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Some new Hilbert type inequalities and applications

Gao Mingzhe[1]

ABSTRACT. In this paper it is shown that some new Hilbert type integral inequalities can be established by introducing a proper logarithm function. And the constant factor is proved to be the best possible. In particular, for case , the classical Hilbert inequality and its equivalent form are obtained. As applications, some new inequalities which they are equivalent each other are built.

1. INTRODUCTION

Let $f(x), g(x) \in L^2(0, +\infty)$. Then

$$\int_0^\infty \int_0^\infty \frac{f(x)\,g(x)}{x+y}\,dxdy \le \pi \left\{ \int_0^\infty f^2(x)\,dx \right\}^{\frac{1}{2}} \left\{ \int_0^\infty g^2(x)\,dx \right\}^{\frac{1}{2}} \tag{1.1}$$

This is the famous Hilbert integral inequality, where the coefficient π is the best possible.

In the papers [1-2], the following inequality of the form

$$\int_0^\infty \int_0^\infty \frac{\left(\ln \frac{x}{y}\right) f(x)\,g(y)}{x-y}\,dxdy \le \pi^2 \left\{ \int_0^\infty f^2(x)\,dx \right\}^{\frac{1}{2}} \left\{ \int_0^\infty g^2(x)\,dx \right\}^{\frac{1}{2}} \tag{1.2}$$

was established, and the coefficient π^2 is also the best possible.

Owing to the importance of the Hilbert inequality and the Hilbert type inequality in analysis and applications, some mathematicians have been studying them. Recently, various improvements and extensions of (1.1) and (1.2) appear in a great deal of papers (see [3]-[8]etc.).

The aim of the present paper is to build some new Hilbert type integral inequalities by introducing a proper integral kernel function and by using the

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Key words and phrases. Hilbert type integral inequality, logarithm function, euler number, the best constant, equivalent inequalities.

technique of analysis, and to discuss the constant factor of which is related to the Euler number, and then to study some equivalent forms of them.

In the sake of convenience, we introduce some notations and define some functions.

Let $0 < \alpha < 1$ and n be a positive integer. Define a function ζ^* by

$$\zeta^*\left(n, \alpha\right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(\alpha + k)^n}. \tag{1.3}$$

And further define the function ζ_2 by

$$\zeta_2 = (2n)! \left\{ 2\zeta^* \left(2n+1, \frac{1}{2} \right) \right\}, (n \in N_0) \tag{1.4}$$

In order to prove our main results, we need the following lemmas.

Lemma 1.1. Let $0 < \alpha < 1$ and n be a nonnegative integer. Then

$$\int\limits_0^1 t^{\alpha-1} \left(\ln \frac{1}{t} \right)^n \frac{1}{1+t} dt = n! \zeta^* \left(n+1, \alpha\right). \tag{1.5}$$

where ζ^* is defined by (1.3).

This result has been given in the paper [9]. Hence its proof is omitted here.

Lemma 1.2. With the assumptions as Lemma 1.1, then

$$\int\limits_0^\infty u^{\alpha-1} \left(\ln \frac{1}{u} \right)^{2n} \frac{1}{1+u} du = (2n)! \left\{ \zeta^* \left(2n+1, \alpha\right) + \zeta^* \left(2n+1, 1-\alpha\right) \right\} \tag{1.6}$$

where ζ^* is defined by (1.3).

Proof. It is easy to deduce that

$$\int\limits_0^\infty u^{\alpha-1} \left(\ln \frac{1}{u} \right)^{2n} \frac{1}{1+u} du = \int\limits_0^1 u^{\alpha-1} \left(\ln \frac{1}{u} \right)^{2n} \frac{1}{1+u} du +$$

$$+ \int\limits_1^\infty u^{\alpha-1} \left(\ln \frac{1}{u} \right)^{2n} \frac{1}{1+u} du = \int\limits_0^1 u^{\alpha-1} \left(\ln \frac{1}{u} \right)^{2n} \frac{1}{1+u} du +$$

$$+\int_0^1 v^{-\alpha}(\ln v)^{2n}\frac{1}{1+v}dv=\int_0^1 u^{\alpha-1}\left(\ln\frac{1}{u}\right)^{2n}\frac{1}{1+u}du+$$

$$+\int_0^1 v^{(1-\alpha)-1}\left(\ln\frac{1}{v}\right)^{2n}\frac{1}{1+v}dv.$$

By using Lemma 1.1, the equality (1.6) is obtained at once.

