + ... + x_{k-1}^m\right) \geq$

$\geq x_1^p x_2^p ... x_n^p \left(x_1^{m-p} + x_2^{m-p} + ... + x_k^{m-p}\right) \left(x_2^{m-p} + x_3^{m-p} + ... + x_{k+1}^{m-p}\right) \cdot ...$
$\cdot \left(x_n^{m-p} + x_1^{m-p} + ... + x_{k-1}^{m-p}\right)$ for all $x_i > 0 \ (i = 1, 2, ..., n)$.

Remark. If $m = 3$, $k = 2$, $p = 2$, then $\frac{x_1^3 + x_2^3}{x_1 x_2} \geq x_1 + x_2 \Leftrightarrow (x_1 - x_2)^2 \geq 0$
therefore $\prod \frac{x_1^3 + x_2^3}{x_1 x_2} \geq \prod (x_1 + x_2) \Rightarrow \prod \left(x_1^3 + x_2^3\right) \geq x_1^2 x_2^2 ... x_n^2 \prod (x_1 + x_2)$ for all $x_i > 0 \ (i = 1, 2, ..., n)$.

Mihály Bencze

OQ. 3399. 1). Prove that exist infinitely many $x, y, z \in R$ for which
$x^2 + y^2 + z^2 \geq x + y + z$

2). Determine all $x, y, z \in R$ for which $x^2 + y^2 + z^2 \geq x + y + z$.

We have the following result. Using the weighted $AM - GM$ inequality, we

get $\frac{2a^2}{3b^2} + \frac{b^2}{6c^2} + \frac{c^2}{6a^2} \geq \frac{a}{b}$ so $\sum \frac{a^2}{b^2} = \sum \left(\frac{2a^2}{3b^2} + \frac{b^2}{6c^2} + \frac{c^2}{6a^2}\right) \geq \sum \frac{a}{b}$. If

$x=\frac{a}{b}, y=\frac{b}{c}, z=\frac{c}{a}$, then $x^2+y^2+z^2 \geq x+y+z$.

3). Determine all $x_k \in R$ $(k=1,2,...,n)$ for which $\sum\limits_{k=1}^{n} x_k^2 \geq \sum\limits_{k=1}^{n} x_k$

4). Determine all $x_k \in R$ $(k=1,2,...,n)$ and all $\alpha, \beta \in R$ for which $\sum\limits_{k=1}^{n} x_k^{\alpha} \geq \sum\limits_{k=1}^{n} x_k^{\beta}$.

Mihály Bencze

OQ. 3400. If $a_i>0$ $(i=1,2,...,n)$, then determine all $k \in \{1,2,...,n\}$ for which $\left(\sum\limits_{i=1}^{n} \frac{1}{a_i}\right)\left(\sum\limits_{cyclic} \frac{a_1^k}{a_1+a_2+...+a_k}\right) \geq \left(\sum\limits_{i=1}^{n} a_i\right)\left(\sum\limits_{cyclic} \frac{a_1}{a_1^k+a_2^k+...+a_k^k}\right)$. For k= 1 we have equality. For $k=2$ we start from the followings:

(1) $\frac{a_1^2}{a_1+a_2} \geq \frac{3a_1-a_2}{4} \Rightarrow \sum \frac{a_1^2}{a_1+a_2} \geq \sum \frac{3a_1-a_2}{4} = \frac{1}{2}\sum\limits_{i=1}^{n} a_i$ and $\frac{a_1}{a_1^2+a_2^2} \leq \frac{1}{2a_2} \Rightarrow$

(2) $\sum \frac{a_1}{a_1^2+a_2^2} \leq \frac{1}{2}\sum\limits_{i=1}^{n} \frac{1}{a_i}$. Multiplying (1) and (2) we get the affirmation for $k=2$.

Mihály Bencze

OQ. 3401. If $a_i>0$ $(i=1,2,...,n)$ and $k \in \{2,3,...,n\}$, then

$$\sum_{i=1}^{n} \frac{1}{a_i} + \frac{n^2}{\sum\limits_{i=1}^{n} a_i} \geq \frac{2nk}{\binom{n}{k}} \sum_{1 \leq i_1 < ... < i_k \leq n} \frac{1}{a_{i_1}+a_{i_2}+...+a_{i_k}}.$$

Mihály Bencze

OQ. 3402. If $z_k \in C$ $(k=1,2,...,n)$ and $\sum\limits_{k=1}^{n} z_k = 0$. Determine all $m \in N$ for which $m\sum\limits_{k=1}^{n} \left(z^k\right)^m \leq \left(\sum\limits_{k=1}^{n} |z_k|\right)^m$.

Mihály Bencze

OQ. 3403. 1). If $z_k \in C$ $(k = 1, 2, ..., n)$ and $z_k \neq z_p$ $(k \neq p)$, then
$$\left(\sum_{cyclic} |z_1 - z_2|\right)\left(\sum_{cyclic} \frac{1}{|z1|-z2}\right) \geq n^2 + 1.$$

2). Determine all $A \geq 1$, $a_k \in R$ $(k = 1, 2, ..., n)$ such that
$$(|a_1z_1 + a_2z_2 + ... + a_nz_n| + |a_1z_2 + a_2z_3 + ... + a_nz_1| + ...$$
$$+ |a_1z_n + a_2z_1 + ... + a_nz_{n-1}|) \cdot$$
$$\cdot \left(\frac{1}{|a_1z_1+a_2z_2+...+a_nz_n|} + \frac{1}{|a_1z_2+a_2z_3+...+a_nz_1|} + ... + \frac{1}{|a_1z_n+a_2z_1+...+a_nz_{n-1}|}\right) \geq$$
$$n^2 + A.$$

Mihály Bencze

OQ. 3404. If $x_k > 0$ $(k = 1, 2, ..., n)$, then determine all $\alpha \in R$ for which
$$\sum_{k=1}^{n} \left(\frac{x_k}{1+x_1^\alpha+x_2^\alpha+...+x_k^\alpha}\right)^\alpha \leq \frac{\sum\limits_{k=1}^{n} x_k^\alpha}{1+\sum\limits_{k=1}^{n} x_k^\alpha}.$$

Mihály Bencze

OQ. 3405. If $a_k > 0$ $(k = 1, 2, ..., n)$, then determine all $p \in \{1, 2, ..., n\}$ for which $\sum\limits_{k=1}^{n} a_k \geq (n-p)\sqrt[n]{\frac{1}{n}\sum\limits_{k=1}^{n} a_k^n} + p\sqrt[n]{\prod\limits_{k=1}^{n} a_k}$.

Mihály Bencze

OQ. 3406. If $a_k > 0$ $(k = 1, 2, ..., n)$ and $\prod\limits_{k=1}^{n} a_k = \alpha^n$, then determine all $\alpha > 0$ for which $\sum\limits_{k=1}^{n} \frac{1}{(x+a_k)^p} \geq \frac{n}{(x+\alpha)^p}$ for all $x > 0$ and $p \geq 2, p \in N$.

Mihály Bencze

OQ. 3407. Determine all $a_k > 0$ $(k = 1, 2, ..., n)$ for which
$$\sqrt{\frac{1}{n}\sum_{k=1}^{n} a_k^2} + \frac{n}{\sum\limits_{k=1}^{n} \frac{1}{a_k}} \geq \frac{1}{n}\sum_{k=1}^{n} a_k + \sqrt[n]{\prod_{k=1}^{n} a_k}.$$

Mihály Bencze

OQ. 3408. Let be $A_p = \{n \in N^* | n = p[\sqrt{n}] + 1, p \in N^*\}$, where $[\cdot]$ denote the integer part. Compute $\sum\limits_{p=3}^{\infty} \sum\limits_{n \in A_p} \frac{1}{n^2}$.

Mihály Bencze

OQ. 3409. Let $S(n)$ be the sum of those positive divisors of n that are less than n. A triple of integers (a, b, c) is a $(x, y)-$ friendly triple if $1 < a \leq b \leq c$ and $xs(a) + ys(b) = (x + y)c$, $xs(b) + ys(c) = (x + y)a$, $xs(c) + ys(a) = (x + y)b$, where $x, y \in N$. Determine all $(x, y)-$ friendly triple. In same way $(a_1, a_2, ..., a_n)$ is $(x_1, x_2, ..., x_{n-1})-$ friendly if $1 < a_1 \leq a_2 \leq ... \leq a_n$ and $x_1 s(a_1) + x_2 s(a_2) + ... + x_{n-1} s(a_{n-1}) = (x_1 + ... + x_{n-1}) a_n, ..., x_1 s(a_n) + x_2 s(a_1) + ... + x_{n-1} s(a_{n-2}) = (x_1 + ... + a_{n-1}) a_{n-1}$. Determine all $(x_1, x_2, ..., x_{n-1})-$ friendly numbers.

Mihály Bencze

OQ. 3410. If $a_k > 0$ $(k = 1, 2, ..., n)$ and $\prod\limits_{k=1}^{n} a_k = 1$, then determine min and max of the expression $\sum\limits_{cyclic} \frac{x+a_1}{y+a_1+a_1a_2+...+a_1a_2...a_p}$, where $x, y > 0$ and $p \in \{2, 3, ..., n-1\}$.

Mihály Bencze

OQ. 3411. Determine all primes $p_1, p_2, ..., p_n$ such that $p_1 | (p_2 + p_3), p_2 | (p_3 + 2p_4), ..., p_n | (p_1 + np_2)$.

Mihály Bencze

OQ. 3412. If $a_k > 0$ $(k = 1, 2, ..., n)$, then determine all $p_k > 0$ $(k = 1, 2, ..., n)$ for which $\frac{\sum\limits_{k=1}^{n} ka_k}{\sum\limits_{k=1}^{n} p_k} \leq \frac{n-1}{2} + \frac{\sum\limits_{k=1}^{n} a_k^k}{\sum\limits_{k=1}^{n} p_k^{p_k}}$.

Mihály Bencze

OQ. 3413. The sequence $(a_n)_{n \geq 1}$ of positive integers is defind for $n \geq 1$ by $a_{n+1} = xa_n + yb_n$, when b_n is obtained from a_n by reversing its digits, and the number b_n may start with zeroes.

1). Determine all positive integers a_1, x, y for which the sequence $(a_n)_{n \geq 1}$ contain infinitely many prime.

2). Exist $a_1, x, y \in N$ for which the sequence $(a_n)_{n\geq 1}$ contain only coposite numbers?
3). Study the sequence if $a_1 \in \{d(k), \sigma(k), \Psi(k), \Phi(k), F_k, L_k, P_k, ...\}$ and k is given.