Throughout the paper, we define $\left(\ln\frac{x}{y}\right)^0=1$, when $x=y$.

2. MAIN RESULTS

We are ready now to formulate our main results.

Theorem 2.1. Let f and g be two real functions, and n be a nonnegative integer, If

$$\int_0^\infty f^2(x)\,dx<+\infty \text{ and } \int_0^\infty g^2(x)\,dx<+\infty, \text{ then}$$

$$\int_0^\infty\int_0^\infty \frac{\left(\ln\frac{x}{y}\right)^{2n}f(x)\,g(y)}{x+y}dxdy\le$$

$$\le\left(\pi^{2n+1}E_n\right)\left\{\int_0^\infty f^2(x)\,dx\right\}^{\frac{1}{2}}\left\{\int_0^\infty g^2(x)\,dx\right\}^{\frac{1}{2}}, \tag{2.1}$$

where the constant factor $\pi^{2n+1}E_n$ is the best possible, and that $E_0=1$ and E_n is the Euler number, viz. $E_1=1$, $E_2=5$, $E_3=61$, $E_4=1385$, $E_5=50521$, etc.

Proof. We may apply the Cauchy inequality to estimate the left-hand side of (2.1) as follows:

$$\int_0^\infty\int_0^\infty \frac{\left(\ln\frac{x}{y}\right)^{2n}f(x)\,g(y)}{x+y}dxdy=$$

$$= \int_0^\infty \int_0^\infty \left(\frac{\left(\ln \frac{x}{y}\right)^{2n}}{x+y} \right)^{\frac{1}{2}} \left(\frac{x}{y}\right)^{\frac{1}{4}} f(x) \left(\frac{\left(\ln \frac{x}{y}\right)^{2n}}{x+y} \right)^{\frac{1}{2}} \left(\frac{y}{x}\right)^{\frac{1}{4}} g y dx dy \leq$$

$$\leq \left\{ \int_0^\infty \int_0^\infty \frac{\left(\ln \frac{x}{y}\right)^{2n}}{x+y} \left(\frac{x}{y}\right)^{\frac{1}{2}} f^2(x) dx dy \right\}^{\frac{1}{2}} \left\{ \int_0^\infty \int_0^\infty \frac{\left(\ln \frac{x}{y}\right)^{2n}}{x+y} \left(\frac{x}{y}\right)^{\frac{1}{2}} g^2(x) dx dy \right\}^{\frac{1}{2}} =$$

$$= \left(\int_0^\infty \omega(x) f^2(x) dx \right)^{\frac{1}{2}} \left(\int_0^\infty \omega(x) g^2(x) dx \right)^{\frac{1}{2}} \tag{2.2}$$

where $\omega(x) = \int\limits_0^\infty \frac{\left(\ln \frac{x}{y}\right)^{2n}}{x+y} \left(\frac{x}{y}\right)^{\frac{1}{2}} dy$,
By using Lemma 1.2, it is easy to deduce that

$$\omega(x) = \int_0^\infty \frac{\left(\ln \frac{x}{y}\right)^{2n}}{x\left(1+\frac{y}{x}\right)} \left(\frac{x}{y}\right)^{\frac{1}{2}} dy = \int_0^\infty u^{-\frac{1}{2}} \left(\ln \frac{1}{u}\right)^{2n} \frac{1}{1+u} du = \zeta_2. \tag{2.3}$$

where ζ_2 is defined by (1.4). Based on (1.3) and (1.4), we have

$$\zeta_2 = (2n)! \left\{ 2\zeta^* \left(2n+1, \frac{1}{2}\right) \right\} = (2n)! 2 \sum_{k=0}^{\infty} \frac{(-1)^k}{\left(\frac{1}{2}+k\right)^{2n+1}} =$$