Mihály Bencze

OQ. 3414. A positive integer n is said to be k-type number if it has the following two properties:
a). n is divisible by at least k distinct primes
b). there exist distinct positive divisors $d_1 = 1, d_2, d_3, ..., d_k$ of n such that $d_1 + d_2 + ... + d_k = n$.
1). Show that there exist infinitely many $k-$type numbers, for all $n \geq 6$
2). Denote S_n the set of k-type numbers. Compute $\sum\limits_{n\in S_n} n$, $\sum\limits_{n\in S_n} \frac{1}{n}$, $\sum\limits_{n\in S_n} \frac{1}{n^\alpha}$, when $\alpha \geq 1$.
3). Compute $\sum\limits_{n\notin S_n} \frac{1}{n^\beta}$ when $\beta \geq 1$.

Mihály Bencze

OQ. 3415. In convex polygon $A_1A_2...A_{3n-1}A_{3n}$ the triangles $A_1A_2A_3, A_4A_5A_6, ..., A_{3n-1}A_{3n}A_1$ are similar. Determine all polygons $A_1A_2...A_{3n-1}A_{3n}$ for which the polygon $A_1A_3A_5...A_{3n-1}$ is regular if and only if $A_2A_4A_6...A_{3n}$ is regular.

Mihály Bencze

OQ. 3416. If $a_i \in R$ $(i = 1, 2, ..., n)$ are distinct, and $b_k = \prod\limits_{\substack{i=1\\i\neq k}}^{n} \left(1 + \frac{a_i a_k}{a_i - a_k}\right)$, then $1 + \left|\sum\limits_{k=1}^{n} a_k b_k\right| \leq \prod\limits_{k=1}^{n} (1 + |a_k|)$.

Mihály Bencze

OQ. 3417. For a positive integer a, let $M_n(c, d)$ be the set of all primes p for which there exists an odd integer b such that $(2c)^{b(2d)^a} - 1$ is divisible by p, when c and d are given positive integers. For any positive integer a, prove that there exist infinitely many primes that are not in $M_n(c, d)$. Denote $k_n(c, d)$ the set of all prime p for which $(2c)^{b(2d)^a} - 1$ is not divisible by p. Prove that there exist infinitely many primes that are not in $k_n(c, d)$.

Mihály Bencze

OQ. 3418. If $d_k = \begin{vmatrix} f(k) & f(k+1) & f(k+2) \\ f(k+3) & f(k+4) & f(k+5) \\ f(k+6) & f(k+7) & f(k+8) \end{vmatrix}$, then compute:

1). $\sum\limits_{k=1}^{n} d_k$

2). $\sum\limits_{k=1}^{n} \frac{1}{d_k}$

3). $\sum\limits_{k=1}^{\infty} \frac{1}{d_k^\alpha}$, where $\alpha \geq 1$.

4). How many prime exist between d_k and d_{k+1}?

5). Solve the equation $2d_n = d_p + d_k$

6). Solve the equation $d_n^2 = d_p d_k$.

We consider the following cases

$f(k) \in \{d(k), \sigma(k), \Phi(k), \Psi(k), L_k, F_k, P_k, ...\}$

Mihály Bencze

OQ. 3419. Let $A = (a_{ij})_{1 \leq i,j \leq k}$ be a magic square matrix, and $d_k = \det(f(a_{ij}))_{1 \leq i,j \leq k}$, where $f(k) \in \{d(k), (k), \Phi(k), \Psi(k), F_k, L_k, ...\}$. Compute

1). $\sum\limits_{k=1}^{n} d_k$ 2). $\sum\limits_{k=1}^{n} \frac{1}{d_k}$ 3). $\sum\limits_{k=1}^{n} \frac{1}{d_k^\alpha}$, where $\alpha \geq 1$.

4). How many prime exist between d_k and d_{k+1}?

5). Solve the equation $2d_n = d_p + d_k$

6). Solve the equation $d_n^2 = d_p d_k$

Mihály Bencze

OQ. 3420. If $p_1, p_2, ..., p_k$ are distinct primes and t a given positive integer, there are infinitely many natural numbers n such that the decimal representation of $p_1^n + p_2^n + ... + p_k^n$ has a block of t consecutive zeroes. If $k = 1$, then for $p_1 \notin \{2, 5\}$ let $r > t$, $r \in Z$ and $t = \Phi(10^r) = \frac{2}{5} \cdot 10^r$ and $p_1^n \equiv 1 \pmod{10^r}$, hence has the desired property. If $p_1 = 2$, let $r > 2t$, $n = \Phi(5^r) + r = 4 \cdot 5^{r-1} + r$, $2^{\Phi(5^r)} \equiv 1 \pmod{5^r}$, therefore $2^t = 2^{\Phi(5^r)} \cdot 2^r \equiv 2^r \pmod{10^r}$. Since $r > 2t > \log_5 10t$ and $\frac{10^r}{2^r} = 5^r > 10^t$, 2^n contains a block of t consecutive zeros to left of the rightmost 2^r digits. If $p_1 = 5$ let $r > 4t$, $n = \Phi(2^r) + r = 2^{r-1} + r$, $5^{\Phi(2^r)} \equiv 1 \pmod{2^r}$, therefore $5^t = 5^{\Phi(2^r)} 5^r \equiv 5^r \pmod{10^r}$. Since $r > 4t > \log_2 10 \cdot t$ and $\frac{10^r}{5^r} = 2^r > 10^t$, thus 5^n contain a block of t consecutive zeroes to the left of the rightmost 5^r digits.

Mihály Bencze

OQ. 3421. If $a_k > 0$ $(k = 1, 2, ..., n)$ and $S = \sum\limits_{k=1}^{n} a_k$, then
$\sum\limits_{cyclic} \frac{a_1}{a_2} \geq \sum\limits_{cyclic} \frac{S-a_2}{S-a_1}$.

Mihály Bencze

OQ. 3422. A positive integer n is f-charming if there are positive integers $a_1, a_2, ..., a_n$ (not necessarily distinct) such that
$f(a_1) + f(a_2) + ... + f(a_n) = f(a_1) f(a_2) ... f(a_n) = n.$

Determine all f-charming integers if $f \in \{d, \Phi, \sigma, \Psi, F, L, p, ...\}$.

Mihály Bencze

OQ. 3423. Let ABC be a triangle and $M \in Int(ABC)$. Prove that exist a constant $\alpha(M) > 0$ for which $2r \leq \alpha(M)(AA_1 + BB_2 + CC_1) \leq R$, where $\{A_1\} = AM \cap BC$, $\{B_1\} = BM \cap CA$, $\{C_1\} = CM \cap AB$.

If $M \equiv H$, then we have $9r \leq h_a + h_b + h_c \leq \frac{9R}{2}$. If $M \equiv G$, then we have $9r \leq m_a + m_b + m_c \leq \frac{9R}{2}$. If $M \equiv I$, then $9r \leq w_a + w_b + w_c \leq \frac{9R}{2}$.

Mihály Bencze

OQ. 3424. Let ABC be a triangle, and $M \in Int(ABC)$,
$\{A_1\} = AM \cap BC$, $\{B_1\} = BM \cap CA$, $\{C_1\} = CM \cap AB$.

Determine all $M \in Int(ABC)$ for which $R + r \leq \max\{AA_1, BB_1, CC_1\}$. If $M \equiv H$, then we reobtain a problem of Pál Erdős, namely
$R + r \leq \max\{h_a, h_b, h_c\}$ (Matematika V Skole, 1962, Nr. 6, 87-88).

Mihály Bencze

OQ. 3425. Let ABC be a triangle, and $M \in Int(ABC)$,
$\{A_1\} = AM \cap BC$, $\{B_1\} = BM \cap CA$, $\{C_1\} = CM \cap AB$, then determine the minimum and maximum of the expressions:

1). $\frac{\sum AA_1}{\sum a}$ 2). $\frac{\sum AA_1 \cdot BB_1}{\sum ab}$
3). $\sum \frac{AA_1^{\alpha}}{\sum a^{\alpha}}$ 4). $\frac{\sum AA_1^{\alpha} \cdot BB_1^{\beta}}{\sum a^{\alpha} b^{\beta}}$, where $\alpha, \beta \in R$.

Mihály Bencze

OQ. 3426. Let ABC be a triangle. Determine all function $f : R^3 \to R$ such that $\frac{\sum a^t(f(a,b,c)-a)}{\sum a^{t-2}} \leq \frac{abc}{2}$ for all $t \in R$. If $f(a,b,c) = \frac{a+b+c}{2}$, then we reobtain a result of R.Z. Djordjevic.

Determine $\min\left\{\frac{\sum a^t(f(a,b,c)-a)}{\sum a^{t-2}}\right\}$.

Mihály Bencze

OQ. 3427. Let ABC be a triangle, $M_k \in Int(ABC)$, $\{A_k\} = AM_k \cap BC$, $\{B_k\} = BM_k \cap CA$, $\{C_k\} = CM_k \cap AB$ $(k = 1, 2)$.

1). Determine all M_1, M_2 such that
$AM_1 + BM_1 + CM_1 \leq AM_2 + BM_2 + CM_2$

2). Determine all M_1, M_2 such that $AM_1 \cdot BM_1 \cdot CM_1 \leq AM_2 \cdot BM_2 \cdot CM_2$

3). Determine all M_1, M_2 such that
$AM_1 \cdot BM_1 + BM_1 \cdot CM_1 + CM_1 \cdot AM_1 \leq AM_2 \cdot BM_2 + BM_2 \cdot CM_2 + CM_2 \cdot AM_2$

4). Determine all M_1, M_2 such that $\sum AM_1 \cdot BM_1 \leq \sum AM_2 \cdot BM_2$ if and only if $\prod AM_1 \leq \prod AM_2$.

Mihály Bencze

OQ. 3428. 1). Let ABC be a triangle. Determine the best constants $x, y > 0$ such that $(2x + y) r \leq \sum h_a \leq xR + yr$. If $x = 2$, $y = 5$ or $x = y = 3$, then we obtain two solutions.

Mihály Bencze

OQ. 3429. Let $A_1A_2...A_n$ be a convex polygon. Determine all $x_k > 0$ $(k = 1, 2, ..., n-1)$ such that if $\left(\sum\limits_{k=1}^{n-1} x_k\right) a_1 \leq \sum\limits_{k=1}^{n-1} x_k a_{k+1}$, then

$\left(\sum\limits_{k=1}^{n-1} x_k\right) A_1 \leq \sum\limits_{k=1}^{n-1} x_k A_{k+1}$, where $a_k = A_kA_{k+1}$ $(k = 1, 2, ..., n)$.

Mihály Bencze

OQ. 3430. Let ABC be a triangle, $M_k \in Int\,(ABC)$, $\{A_k\} = AM_k \cap BC$, $\{B_k\} = BM_k \cap CA$, $\{C_k\} = CM_k \cap AB$ $(k = 1, 2, 3)$.

1). Determine all $x, y > 0$, M_1, M_2, M_3 such that
$(x+y)\,AM_1 = xBM_2 + yCM_3$

2). Determine all $x, y > 0$, M_1, M_2, M_3 such that $(AM_1)^{x+y} = BM_2^x \cdot CM_3^y$

3). Determine all $x, y > 0$, M_1, M_2, M_3 such that $\frac{x+y}{AM_1} = \frac{x}{BM_2} + \frac{y}{CM_3}$

Mihály Bencze

OQ. 3431. Let ABC be a triangle, $M_k \in Int\,(ABC)$, $\{A_k\} = AM_k \cap BC$, $\{B_k\} = BM_k \cap CA$, $\{C_k\} = CM_k \cap AB$ $(k = 1, 2)$.

Determine all M_1, M_2 such that
$AM_1^2 + BM_1^2 + CM_1^2 \le s^2 \le AM_2^2 + BM_2^2 + CM_2^2$. If $M_1 \equiv I$, $M_2 \equiv G$, then we obtain $\sum w_a^2 \le s^2 \le \sum m_a^2$.

Mihály Bencze

OQ. 3432. Let ABC be a triangle, $M \in Int\,(ABC)$. Denote R_1, R_2, R_3 the radii of the circles inscribed in the sectors AMB, BMC, CMA.

Determine all points M such that $\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \ge \frac{3+2\sqrt{3}}{R}$.

If $M \equiv O$, then we obtain a solution (J.I. Gerasimov, Matematica v. Scole, 1967, Nr. 3).

Mihály Bencze

OQ. 3433. Let $A_1A_2A_3A_4$ be a convex quadrilateral, and let M be an arbitrary point in its plane. Determine all $x, y > 0$ such that
$\sum\limits_{i=1}^{4} PA_i \ge x \min\{PA_i | i \in \{1,2,3,4\}\} + y \max\{PA_i | i \in \{1,2,3,4\}\}$. If $A_1A_2A_3A_4$ is square, then I.S. Gál and L. Bankoff (Problem E. 1308 AMM, 65(1958)) have obtained $x = 1 + \sqrt{2}$ and $y = 1$.

Mihály Bencze

OQ. 3434. Let ABC be a triangle and consider the points M, N, K in plane of the given triangle.
Determine all points M, N, K such that $4 \leq \frac{area(MNK)}{area(ABC)} \leq \left(\frac{R}{r}\right)^2$. If $M = I_a$, $N = I_b$, $K = I_c$, then we obtain a result of M.S.Klamkin, Math. Teacher 60(1967), 323-328.

Mihály Bencze

OQ. 3435. Let ABC be a triangle, $M \in Int\,(ABC)$ and denote x, y, z the distances of point M to the sides of triangle. Determine minimum and maximum of the expressions:

1). $\frac{\sum h_a}{\sum x}$ 2). $\frac{\sum m_a}{\sum x}$ 3). $\frac{\sum w_a}{\sum x}$

4). $\frac{\sum h_a m_a}{\sum x^2}$ 5). $\frac{\sum h_a w_a}{\sum x^2}$ 6). $\frac{\sum m_a w_a}{\sum x^2}$

7). $\frac{\sum h_a^x}{\sum m_a^x}$ 8). $\frac{\sum h_a^x}{\sum w_a^x}$ 9). $\frac{\sum m_a^x}{\sum w_a^x}$

10). $\frac{\sum x h_a}{\sum x m_a}$ 11). $\frac{\sum x h_a}{\sum x w_a}$ 12). $\frac{\sum x m_a}{\sum x w_a}$

If $M \equiv 0$ then $\frac{\sum w_a}{\sum x} \leq 3$ this is a result of F. Leuenberger (Problem E. 1579, AMM 1963).

Mihály Bencze

OQ. 3436. If $a_k > 0$ $(k = 1, 2, ..., n)$ and $\alpha \in [0, n^2)$, then

$$\alpha\left(n^2+1\right) + \sqrt{\left(n^2-\alpha\right)\left(\left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} \frac{1}{a_k^2}\right) - \alpha\right)} \geq$$

$$\geq (\alpha+1)\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} \frac{1}{a_k}\right).$$

Mihály Bencze

OQ. 3437. If $a_i > 0$ $(i = 1, 2, ..., n)$ and $\prod_{i=1}^{n} a_i = 1$, then determine all $k \in \{2, ..., n\}$ such that $\prod_{cyclic} (a_1 + a_2 + ... + a_k) \geq \prod_{i=1}^{n} (a_i + k - 1)$.

Mihály Bencze

OQ. 3438. If $a_{ij} > 0$ $(i = 1, 2, ..., n; j = 1, 2, ..., k)$, then determine all $1 \le k \le n$ such that $\sum\limits_{i=1}^{n} \prod\limits_{j=1}^{k} a_{ij} + \sqrt[k]{\prod\limits_{j=1}^{k} \left(\sum\limits_{i=1}^{n} a_{ij}^{k} \right)} \ge \frac{2}{n^{k-1}} \prod\limits_{j=1}^{k} \left(\sum\limits_{i=1}^{n} a_{ij} \right).$

Mihály Bencze

OQ. 3439. If $a_i > 0$ $(i = 1, 2, ..., n)$ and $\sum\limits_{cyclic} a_1 a_2 ... a_k \ge n$, then determine all $2 \le k \le n$ such that $\prod\limits_{cyclic} (a_1 + a_2 + ... + a_k) \ge \prod\limits_{i=1}^{n} (a_i + k - 1).$

Mihály Bencze

OQ. 3440. If $a_i > 0$ $(i = 1, 2, ..., n)$ and $\sum\limits_{i=1}^{n} a_i^k = n$, then determine all $2 \le k \le n$ such that $\sum\limits_{cyclic} (a_1 + a_2 + ... + a_{k+1})^{k+1} \le n.$

Mihály Bencze

OQ. 3441. If $a_k > 0$ $(k = 1, 2, ..., n)$, then determine all $\alpha > 0$ such that $\left(\prod\limits_{k=1}^{n} a_k \right) \left(\frac{1}{n} \left(\sum\limits_{k=1}^{n} a_k \right) \left(\sum\limits_{k=1}^{n} \frac{1}{a_k} \right) - n + \alpha \right) \le \frac{\alpha}{n^n} \left(\sum\limits_{k=1}^{n} a_k \right)^n.$

Mihály Bencze

OQ. 3442. 1). If $a_i > 0$ $(i = 1, 2, ..., n)$ such that $\sum\limits_{cyclic} a_1 a_2 ... a_k = n$, then determine all $k \in \{1, 2, ..., n\}$ such that $\sum\limits_{cyclic} \frac{(1+a_1^3)(1+a_2^3)...(1+a_k^3)}{(1+a_1^2)(1+a_2^2)...(1+a_k^2)} \ge n.$

2). If $a_i > 0$ $(i = 1, 2, ..., n)$ such that $\sum\limits_{cyclic} a_1 a_2 ... a_k = n$, then determine all $k, p \in N$ such that $\sum\limits_{cyclic} \frac{(1+a_1^{p+1})(1+a_2^{p+1})...(1+a_k^{p+1})}{(1+a_1^p)(1+a_2^p)...(1+a_k^p)} \ge n.$

Mihály Bencze

OQ. 3443. If $a, b, c > 0$ and $k \in N^*$, then $\sum\limits_{cyclic} \frac{a^k}{b+c} \geq \frac{3 \sum\limits_{cyclic} a^{k+1}}{2 \sum\limits_{cyclic} a^2}$

2). If $a_i > 0$ $(i = 1, 2, ..., n)$ and $k \in N^*$, then $\sum\limits_{cyclic} \frac{a_1^k}{a_2+a_3+...+a_n} \geq \frac{n \sum\limits_{i=1}^{n} a_i^{k+1}}{(n-1) \sum\limits_{i=1}^{n} a_i^2}$

3). If $a_i > 0$ $(i = 1, 2, ..., n)$ and $k, p \in N^*$, then
$\sum\limits_{cyclic} \frac{a_1^k}{a_2^p+a_3^p+...+a_n^p} \geq \frac{n \sum\limits_{i=1}^{n} a_i^{k+1}}{(n-1) \sum\limits_{i=1}^{n} a_i^{p+1}}.$

Mihály Bencze

OQ. 3444. If $a_i > 0$ $(i = 1, 2, ..., n)$ and $\sum\limits_{i=1}^{n} a_i^k = n$, then determine all

$k \in N^*$ such that $\sum\limits_{cyclic} a_1 a_2 ... a_k \geq 1 + (n-1) \prod\limits_{i=1}^{n} a_i.$

Mihály Bencze

OQ. 3445. If $x_i > 0$ $(i = 1, 2, ..., n)$ then determine all $k \in N^*$ such that

$$\sum_{i=1}^{n} x_i \geq (n-k) \sqrt[n]{\prod_{i=1}^{n} x_i} + \sum_{j=1}^{k} \sqrt[j+1]{\frac{1}{n} \sum_{i=1}^{n} x_i^{j+1}}.$$

Mihály Bencze

OQ. 3446. If $a_i > 0$ $(i = 1, 2, ..., n)$ and $\prod\limits_{i=1}^{n} a_i = 1$, then determine all

$k \in N^*$ such that $\prod\limits_{i=1}^{n} \left(1 + a_i^k\right) \geq \left(\sum\limits_{i=1}^{n} a_i\right)^k.$

Mihály Bencze

OQ. 3447. If $a_i > 0$ $(i = 1, 2, ..., n)$ and $\sum\limits_{i=1}^{n} a_i = 1$, then determine all

$k, p \in N^*$ such that $\sum\limits_{cyclic} \sqrt[p]{a_1 + a_2^2 + ... + a_k^k} \geq n \sqrt[p]{\frac{3^k-1}{2 \cdot 3^k}}.$

Mihály Bencze

OQ. 3448. If $a_i > 0$ $(i = 1, 2, ..., n)$ then determine all $k \in N^*$ such that

$$k\left(\sum_{i=1}^{n} a_i^n + n\prod_{i=1}^{n} a_i\right) \geq 2\sum_{cyclic} a_2a_3...a_n\sqrt[k]{k^{k-1}\left(a_1^k + a_2^k + ... + a_k^k\right)}.$$

Mihály Bencze

OQ. 3449. If $x_k > 0$ $(k = 1, 2, ..., n)$, then determine the best constant

$\lambda > 0$ such that $\sum_{k=1}^{n}\left(\prod_{i=1}^{k} x_i\right)^{\frac{1}{k}} \leq \lambda\sum_{k=1}^{n} x_k$.