$$= (2n)! 2^{2n+2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2n+1}}.$$

It is known from the paper [10] that

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{\left(\frac{1}{2}+k\right)^{2n+1}} = \frac{\pi^{2n+1}}{2^{2n+2}(2n)!} E_n. \tag{2.4}$$

where E_n is the Euler number,viz. $E_1 = 1$, $E_2 = 5$, $E_3 = 61$, $E_4 = 1385$, $E_5 = 50521$, etc.
Since $\sum\limits_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = \frac{\pi}{4}$, we can define $E_0 = 1$. As thus, the relation (2.4) is also valid when $n = 0$. So, we get from (2.3) and (2.4) that

$$\omega(x) = \pi^{2n+1}E_n, \tag{2.5}$$

It follows from (2.2) and (2.5) that the inequality (2.1) is valid.
It remains to need only to show that $\pi^{2n+1}E_n$ in (2.1) is the best possible.
$\forall\varepsilon > 0$. Define two functions by $\widetilde{f}(x) = \begin{cases} 0 \text{ if } x \in (0,1) \\ x^{-\frac{1+\varepsilon}{2}} \text{ if } x \in [1,\infty) \end{cases}$ and $\widetilde{g}(y) = \begin{cases} 0 \text{ if } y \in (0,1) \\ y^{-\frac{1+\varepsilon}{2}} \text{ if } y \in [1,\infty) \end{cases}$. It is easy to deduce that

$$\int_0^{+\infty} \widetilde{f^2}(x)\,dx = \int_0^{+\infty} \widetilde{g}^2(y)\,dy = \frac{1}{\varepsilon}.$$

If $\pi^{2n+1}E_n$ is not the best possible, then there exists $C > 0$, such that $C < \pi^{2n+1}E_n$ and

$$S\left(\widetilde{f},\widetilde{g}\right) = \int_0^\infty\int_0^\infty \frac{\left(\ln\frac{x}{y}\right)^{2n}\widetilde{f}(x)\,\widetilde{g}(y)}{x+y}dxdy \le$$

$$\le C\left(\int_0^\infty \widetilde{f^2}(x)\,dx\right)^{\frac{1}{2}}\left(\int_0^\infty \widetilde{g}^2(y)\,dy\right)^{\frac{1}{2}} = \frac{C}{\varepsilon}. \tag{2.6}$$

On the other hand, we have

$$S\left(\widetilde{f},\widetilde{g}\right) = \int_0^\infty\int_0^\infty \frac{\left\{x^{-\frac{1+\varepsilon}{2}}\right\}\left\{\left(\ln\frac{x}{y}\right)^{2n}y^{-\frac{1+\varepsilon}{2}}\right\}}{x+y}dxdy =$$

$$= \int_0^\infty\left\{\int_0^\infty \frac{\left(\ln\frac{x}{y}\right)^{2n}y^{-\frac{1+\varepsilon}{2}}}{x+y}dy\right\}\left\{x^{-\frac{1+\varepsilon}{2}}\right\}dx =$$

$$= \int_0^\infty\left\{\int_0^\infty \frac{\left(\ln\frac{1}{u}\right)^{2n}u^{-\frac{1+\varepsilon}{2}}}{1+u}du\right\}\left\{x^{-1-\varepsilon}\right\}dx =$$

$$= \frac{1}{\varepsilon}\int_0^\infty u^{-\frac{1+\varepsilon}{2}}\left(\ln\frac{1}{u}\right)^{2n}\frac{1}{1+u}du. \tag{2.7}$$

When ε is small enough, based on (2.3) and (2.5) we can write the integral of (2.7) in the following form:

$$\int_0^\infty u^{-\frac{1+\varepsilon}{2}} \left(\ln \frac{1}{u}\right)^{2n} \frac{1}{1+u} du = \pi^{2n+1} E_n + \circ (1) . \ (\varepsilon \to 0) \tag{2.8}$$