Mihály Bencze

OQ. 3450. If $a_{ij} > 0$ $(i = 1, 2, ..., n; j = 1, 2, ..., m)$, then determine all

$\alpha, \beta > 0$ such that $\prod_{j=1}^{m}\left(\sum_{i=1}^{n} a_{ij}\right) \geq \left(\sum_{i=1}^{n}\left(\sum_{j=1}^{m} a_{ij}\right)\right)^{\alpha}\left(\sum_{i=1}^{n}\frac{\prod_{j=1}^{m} a_{ij}}{\sum_{j=1}^{m} a_{ij}}\right)^{\beta}$.

Mihály Bencze

OQ. 3451. If $x_k > 0$ $(k = 1, 2, ..., n)$ and $\alpha > 0$, then determine the best constant $\lambda_\alpha > 0$ such that $\sum_{i=1}^{n}\left(\frac{1}{i}\sum_{k=1}^{i} x_k\right)^{\alpha} \leq \lambda_\alpha\sum_{i=1}^{n} x_i^\alpha$. If $\alpha = 2$, then $\lambda_2 = 4$, and if $\alpha = 3$, then $\lambda_3 = \frac{27}{8}$.

Mihály Bencze

OQ. 3452. If $\lambda_k > 0$ $(k = 1, 2, ..., n)$ and $\sum_{k=1}^{n}\lambda_k = 1$, then determine all $x_k > 0$ $(k = 1, 2, ..., n)$ such that $\sum_{k=1}^{n}\frac{\lambda_k}{1+x_k} \leq \frac{1}{1+\prod_{k=1}^{n} x_k^{\lambda_k}}$.

Mihály Bencze

OQ. 3453. 1). Let be $M, N, K \in Int(ABC)$, where ABC is a triangle. AM intersect the side BC in M_1, BM intersect the sides CA in M_2, CM intersect the sides AB in M_3, in same way define the points N_1, N_2, N_3 and K_1, K_2, K_3. Denote:

$U = \{AM_1, BM_2, CM_3\}$, $V = \{AN_1, BN_2, CN_3\}$, $W = \{AK_1, BK_2, CK_3\}$. Prove that exist $x \in U, y \in V, z \in W$ for which x, y, z are the sides of a triangle.

2). Extend in space
3). Extend for simplexes.

Mihály Bencze

OQ. 3454. If $a_i > 0$ $(i = 1, 2, ..., n)$ and $\sum\limits_{i=1}^{n} a_i = n$, $F(k) = \sum\limits_{i=1}^{n} \frac{a_i^k}{a_i^{k+1}+1}$, then

$F(k) \leq F(k-1) \leq ... \leq F(0)$.

Mihály Bencze

OQ. 3455. Determine all $x, y \in Q$ for which is there an infinite sequence of prime $p_1, p_2, p_3, ...$ such that $|xp_{n+1} - yp_n| = 1$ for each $n \geq 1$.

Mihály Bencze

OQ. 3456. If $\binom{n}{k}_a = \frac{(a^n-1)(a^{n-1}-1)...(a^{n-k+1}-1)}{(a^k-1)(a^{k-1}-1)...(a-1)}$, then determine all m, n, k, p integers such that $0 < k < n < m < p$ and all $a > 1$ integers such that $\binom{m}{k}_a$ is divisible by $\binom{n}{p}_a$.

Mihály Bencze

OQ. 3457. Prove that exist infinitely many $x_k, y_k \in R$ $(k = 1, 2, ..., 2n+1)$ such that

1). $\sum\limits_{k=1}^{2n} (-1)^k (\cos x_k)^{2n+1} > 0$

2). $\sum\limits_{k=1}^{2n} (-1)^k (\cos y_k)^{2n+1} < 0$

3). Determine all $z_k \in R$ $(k = 1, 2, ..., 2n+1)$ such that $\sum\limits_{k=1}^{2n+1} (-1)^k (\cos z_k)^{2n+1} = 0$.

Mihály Bencze

OQ. 3458. If $a_1 < a_2 < ... < a_n$ are integers, then denote $F(n)$ the number for which $a_j - a_i$ $(1 \leq i < j \leq n)$ are of the form p^m, where p is a positive integer, and $G(n)$ the number for which $j - i$ $(1 \leq i < j \leq n)$ are of the form q^r, where q is a positive integer.
Determine all p, q, m, r for which $F(n) \leq G(n)$ for all $n \in N^*$.

Mihály Bencze

OQ. 3459. 1). Determine all $\binom{n_k}{p_k}$ $(k = 1, 2, ..., m)$ such that $\left(\binom{n_1}{p_1}; \binom{n_2}{p_2}; ...; \binom{n_m}{p_m}\right) = 1$. A solution is $\left(\binom{n}{k}; \binom{n+1}{k}; ...; \binom{n+k}{k}\right) = 1$

2). Determine all $\binom{n_k}{p_k}$ $(k = 1, 2, ..., m)$ such that $\left(\binom{n_1}{p_1}; \binom{n_2}{p_2}; ...; \binom{n_m}{p_m}\right) = \binom{n_1+n_2+...+n_m}{p_1+p_2+...+p_m}$.

Mihály Bencze

OQ. 3460. 1). Determine all solutions of the equation $x_1^2 + x_2^2 + ... + x_n^2 = y^2$ for which $x_1, x_2, ..., x_n, y$ are in arithmetical progression

2). Determine all solutions of the equation $x_1^2 + x_2^2 + ... + x_n^2 = y^2$ for which $x_1, x_2, ..., x_n, y$ are in geometrical progression.

Mihály Bencze

OQ. 3461. 1). Determine all n composite numbers for which $k^n - k$ is divisible by n for all $k \geq 2$, $k \in N$. If $k = 2$, $n = \frac{4^p - 1}{3}$ when $p > 3$ is a prime, then $2^n - 2$ is divisible by n.

2). Determine all n, k positive integers for which $k^n + k$ is divisible by n.

Mihály Bencze

OQ. 3462. Prove that in all triangle ABC exist a choise of the median, bisector and altitude (for example m_a, w_b, h_c) where are the sides of a triangle.

Nica Nicolae and Nica Cristina-Paula

OQ. 3463. Solve the equations:

1). $\sum\limits_{k=1}^{n} \Phi\left(k^{\alpha}\right) = \frac{n(n+1)}{2}\Phi^{\alpha}\left(n\right)$

2). $\sum\limits_{k=1}^{n} \Psi\left(k^{\alpha}\right) = \frac{n(n+1)}{2}\Psi^{\alpha}\left(n\right)$, where $\alpha \in N$.

Mihály Bencze

OQ. 3464. Let ABC be a triangle, and denote M_k the set of cevians of rank k. Prove that exist a choise $x \in M_k$, $y \in M_{k+1}$, $z \in M_{k+2}$ such that x, y, z are the sides of a triangle.

Mihály Bencze and Nica Nicolae

OQ. 3465. Determine all $a < b$, $c < d$ rational numbers for which the equation $(1 - x^c)^a = \left(1 - x^d\right)^b$ have rational roots, different from 0 and 1.

Mihály Bencze

OQ. 3466. If $\sum\limits_{k=1}^{n} \lambda_k = 1$, where $\lambda_k > 0$ $(k = 1, 2, ..., n)$ and $\mu_k > 0$ $(k = 1, 2, ..., n)$ then $(1 - \lambda_1^{\mu_1})^{\mu_2} + (1 - \lambda_2^{\mu_2})^{\mu_3} + ... + (1 - \lambda_n^{\mu_n})^{\mu_1} \geq n - 1$.

Mihály Bencze

OQ. 3467. If $a_k > 0$ $(k = 1, 2, ..., n)$ and $\prod\limits_{k=1}^{n} a_k = 1$, then $\sum\limits_{k=1}^{n} \frac{1}{1+a_k+a_k^2+...+a_k^{n-1}} \geq 1$.

Mihály Bencze

OQ. 3468. 10. Determine all $k \in N$ such that $\sum \frac{1}{ka^2+bc} \geq \frac{27}{(k+1)(a+b+c)^2}$ for all $a, b, c > 0$

2). Determine all $k \in N$ such that $\sum \frac{1}{ka_1^k+a_2a_3...a_{k+1}} \geq \frac{n^{k+1}}{(k+1)(a_1+a_2+...+a_n)^k}$ for all $a_i > 0$ $(i = 1, 2, ..., n)$.

Mihály Bencze

OQ. 3469. 1). If $x_k > 0$ $(k = 1, 2, ..., n)$ then $\frac{\sum\limits_{k=1}^{n} x_k^2}{\sum\limits_{k=1}^{n} x_k} \geq \sqrt[3]{\frac{1}{n}\sum\limits_{k=1}^{n} x_k^3} \geq \frac{1}{n}\sum\limits_{k=1}^{n} x_k.$

2). If $x_k > 0$ $(k = 1, 2, ..., n)$ and $p \in N^*$ $(p \geq 2)$, then
$\frac{\sum\limits_{k=1}^{n} x_k^p}{\sum\limits_{k=1}^{n} x_k^{p-1}} \geq \sqrt[p+1]{\frac{1}{n}\sum\limits_{k=1}^{n} x_k^{p+1}} \geq \frac{1}{n}\sum\limits_{k=1}^{n} x_k.$

Mihály Bencze

OQ. 3470. 1). If $x, y, z > 0$ then determine all $a, b > 0$ such that
$\sqrt{a+b}\,(x+y+z) \geq \sqrt{ax^2+byz} + \sqrt{ay^2+bzx} + \sqrt{az^2+bxy}$ (A solution is $a = 1,\ b = 2$).

2). If $x, y, z > 0$, then determine all $a, b, c, d > 0$ such that
$\sqrt[3]{a+b+c+d}\,(x+y+z) \geq \sqrt[3]{ax^3+bxy^2+czy^2+dxyz}+$

$+\sqrt[3]{ay^3+byz^2+cxz^2+dxyz} + \sqrt[3]{az^3+bzx^2+cyx^2+dxyz}$ (A solution is $a = 1, b = c = 3, d = 2$).

Mihály Bencze

OQ. 3471. Determine all $\alpha > 0$ such that $\sum\limits_{cyclic} \frac{x_1^\alpha(x_2+x_3)}{x_2^\alpha+x_3^\alpha} \geq \sum\limits_{k=1}^{n} x_k$ for all $x_k > 0$ $(k = 1, 2, ..., n)$.

Mihály Bencze

OQ. 3472. Determine all $k \in N^*$ such that

$\sum x^k(x-y)(x-2y)...(x-ky) \geq 0$ for all $x, y, z \geq 0$.

Mihály Bencze

OQ. 3473. If $a_k > 0$ $(k = 1, 2, ..., n)$ then determine all $x, y, z > 0$ such that
$\prod\limits_{k=1}^{n}\left(1-a_k+a_k^2\right) \geq \frac{x+y\prod\limits_{k=1}^{n} a_k+z\prod\limits_{k=1}^{n} a_k^2}{x+y+z} \geq \prod\limits_{k=1}^{n} a_k.$

Mihály Bencze

OQ. 3474. If $a, b, c > 0$ then determine all $x, y > 0$ such that

$$9(xa+yb)(xb+yc)(xc+ya) \geq (x+y)^3 (a+b+c)(ab+bc+ca).$$

Mihály Bencze

OQ. 3475. Determine all $k \in N$ such that $\sum (x-y)((k+1)x+ky)^k \geq 0$ for all $x, y, z > 0$.

Mihály Bencze

OQ. 3476. Determine all $\alpha > 0$ such that

$\sum \sqrt[2k]{\frac{b^k c^k + \alpha a^k (b^k + c^k)}{b+c}} \geq \frac{1}{2} \sqrt[2k]{2\alpha + 1}$ for all $a, b, c > 0$, where $k \in N^*$.

Mihály Bencze

OQ. 3477. Determine all $k \in N$ such that $\sum x (x^k - y^k)(x - ky) \geq 0$ for all $x, y, z > 0$.

Mihály Bencze

OQ. 3478. Determine all $k \in N$ such that $\sum (x-y)(kx^{k+1} + y^{k+1}) \geq 0$ for all $x, y, z > 0$.

Mihály Bencze

OQ. 3479. If $a_i > 0$ $(i = 1, 2, ..., n), k \in N^*$, then

$$(k+1)^2 \prod_{i=1}^{n} (a_i^{2k} + 1) \geq 2^n \left(1 + \prod_{i=1}^{n} a_i + \prod_{i=1}^{n} a_i^2 + ... + \prod_{i=1}^{n} a_i^k\right)^2.$$

Mihály Bencze

OQ. 3480. If $a_k > 0$ $(k = 1, 2, ..., n)$ and $\prod_{k=1}^{n} a_k = 1$, then

$$\sum_{k=1}^{n} \frac{1}{(1+a_k)^{n-1}} + \frac{2(2^{n-1}-n)}{\prod_{k=1}^{n} (1+a_k)} \geq 1.$$

Mihály Bencze

OQ. 3481. Compute the following sums

1). $S_k^t = \sum\limits_{0\le i_1<...<i_k\le n} (i_1+i_2+...+i_k)^t \binom{n}{i_1}\binom{n}{i_2}...\binom{n}{i_k}$

2). $R_t^t = \sum\limits_{0\le i_1<...<i_k\le n} (-1)^{i_1+...+i_k-1} (i_1+i_2+...+i_k)^t \binom{n}{i_1}\binom{n}{i_2}...\binom{n}{i_k}$

Starting from $\sum\limits_{i=0}^{n} x^i\binom{n}{i} = (1+x)^n$ after derivation we get
$\sum\limits_{i=0}^{n} ix^{i-1}\binom{n}{i} = n(1+x)^{n-1}$.
If we take $x=1$ and $x=-1$ we get $S_1^1 = n2^{n-1}$ and $R_1^1 = 0$.
From the identity $\sum\limits_{i=0}^{n} ix^i\binom{n}{i} = nx(1+x)^{n-1}$ after derivation we get
$\sum\limits_{i=0}^{n} i^2x^{i-1}\binom{n}{i} = n(1+x)^{n-1} + n(n-1)x(1+x)^{n-2}$.
If $x=1$ and $x=-1$ then we get $S_1^2 = n(n+1)2^{n-2}$ and $R_1^2 = 0$.
We can continue iterativ with this method starting from:
$2\sum\limits_{0\le i_1<i_2\le n} x^{i_1+i_2}\binom{n}{i_1}\binom{n}{i_2} = (1+x)^{2n} - \sum\limits_{i=0}^{n} x^{2i}\binom{n}{i}^2$ after derivation we obtain
$\sum\limits_{0\le i_1<...<i_k\le n} (i_1+i_2)x^{i_1+i_2-1}\binom{n}{i_1}\binom{n}{i_2} = n(1+x)^{2n-1} - \sum\limits_{i=0}^{n} ix^{2i-1}\binom{n}{i}^2$.
If we take $x=1$ and $x=-1$, then we obtain: $S_2^1 = n\left(2^{n-1} - \binom{2n}{n}\right)$ and
$R_2^1 = n\binom{2n-1}{n-1}$.
Multiplying by x and after then derivating, this offer an iterative method to obtain new results.

Mihály Bencze

OQ. 3482. If $b_1, b_2, ..., b_n$ are a rearrengemant of positive real numbers $a_1, a_2, ..., a_n$, then determine all $x, y \in R$ such that

$$\min\left\{\prod_{k=1}^{n}\left(a_k^x + b_k^y\right); \prod_{k=1}^{n}\left(a_k^y + b_k^x\right)\right\} \ge$$
$$\ge \max\left\{\prod_{k=1}^{n}\left(a_k^x + a_{n-k+1}^y\right); \prod_{k=1}^{n}\left(b_k^x + b_{n-k+1}^y\right)\right\}.$$

Mihály Bencze

OQ. 3483. 1). If $p, q \geq 3$ are two primes, then the sequences $(pn+q)_{n\geq 0}$ and $(qn+p)_{n\geq 0}$ contains infinitely many palindromes numbers.
2). If $p_k \geq 3\,(k = 1, 2, ..., r)$ are prime numbers then the sequence $\left(p_1 n^{r-1} + p_2 n^{r-2} + ... + p_{r-1} n + p_r\right)_{n\geq 0}$ contain infinitely many palindromes numbers.

Mihály Bencze

OQ. 3484. If $a_k > 0$ $(k = 1, 2, ..., n)$ then

$$\frac{1}{n}\sum_{k=1}^{n} a_k \leq \left(\frac{\sum_{k=1}^{n} a_k^{\frac{\alpha}{\alpha+1}}}{\sum_{k=1}^{n} a_k^{\frac{1}{\alpha+1}}}\right)^{\frac{\alpha+1}{\alpha-1}} \leq \sqrt{\frac{1}{n}\sum_{k=1}^{n} a_k^2} \text{ for all } \alpha > 1.$$

Mihály Bencze

OQ. 3485. If $g(x) = \sum_{k=1}^{\infty} \frac{x^{f(k)}}{(f(k))!}$ and $h(x) = \sum_{k=1}^{\infty} (-1)^{f(k)} \frac{x^{f(k)}}{(f(k))!}$, where $f : N \to N$, then determine $\sum_{k=1}^{\infty} \frac{x^{f(k)}}{f(k)}$ and $\sum_{k=1}^{\infty} (-1)^k \frac{x^{f(k)}}{f(k)}$ in function of g and h.

Mihály Bencze

OQ. 3486. 1). Determine all $x_k \in Z\,(k = 1, 2, ..., n)$ such that

$$\left(\prod_{k=1}^{n} x_k\right)^{\alpha} \left(\sum_{k=1}^{n} x_k\right)^{-\beta} \in Z, \text{ when } \alpha, \beta \in Z$$

2). Determine all $x_k \in Z$ $(k = 1, 2, ..., n)$ such that $\left(\sum_{k=1}^{n} x_k^{\alpha}\right)\left(\sum_{k=1}^{n} x_k^{-\beta}\right) \in Z$ when $\alpha, \beta \in Z$.

Mihály Bencze

OQ. 3487. Determine all functions $f : R \to (0, +\infty)$ for which

$$\left(\sum_{k=1}^{n} \frac{1}{f(x_k)}\right) f\left(\frac{1}{n}\sum_{k=1}^{n} x_k\right) \geq n - 3 + \frac{3n \sum_{k=1}^{n} f^2(x_k)}{\left(\sum_{k=1}^{n} f(x_k)\right)^2} \text{ for all } x_k \in I \subseteq R$$

$(k = 1, 2, ..., n)$.

Mihály Bencze

OQ. 3488. Determine all $\lambda > 0$ and all $n \in N^*$ for which
$\frac{1}{n}\left(\sum\limits_{k=1}^{n} x_k\right)\left(\sum\limits_{k=1}^{n} \frac{1}{x_k}\right) \geq n - \lambda + \lambda n \max\left\{\frac{\sum\limits_{k=1}^{n} x_k^2}{\left(\sum\limits_{k=1}^{n} x_k\right)^2}; \frac{\sum\limits_{k=1}^{n} \frac{1}{x_k^2}}{\left(\sum\limits_{k=1}^{n} \frac{1}{x_k}\right)^2}\right\} \geq n$ for all
$x_k > 0$ $(k = 1, 2, ..., n)$.

Mihály Bencze

OQ. 3489. Determine all $k \in N^*$ for which the set $\left\{\left[\frac{k^n}{n}\right] | n \in N^*\right\}$ have infinitely many elements divisible by p, infinitely many elements divisible by q, and infinitely many elements divisible by r, where $[\cdot]$ denote the integer part, $p < q < r$ are three giving prime.

Mihály Bencze

OQ. 3490. 1). If $x, y > 0$ and $a, b, c > 0$, then
$x \sum a^4 + y \sum a^3 b \geq \frac{x+y}{27}\left(\sum a\right)^4$

2). If $x, y > 0$ and $a, b, c > 0$ then determine all $n \in N$ such that
$x \sum a^n + y \sum a^{n-1} b \geq \frac{x+y}{3^{n-1}}\left(\sum a\right)^n$.

Mihály Bencze

OQ. 3491. Let $p_1, p_2, ..., p_k$ prime numbers, and $m, n \in N^*$ such that
$n^m = p_1^m + p_2^m + ... + p_k^m$. Determine all $m, p_1, ..., p_k$ for which n is prime. It‘s easy to prove, that if p, q are prime and $n^2 = p^2 + q^2 + 1$, then n is prime.

Mihály Bencze

OQ. 3492. If $a_i > 0$ $(i = 1, 2, ..., n)$ and $k \in \{2, 3, ..., n\}$, then
$\sum\limits_{cyclic} \frac{(a_1+a_2+...+a_k)^k}{a_1^k + a_2 a_3 ... a_{k+1}} \geq \frac{nk^k}{2}$.

Mihály Bencze

OQ. 3493. If $a_i > 0$ $(i = 1, 2, ..., n)$ and $\sum\limits_{i=1}^{n} a_i = n$, then
$\sum\limits_{cyclic} \frac{1}{k+2-a_1 a_2 ... a_k} \leq \frac{n}{k+1}$ for all $k \in \{2, 3, ..., n\}$.