It follows from (2.7) and (2.8) that

$$S\left(\widetilde{f}, \widetilde{g}\right) = \frac{1}{\varepsilon} \left\{ \left(\pi^{2n+1} E_n\right) + \circ (1) \right\}, \ (\varepsilon \to 0) \tag{2.9}$$

When ε is small enough, it is obvious that the inequality (2.6) is in contradiction with (2.9). Therefore, the constant factor $\pi^{2n+1}E_n$ in (2.1)is the best possible. Thus the proof of Theorem is completed.
In particular, when $n = 0$, the inequality (2.1) is reduced to (1.1). Thereby the inequality (2.1) is an extension of (1.1).
Notice that the constant factor $\pi^{2n+1}E_n$ in (2.1) can be reduced to π^3, if $n = 1$. Hence we have the following important result.

Corollary 2.2. With the assumptions as Theorem 2.1, then

$$\int_0^\infty \int_0^\infty \frac{\left(\ln \frac{x}{y}\right)^2 f(x) g(y)}{x+y} dxdy \le \pi^3 \left\{ \int_0^\infty f^2(x) dx \right\}^{\frac{1}{2}} \left\{ \int_0^\infty g^2(x) dx \right\}^{\frac{1}{2}} \tag{2.10}$$

where the constant factor π^2 is the best possible.

Corollary 2.3. Let $f(x)$ be a real functions, and n be a nonnegative integer, If $\int_0^\infty f^2(x) dx < +\infty$, then

$$\int_0^\infty \int_0^\infty \frac{\left(\ln \frac{x}{y}\right)^2 f(x) g(y)}{x+y} dxdy \le \left(\pi^{2n+1} E_n\right) \int_0^\infty f^2(x) dx, \tag{2.11}$$

where the constant factor $\pi^{2n+1}E_n$ is the best possible, and that $E_0 = 1$ and E_n is the Euler number, viz. $E_1 = 1$, $E_2 = 5$, $E_3 = 61$, $E_4 = 1385$, $E_5 = 50521$, etc.

Corollary 2.4. Let $f(x)$ be a real functions, and n be a nonnegative integer, If $\int_0^\infty f^2(x) dx < +\infty$, then

$$\int\limits_0^\infty\int\limits_0^\infty \frac{\left(\ln \frac{x}{y}\right)^2 f(x)\, g(y)}{x+y} dxdy \le \pi^3 \int\limits_0^\infty f^2(x)\, dx \tag{2.12}$$

where the constant factor π^3 is the best possible.
Notice that $E_0 = 1$, so we obtain (1.1) from (2.1) immediately when $n = 0$. Thereby the
inequality (2.1) is an extension of (1.1).

3. SOME APPLICATIONS

As applications, we will build the following inequalities.

Theorem 3.1. Let n be a nonnegative integer. If $\int\limits_0^\infty f^2(x)\, dx < +\infty$, then

$$\int\limits_0^\infty \left\{ \int\limits_0^\infty \frac{\left(\ln \frac{x}{y}\right)^{2n}}{x+y} f(x)\, dx \right\}^2 dy \le \left(\pi^{2n+1} E_n\right)^2 \int\limits_0^\infty f^2(x)\, dx, \tag{3.1}$$

where $\left(\pi^{2n+1}E_n\right)^2$ in (3.1) is the best possible, and that $E_0 = 1$ and E_n is the Euler number,viz. $E_1 = 1$, $E_2 = 5$, $E_3 = 61$, $E_4 = 1385$, $E_5 = 50521$, etc. And the inequality (3.1) is equivalent to (2.1).
Proof. Assume that the inequality (2.1) is valid. Setting a real function $g(y)$ as

$$g(y) = \int\limits_0^\infty \frac{\left(\ln \frac{x}{y}\right)^{2n}}{x+y} f(x)\, dx,\ y \in (0, +\infty)$$