Mihály Bencze

OQ. 3494. If $a_i > 0$ $(i = 1, 2, ..., n)$ and $k \in N$, then
$\left(\sum\limits_{i=1}^{n} \frac{1}{a_i}\right)^k \geq \sum\limits_{i=1}^{n} \frac{i^k}{a_1^k + a_2^k + ... + a_i^k}.$

Mihály Bencze

OQ. 3495. If $a_i > 0$ $(i = 1, 2, ..., n)$ and $k \in N$, then
$\sum\limits_{cyclic} \frac{a_1}{a_2^k + a_3^k + ... + a_n^k} \geq \frac{k^2}{\sum\limits_{i=1}^{n} a_i}.$

Mihály Bencze

OQ. 3496. If $a_i > 0$ $(i = 1, 2, ..., n)$ and $k \in \{2, 3, ..., n\}$, then

$$\sqrt[k]{\frac{a_1 a_2 ... a_k}{\lambda_1 a_1^k + \lambda_2 a_2^k + ... + \lambda_n a_n^k}} + \sqrt[k]{\frac{a_2 a_3 ... a_{k+1}}{\lambda_1 a_2^k + \lambda_2 a_3^k + ... + \lambda_n a_1^k}} + ...$$
$$+ \sqrt[k]{\frac{a_n a_1 ... a_{k-1}}{\lambda_1 a_n^k + \lambda_2 a_1^k + ... + \lambda_n a_{n-1}^k}} \leq \frac{n}{\sqrt[k]{\sum\limits_{i=1}^{n} \lambda_i}}, \text{ where } \lambda_i > 0 \ (i = 1, 2, ..., n).$$

Mihály Bencze

OQ. 3497. If $a_i > 0$, $\prod\limits_{i=1}^{n} a_i = 1$, then determine all $\alpha \geq 1$ such that
$\sum\limits_{cyclic} \frac{a_1}{n - 1 + a_2^\alpha} \geq 1.$

Mihály Bencze

OQ. 3498. If $a_i > 0$ $(i = 1, 2, ..., n)$ and $k \in \{2, 3, ..., n\}$, then
$\sum\limits_{cyclic} \frac{1}{(ka_1 + a_2)^k} \geq \frac{n^2}{(k+1)^k \sum\limits_{cyclic} a_1 a_2 ... a_k}.$

Mihály Bencze

OQ. 3499. If $a_i > 0$ $(i = 1, 2, ..., n)$, $\alpha \geq 1$, $k \in \{2, 3, ..., n\}$, then
$\sum\limits_{cyclic} \left(a_1 + \frac{a_2^k}{a_3 a_4 ... a_{k+1}}\right)^\alpha \geq \frac{n \cdot 2^\alpha \sum\limits_{i=1}^{n} a_i^{k+\alpha-1}}{\sum\limits_{i=1}^{n} a_i^{k-1}}.$

Mihály Bencze

OQ. 3500. If $a_i > 0$ $(i = 1, 2, ..., n)$, then

$$\sum_{cyclic}\left(\frac{a_1}{a_1+a_2+...+a_{n-1}}\right)^n + \frac{((n-1)^n-n)\prod\limits_{k=1}^{n} a_k}{\prod\limits_{cyclic}(a_1+a_2+...+a_{n-1})} \geq 1.$$

Mihály Bencze

OQ. 3501. If $a_i > 0$ $(i = 1, 2, ..., n)$, then determine all $k \in \{2, 3, ..., n\}$, for which $\sum\limits_{cyclic} \sqrt[k]{\frac{a_1^{k+1}}{\sum\limits_{i=0}^{n} a_1^{k-i}a_2^i}} \geq \frac{1}{\sqrt[k]{k+1}} \sum\limits_{i=1}^{n} \sqrt[k]{a_i}$.

Mihály Bencze

OQ. 3502. If $a_i > 0$ $(i = 1, 2, ..., n)$ and $\sum\limits_{i=1}^{n} a_i = n - 1$, then determine all $k \in \{2, 3, ..., n\}$ such that $\sum\limits_{cyclic} \frac{a_1a_2...a_k}{1+a_{k+1}^k} \leq \frac{n(n-1)^k}{n^k+(n-1)^k}$.

Mihály Bencze

OQ. 3503. If $A(a_1, a_2, ..., a_n) = \frac{1}{n}\sum\limits_{k=1}^{n} a_k$, $G(a_1, a_2, ..., a_n) = \sqrt[n]{\prod\limits_{k=1}^{n} a_k}$, $H(a_1, a_2, ..., a_n) = \frac{n}{\sum\limits_{k=1}^{n}\frac{1}{a_k}}$, then

1). $\sum\limits_{cyclic} G(a_1, a_2, ..., a_k) \leq \sum\limits_{cyclic} G(a_1, a_2, ..., a_{k+1})$

2). $\prod\limits_{cyclic} A(a_1, a_2, ..., a_k) \leq \prod\limits_{cyclic} A(a_1, a_2, ..., a_{k+1})$

3). $\prod\limits_{cyclic} H(a_1, a_2, ..., a_k) \leq \prod\limits_{cyclic} H(a_1, a_2, ..., a_{k+1})$ for all $k \in \{1, 2, ..., n-1\}$.

Mihály Bencze

OQ. 3504. If $x_i > 0$ $(i = 1, 2, ..., n)$ and $\sum\limits_{i=1}^{n} x_i = 1$, then determine all $k \in \{2, 3, ..., n\}$ such that $\sum\limits_{cyclic} \sqrt[k]{x_1 + x_2^2 + x_3^3 + ... + x_k^k} \geq \sqrt[k]{\frac{n^k-1}{n-1}}$.

Mihály Bencze

OQ. 3505. If $a_i > 0$ $(i = 1, 2, ..., n)$ and $\sum\limits_{i=1}^{n} a_i = n$, then determine all $k \in \{2, 3, ..., n\}$ such that $\sum\limits_{cyclic} \frac{1}{n-1+a_1^k a_2^k ... a_k^k} \geq 1$.

Mihály Bencze

OQ. 3506. If $a_i > 0$ $(i = 1, 2, ..., n)$ then determine all $k \in \{2, 3, ..., n\}$ such that $\sum\limits_{cyclic} \frac{a_1^2}{a_2} \geq n \sqrt[k]{\frac{1}{n} \sum\limits_{i=1}^{n} a_i^k}$.

Mihály Bencze

OQ. 3507. Determine the best constant $\alpha(k, n) > 0$ such that $\sum\limits_{i=1}^{n} (x_1 + x_2 + ... + x_i)^k \geq \alpha(n) \left(\sum\limits_{i=1}^{n} x_i \right)^k$ for all $x_i > 0$ $(i = 1, 2, ..., n)$ and $k, n \in N$.

Mihály Bencze

OQ. 3508. If $a_i > 0$ $(i = 1, 2, ..., n)$ such that $\sum\limits_{i=1}^{n} a_i^k = n$, then determine all $k \in \{2, 3, ..., n\}$, for which $\sum\limits_{cyclic} \frac{a_1}{a_2 + k} \leq \frac{n}{k+1}$.

Mihály Bencze

OQ. 3509. If $a_i > 0$ $(i = 1, 2, ..., n)$ then determine all $k \in \{2, 3, ..., n\}$ such that $\sum\limits_{cyclic} \frac{1}{\sqrt[k]{a_1^k + a_2 a_3 ... a_{k+1}}} \geq \frac{n^2}{\sqrt[k]{2} \sum\limits_{i=1}^{n} a_i}$.

Mihály Bencze

OQ. 3510. If $a_i > 0$ $(i = 1, 2, ..., n)$ and $\sum\limits_{i=1}^{n} a_i^k = n$, then determine all $k \in \{2, 3, ..., n\}$, such that $\sum\limits_{cyclic} \frac{1}{k+1-a_1 a_2 ... a_k} \leq \frac{n}{k}$.

Mihály Bencze

OQ. 3511. If $a_i > 0$ $(i = 1, 2, ..., n)$ then determine all $k \in \{2, 3, ..., n\}$ such that $\sum\limits_{cyclic} \frac{a_1}{a_2} \geq n \sqrt[k]{\frac{\sum\limits_{i=1}^{n} a_i^k}{\sum\limits_{cyclic} a_1 a_2 ... a_k}}$.

Mihály Bencze

OQ. 3512. If $a_i > 0$ $(i = 1, 2, ..., n)$ then determine all $k \in \{2, 3, ..., n\}$ such that $\sum\limits_{cyclic} \left(\frac{a_1}{a_2}\right)^k + \frac{n^k \sum\limits_{cyclic} a_1 a_2 ... a_k}{\sum\limits_{i=1}^{n} a_i^k} \geq n^k + n$.

Mihály Bencze

OQ. 3513. If $\lambda, a_i > 0$ $(i = 1, 2, ..., n)$, $k \in \{2, 3, ..., n\}$ and $\sum\limits_{i=1}^{n} x_i = n$, then $\sum\limits_{cyclic} \frac{x_1}{\lambda + x_1 x_2 ... x_k} \geq \frac{n}{\lambda + 1}$.

Mihály Bencze

OQ. 3514. If $a_i > 0$ $(i = 1, 2, ..., n)$ and $\sum\limits_{i=1}^{n} a_i = n$, then determine all $\lambda > 0$, $\alpha \geq \beta \geq 0$ such that $\prod\limits_{i=1}^{n} \frac{\lambda^\alpha + a_i^\lambda}{\lambda^\beta + a_i^\beta} \geq 1$.

Mihály Bencze

OQ. 3515. If $a_i > 0$ $(i = 1, 2, ..., n)$ then

$$\sum \frac{1}{a_1 \sqrt[k]{a_1 + a_2 + ... + a_k}} \geq \frac{n}{\sqrt[k]{k \left(\prod\limits_{i=1}^{n} a_i\right)^{\frac{k+1}{n}}}}, \text{ where } k \in \{2, 3, ..., n\}.$$

Mihály Bencze

OQ. 3516. If $\lambda, a_i > 0$ $(i = 1, 2, ..., n)$ and $\sum\limits_{i=1}^{n} a_i = n$, then $\sum\limits_{cyclic} \sqrt[k]{\frac{a_1 + \lambda}{a_1 a_2 + \lambda}} \geq n$.

Mihály Bencze

OQ. 3517. If $a_i > 0$ $(i = 1, 2, ..., n)$ and $\sum\limits_{i=1}^{n} a_i = 1$, then determine all $\alpha > 0$ and $k \in N^*$ such that $\sum\limits_{cyclic} \sqrt[k]{x_1 + \alpha (x_2 - x_3)^2} \le \sqrt[k]{n^{k-1}}$.

Mihály Bencze

OQ. 3518. If $a_i > 0$ $(i = 1, 2, ..., n)$ then determine all $\alpha > 0$ and $k \in \{2, 3, ..., n\}$ such that $\sum\limits_{cyclic} \left(\alpha + \frac{a_1}{a_2}\right)^k \ge \frac{(\alpha+1)^k}{n^{k-2}} \cdot \frac{\left(\sum\limits_{i=1}^{n} a_i\right)^k}{\sum\limits_{cyclic} a_1 a_2 ... a_k}$.

Mihály Bencze

OQ. 3519. If $a_i > 0$ $(i = 1, 2, ..., n)$ and $k \in \{2, 3, ..., n\}$ then

$$\sum_{cyclic} \frac{a_1+a_2+...+a_k}{\sqrt[k]{a_1 a_2 ... a_k + a_{k+1}^k}} \ge \frac{nk}{\sqrt[k]{2}}.$$

Mihály Bencze

OQ. 3520. If $a_i > 0$ $(i = 1, 2, ..., n)$ and $\sum\limits_{i=1}^{n} a_i = n - 1$, then

$$\sum_{cyclic} \frac{a_1}{\sqrt{n+a_2^2+...+a_n^2}} \ge \frac{n(n-1)}{\sqrt{n^3+(n-1)^3}}.$$

Mihály Bencze

OQ. 3521. If $a_i > 0$ $(i = 1, 2, ..., n)$ and $\sum\limits_{i=1}^{n} a_i = 1$, then

$$\sum_{cyclic} \frac{a_1}{\sqrt[k]{\frac{1}{a_2}-1}} \le \sqrt[k]{\frac{n^n}{(n-1)^{n+1}} \prod_{i=1}^{n} (1 - a_i)}.$$

Mihály Bencze

OQ. 3522. If $a_i > 0$ $(i = 1, 2, ..., n)$, $k \in \{2, 3, ..., n\}$ then

$$\sum_{cyclic} \frac{a_1}{a_1^k + a_2 a_3 ... a_{k+1}} \ge \frac{n^2}{2\sqrt[k]{n}}.$$

Mihály Bencze

OQ. 3523. If $a_i > 0$ $(i = 1, 2, ..., n)$, $x, y, z > 0$, then

$$\sum_{i=1}^{n} a_i^2 + x \prod_{i=1}^{n} a_i + y \geq \frac{n+x+y}{(z+1)^n} \prod_{i=1}^{n} (z + a_i).$$

Mihály Bencze

OQ. 3524. If $a, b, c > 0$, then $\sum \frac{1}{a^2+b^2} + \frac{8}{\sum a^2} \geq \frac{25}{2\sum ab}$.

Mihály Bencze

OQ. 3525. If $a_i > 0$ $(i = 1, 2, ..., n)$, $k, p \in \{2, 3, ..., n\}$, $k \leq p$, then
$$\prod_{cyclic} (a_1 + a_2 + ... + a_k) \leq \left(\frac{k}{p}\right)^n \prod_{cyclic} (a_1 + a_2 + ... + a_p).$$

Mihály Bencze

OQ. 3526. If $a_i > 0$ $(i = 1, 2, ..., n)$, $k \in \{2, 3, ..., n\}$ and $\sum\limits_{cyclic} a_1 a_2 ... a_k = 1$, then

1). $\sum\limits_{cyclic} \frac{1}{a_1+a_2+...+a_k} \geq \frac{n\sqrt[k]{n}}{k}$

2). $\sum\limits_{cyclic} \frac{1}{a_1+a_2+...+a_k} + \frac{1}{\sum\limits_{i=1}^{n} a_i} \geq \left(\frac{n}{k} + \frac{1}{n}\right) \sqrt[k]{n}$

Mihály Bencze

OQ. 3527. If $a_i > 0$ $(i = 1, 2, ..., n)$, then

$$\sum_{cyclic} \frac{1}{a_1^k+a_2^k+...+a_{n-1}^k} \geq \frac{n^3}{(n-1)\left(\sum\limits_{i=1}^{n} a_i\right)^k}, \text{ where } k \in N^*.$$

Mihály Bencze

OQ. 3528. If $x_k \in [-1, 1]$ $(k = 1, 2, ..., n)$ and $\sum\limits_{k=1}^{n} x_k = 0$, then
$$\sum_{cyclic} \sqrt{1 + x_1 + x_2^2} \geq n.$$

Mihály Bencze

OQ. 3529. 1). If $a_i \in R\ (i = 1, 2, ..., n)$, then
$$\sum_{i_1, i_2, ..., i_k = 1}^{n} |a_{i_1} + a_{i_2} + ... + a_{i_k}| \geq \binom{n}{k-1} \sum_{i=1}^{n} |a_i|$$

2). What happen when $a_i \in C\ (i = 1, 2, ..., n)$?

Mihály Bencze

OQ. 3530. Determine all $a, b \in N$ such that $\sum\limits_{k=1}^{n} \left(\left\{\frac{m+k}{n}\right\}\right)^a = \left(\frac{n-1}{2}\right)^b$, where $m \in N^*$, and $\{\cdot\}$ denote the fractional part.

Mihály Bencze

OQ. 3531. Let ABC be a triangle, and $M \in Int\,(ABC)$. The lines AM, BM, CM intersect the circumcircle in points A_1, B_1, C_1. Determine all points $M \in Int\,(ABC)$ such that
$\sum a \cdot AA_1 \geq (\max\{a, b, c\})^2 - (\min\{a, b, c\})^2$.

Mihály Bencze

OQ. 3532. 1). A positive integer is called carpathian prime if the number of ones in its binary expansion and the number of zeros in its binary expansion are twin primes. Determine all carpathian primes.

2). Determine all positive integers for which the number of ones in its binary expansion is prime.

3). Determine all positive integers for which the number of zeros in its binary expansion is prime.

4). Determine all positive integers for which the number of zeros and of ones in its binary expansion are prime.

Mihály Bencze

OQ. 3533. 1). Determine all perfect squares $n \in N$ for which $n - 1$ and $n + 1$ are prime.
2). Determine all $n, m, r \in N$ for which $n^m - r$, $n^m + r$ are prime.

Mihály Bencze

OQ. 3534. Let F_n and L_n denote the n^{th} Fibonacci respective Lucas nubers. Determine all $n, m \in N$ such that $F_n - L_m$ and $F_m + L_n$ are prime numbers.

Mihály Bencze

OQ. 3535. If $x_i > 0$ $(i = 1, 2, ..., n)$, then determine all $k \in N^*$ such that

$$\frac{1}{2^k}\left(n \cdot 2^k + 1 - \frac{n\prod\limits_{i=1}^{n} x_i}{\sum\limits_{i=1}^{n} x_i}\right) \le \sum_{cyclic} \sqrt[k]{\frac{x_1}{x_2}} \le \left(\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} \frac{1}{x_i}\right)\right)^{\frac{1}{k-1}}.$$

Mihály Bencze

OQ. 3536. If $a_{ij} > 0$ $(i = 1, 2, ..., n; j = 1, 2, ..., m)$, then

$$\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{m} a_{ij}\right)\right)\left(\sum_{i=1}^{n}\frac{\prod\limits_{j=1}^{m} a_{ij}}{\sum\limits_{j=1}^{m} a_{ij}}\right) \le \prod_{j=1}^{m}\left(\sum_{i=1}^{n} a_{ij}\right).$$

Mihály Bencze

OQ. 3537. Determine all $f : [a, b] \to R$ continuous and differentiable functions such that

$$\left(\int_0^1 f(a + (b-a)x)\,dx\right)\left(\int_0^1 \sqrt{1 + (f'(x))^2}dx\right) = \int_a^b f(x)\sqrt{1 + (f'(x))^2}\,dx.$$

Mihály Bencze

OQ. 3538. Let $D(x)$ denote the distance from the real number x to the nearest integer, for example $D(3,9) = D(4,1) = 0,1$. Compute

$d_\lambda = \lim\limits_{n\to\infty} \frac{1}{n}\int\limits_1^n \left(D\left(\frac{n}{x}\right)\right)^\lambda dx.$

We have the following

$$S_n(\lambda) = \int_1^n \left(D\left(\frac{n}{x}\right)\right)^\lambda dx = \sum_{k=1}^{n}\frac{1}{n}\left(\int_{\frac{2n}{2k+1}}^{\frac{n}{k}}\left(\frac{n}{x} - k\right)^\lambda dx + \int_{\frac{n}{k+1}}^{\frac{2n}{2k+1}}\left(k + 1 - \frac{n}{x}\right)^\lambda dx\right).$$

If $\lambda = 1$ then we get $S_n(1) = \ln\prod\limits_{k=1}^{n}\frac{(2k+1)^2}{2k(2k+2)}$, therefore $d_1 = \ln\frac{4}{\pi}$.

Mihály Bencze

OQ. 3539. If $a, b \in R$, $a < b$, $f_k : [a, b] \to R$ $(k = 1, 2, ..., n)$ are integrable functions, then $\sum\limits_{k=1}^{n} \left(\int\limits_a^b f_k(x)\, dx \right)^n \leq \left(\int\limits_a^b \sqrt[n]{\sum\limits_{k=1}^{n} f_k^n(x) dx} \right)^n$.

Mihály Bencze

OQ. 3540. If $f_k, g_k : R \to (0, +\infty)$ $(k = 1, 2, ..., n)$ are integrable and $\alpha > 0$, then $\frac{\int\limits_a^b \left(\sum\limits_{k=1}^{n} f_k(x) \right)^{\alpha+1} dx}{\int\limits_a^b \left(\sum\limits_{k=1}^{n} g_k(x) \right)^{\alpha}} \leq \sum\limits_{k=1}^{n} \frac{\int\limits_a^b f_k^{\alpha+1}(x) dx}{\int\limits_a^b g_k^{\alpha}(x) dx}$.

Mihály Bencze

OQ. 3541. Let $A_1A_2...A_n$ be a simplex and $B_k \in (A_1...A_{k-1}A_{k+1}...A_n)$ $(k = 1, 2, ..., n)$. Determine all $B_1, B_2, ..., B_n$ for which $A_1A_2...A_n$ and $B_1B_2...B_n$ have the same centroid.

Mihály Bencze

OQ. 3542. 1). If $a_k > 0$ $(k = 1, 2, ..., n)$, then
$\sum\limits_{k=1}^{n} a_k \geq \frac{n^2-1}{n} \left(\prod\limits_{k=1}^{n} a_k \right)^{\frac{1}{n}} + \frac{1}{n} \left(\frac{1}{n} \sum\limits_{k=1}^{n} a_k^n \right)^{\frac{1}{n}}$
2). Determine the best constants $x, y > 0$, $x + y = 1$ such that
$\sum\limits_{k=1}^{n} a_k \geq \frac{x}{n} \left(\prod\limits_{k=1}^{n} a_k \right)^{\frac{1}{n}} + \frac{y}{n} \left(\frac{1}{n} \sum\limits_{k=1}^{n} a_k^n \right)^{\frac{1}{n}}$.

Mihály Bencze

OQ. 3543. If $a \geq b \geq 1$, $x_k > 0$ $(k = 1, 2, ..., n)$ and $\sum\limits_{k=1}^{n} x_k = 1$, then
$\prod\limits_{k=1}^{n} \left(\frac{a}{\sqrt{x_k}} - b\sqrt{x_k} \right) \geq \left(\frac{an-b}{\sqrt{n}} \right)^n$.

Mihály Bencze

An application of the Catalan equation (OQ. 3090)

József Sándor[39]

ABSTRACT. We infirm a conjecture to the effect that if p, q are given prime, then there are infinitely many n with the property that $n + p$ and $n + q$ have only one prime divisor (see [1])

Put $p = 2$, $q = 3$. Then $n + 2$ and $n + 3$ have only one prime divisor, if $n + 2 = a^m$, $n + 3 = b^n$, where a and b are prime. This means that $a^m - 2 = b^n - 3 = n$, i.e.

$$b^n - a^m = 1 \tag{1}$$

Equation (1) was introduced in 1844 by E. Catalan [2], who conjectured that the only solutions in positive integers of (1) are $b = m = 3$ and $a = n = 2$.

The first nontrivial result related to equation (1) is due to C. Siegel (see [5]) who showed in 1929 that the equation can have at most a finite number of solutions. This result settles (in negative) the problem from [1].

The first effective result on Catalan‘s equation was obtained by R. Tijdeman [4] in 1976, who showed that there exists an effectively computable constant C such that for all solutions a, b, n, m of (1) (a, b not necessarily primes)

$$\max\{a, b, n, m\} < C$$

Finally, in 2002, P. Mihailescu [3] (who is an Editor to this journal) completely settled Catalan‘s conjecture.

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On OQ. 3083

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As written there, we must solve the system

$$2\varphi(n) = 1 + \Psi(n)\,;\ 3\Psi(k) = 2 + \varphi(n) \tag{1}$$

As for $n \geq 3$; $\Psi(n)$ is always even, the first equation is impossible. For $n = 1$ we get $\Psi(k) = 1$ so $k = 1$; while for $n = 2$ we obtain however $2 \cdot 1 = 1 + 3$, which is impossible.
Thus $n = 1, k = 1$ is the only solution.

As there may be some misprint (s) in the problem, we can solve similarly the system:

$$2\varphi(k) = 1 + \Psi(n)\,;\ 3\Psi(k) = 2 + \varphi(n) \tag{2}$$

or

$$2\varphi(n) = 1 + \Psi(k)\,;\ 3\Psi(k) = 2 + \varphi(n) \tag{3}$$

By the shown method, it is easy to see that (2) has the only solution $n = 1$, $k = 1$, while [3] again has this only solution again.

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Stirling formulas solves difficult inequalities (OQ. 3073)

Laurenţiu Modan[41]

ABSTRACT. An elegant condition for a difficult double inequality will be given using the famous Stirling formulas for factorial.

In this short note, we answer to the open question OQ. 3073 (see [1]), in which was proposed to find the best constants $a, b, c > 0$ such that:

$$\frac{1}{n-a} \le \left(\frac{a^n n!}{\sqrt{2\pi} n^{n+1}}\right)^2 \le \frac{1}{n-b},\ (\forall)\, n \in N^* \tag{1}$$

Firstly, we notice that $n \notin \{a, b\}$.

As in [2] and [3] we shall use the same way, namely, our starting point will be the next Stirling formulas:

$$n! = \sqrt{2\pi n}\ (n/e)^n\, e^{\theta/2n},\ \theta \in (0,1) \tag{2}$$

With (2) in (1), we obtain:

$$\frac{1}{n-a} \le \left(\frac{a}{e}\right)^{2n} \frac{e^{\theta/6n}}{n} \le \frac{1}{n-b},$$

or equivalently:

$$\frac{n}{n-a} \le \left(\frac{a}{e}\right)^{2n} e^{\theta/6n} \le \frac{n}{n-b},\ (\forall)\, n \in N^*,\ n \notin \{a, b\},\ \theta \in (0,1) \tag{3}$$

At limit in (3), it occurs:

$$\lim_{n\to\infty} \left(\frac{a}{e}\right)^{2n} = 1,$$

from where, obviously:

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$$a = e. \tag{4}$$

We go on, searching conditions for $b > 0$. Keeping account that $n - a, n - b$ have different signs, the following study will impose two cases:

i). With (4), when $n - a < 0$, $n - b > 0$ it occurs $n \in \{1, 2\}$.
In these conditions, using also the right part of (3), we obtain:

$$e^{\frac{\theta}{6}} \leq 1/ (1 - b)\,, \text{ or } e^{\theta/12} < 2/ (2 - b)\,,$$

from where, obviously $b \in (0, 1)$ or $b \in (0, 2)$. In our last inequality, if $\theta = 1/2 \in (0, 1)$,it occurs:

$$b > 1 - 1/e^{1/12}, \text{ or } b > 2 - 2/e^{1/24} \tag{5}$$

So, the relation (1) is true for $n \in \{1, 2\}$, only if:

$$a = e, b \in \left(1 - 1/e^{1/12}, 1\right), \text{ or } b \in \left(2, 2 - 2/e^{1/24}\right) \tag{6}$$

ii). We consider now, that $n - a > 0$, $n - b > 0$, namely $n > e$ with (4), and $n > b$. Because $1/ (n - a) \in 1/ (n - b)$, it normally follows $a \leq b$, namely $b \geq e$. If $n = 3$, the right side of (3) becomes:

$$e^{\theta/18} \leq \frac{3}{3 - b}, \tag{7}$$

from where $b < 3$.
So, the relation (1) is true for $n \geq 3$, only if:

$$a = e, b \in [e, 3) \tag{8}$$

Remark. *The right limit, for the interval of b, could be better, using (7), when* $\theta \in (0, 1)$.

At the end of this note, we propose to the reader, finding the best constants $a, b > 0$ which satisfy the next double inequalities:

i). $\frac{1}{n-a} \leq \frac{a^n n!}{\sqrt{2\pi} n^{n+1}} \leq \frac{1}{n-b}$,

ii). $\frac{1}{n-a} \leq \frac{b^n n!}{\sqrt{2\pi} n^{n+1}} \leq \frac{1}{n-b}$, $n \in N^*$, $n \notin \{a, b\}$

iii). $\frac{1}{n-a} \leq \frac{(ab)^n n!}{\sqrt{2\pi} n^{n+1}} \leq \frac{1}{n-b}$

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On the equation $\varphi^2(n) + \Psi(k) + 1 = 3\sigma^2(p)$ (OQ. 3132)

József Sándor[42]

We will show that when $n \geq 3$ and 3 divides k, then the equation of the title is not solvable.
Since $\varphi(n)$ is even for $n \geq 2$, clearly

$$4|\varphi^2(n)\,. \tag{1}$$

On the other hand,if $3|k$, then

$$\Psi(3)\,|\Psi(k)\,,\ \text{so}\ 4|\Psi(k) \tag{2}$$

Since the left side is odd, $z = \sigma(p)$ must be an odd number. Then $z^2 = 8M + 1$ (as the square of an odd number).
Thus, the left side of the equation is $\equiv 1 \pmod 4$; while the right side will be $\equiv 3 \pmod 4$; which is impossible.
More generally, any equation of the type

$$x^2 + y + 1 = 3z^2 \tag{3}$$

is not solvable,if $2|x$; $4|y$.

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On the equation $\left([e^x]\right)^2+\left([e^y]\right)^2=\left[e^{x^2+y^2}\right]$ (OQ. 3149)

József Sándor[43]

This is the second equation of [1].
Let $x \geq 2$. Then $x^2 \geq 2$, so if $y \geq 2$, too then

$$e^{x^2}\cdot e^{y^2} \geq e^{2x}\cdot e^{2y} > e^{2x}+e^{2y}+1 > \left([e^x]\right)^2+\left([e^y]\right)^2 \tag{1}$$

The last inequality of (1) follows by

$$\left([e^x]\right)^2 \leq \left(e^x\right)^2 = e^{2x} \text{ and } \left([e^y]\right)^2 \leq e^{2y};$$

while the second inequality of (1) follows by

$$(a-1)(b-1) = ab-a-b+1 > 2, \text{ as}$$

$$a-1>6,\ b-1>6$$

(here $a=e^{2x}>e^2>7$).
This means that for $x \geq 2, y \geq 2$ the equation of the title is not solvable.
For $x=1$ we get the equation

$$4+[e^y]^2 = \left[e\cdot e^{y^2}\right] \tag{2}$$

If $y \geq 2$, then the left side of (2) is $\leq 4+e^{2y}$, while the right side is $> e\cdot e^{y^2}-1 > 4+e^{2y}$, if

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$$e^{2y}\left(e\cdot e^{y^2-2y}-1\right)>5 \tag{3}$$

For $y\geq 2$, inequality (3) is valid; as $e^4(e-1)>5$. This means that we should have $y=1$.
However, when $x=1, y=1$ we get

$$4+4=\left[e^2\right]=7 \tag{4}$$

which is imposible.
The equation has no solutions.

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On the equation $2\left(\left[e^x\right]+\left[e^y\right]\right)=\left[e^{x+y}\right]$ (OQ. 3149)

József Sándor[44]

This is the first equation of [1].
Put $e^x=a, e^y=b$. Here $x,y\in N$. Remark that for $x\geq 2$ one has $a\geq e^2=7,45...$ Similarly if $y\geq 2$, then $b\geq e^2$, so as

$$(a-2)(b-2)=ab-2a-2b+4>5$$

(as $a-2>5$, $b-2>5$). We get

$$ab>2(a+b)+1 \tag{1}$$

Thus

$$2([a]+[b])\leq 2(a+b)<ab-1<[ab]$$

which means that the equation cannot have solutions for $x\geq 2, y\geq 2$.
Let now $x=1$. If $y\geq 2$, then $b=e^y>7$, so for the equation

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$$2\left(2+[b]\right)=[e\cdot b] \tag{2}$$

we have that

$$2\left(2+[b]\right)\leq 4+2b<eb-1<[eb]$$

as $(e-2)\,b>5$ holds true by $b>\frac{5}{e-2}=\frac{5}{0,73...}=6,84.,..$; so (2) is impossible. Finally, when $y=1$ we get $a=e$, $b=e$. We get that

$$4\,[e]=\left[e^2\right],\text{ i.e. } 8=7,$$

which is impossible.
Thus the equation of the title is not solvable in positive integers.

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