By using (2.1), we have

$$\int\limits_0^\infty \left\{ \int\limits_0^\infty \frac{\left(\ln \frac{x}{y}\right)^{2n}}{x+y} f(x)\, dx \right\}^2 dy = \int\limits_0^\infty\int\limits_0^\infty \frac{\left(\ln \frac{x}{y}\right)^{2n}}{x+y} f(x)\, g(y)\, dxdy \le$$

$$\le \left(\pi^{2n+1}E_n\right) \left\{ \int\limits_0^\infty f^2(x)\, dx \right\}^{\frac{1}{2}} \left\{ \int\limits_0^\infty g^2(y)\, dy \right\}^{\frac{1}{2}} =$$

$$= \left(\pi^{2n+1}E_n\right)\left\{\int_0^\infty f^2(x)\,dx\right\}^{\frac{1}{2}}\left\{\int_0^\infty\left(\int_0^\infty \frac{\left(\ln\frac{x}{y}\right)^{2n}}{x+y}f(x)\,dx\right)^2 dy\right\}^{\frac{1}{2}} \tag{3.2}$$

where $E_0 = 1$ and E_n is the Euler number,viz. $E_1 = 1$, $E_2 = 5$, $E_3 = 61$, $E_4 = 1385$, $E_5 = 50521$, etc.

It follows from (3.2) that the inequality (3.1) is valid after some simplifications.

On the other hand, assume that the inequality (3.1) keeps valid, by applying in turn Cauchys inequality and (3.1), we have

$$\int_0^\infty\int_0^\infty \frac{\left(\ln\frac{x}{y}\right)^{2n}}{x+y}f(x)\,g(y)\,dxdy = \int_0^\infty\left\{\int_0^\infty \frac{\left(\ln\frac{x}{y}\right)^{2n}}{x+y}f(x)\,dx\right\}g(y)\,dy \leq$$

$$\leq \left\{\int_0^\infty\left(\int_0^\infty \frac{\left(\ln\frac{x}{y}\right)^{2n}}{x+y}f(x)\,dx\right)^2 dy\right\}^{\frac{1}{2}}\left\{\int_0^\infty g^2(y)\,dy\right\}^{\frac{1}{2}} \leq$$

$$\leq \left\{\left(\pi^{2n+1}E_n\right)^2\int_0^\infty f^2(x)\,dx\right\}^{\frac{1}{2}}\left\{\int_0^\infty g^2(y)\,dy\right\}^{\frac{1}{2}} =$$

$$= \left(\pi^{2n+1}E_n\right)^2\left\{\int_0^\infty f^2(x)\,dx\right\}^{\frac{1}{2}}\left\{\int_0^\infty g^2(y)\,dy\right\}^{\frac{1}{2}} \tag{3.3}$$

where $E_0 = 1$ and E_n is the Euler number,viz. $E_1 = 1$, $E_2 = 5$, $E_3 = 61$, $E_4 = 1385$, $E_5 = 50521$, etc.

If the constant factor $\left(\pi^{2n+1}E_n\right)^2$ in (3.1) is not the best possible, then it is known from (3.3) that the constant factor $\pi^{2n+1}E_n$ in (2.1) is also not the best possible. This is a contradiction. Theorem is proved.

Corollary 3.2. With the assumptions as Theorem 3.1, then

$$\int_0^\infty\left\{\int_0^\infty \frac{\left(\ln\frac{x}{y}\right)^{2n}}{x+y}f(x)\,dx\right\}^2 dy \leq \pi^6\int_0^\infty f^2(x)\,dx, \tag{3.4}$$

where the constant factor π^6 is the best possible. Inequality (3.4) is equivalent to (2.10).

In particular, for case $n = 0$, based on Theorem 3.1 we have the following result.

Corollary 3.3. If $\int\limits_0^\infty f^2(x)\,dx < +\infty$, then

$$\int\limits_0^\infty \left\{ \int\limits_0^\infty \frac{1}{x+y} f(x)\,dx \right\}^2 \le \pi^2 \int\limits_0^\infty f^2(x)\,dx. \tag{3.5}$$

where π^2 in (3.5) is the best possible, And the inequality (3.5) is equivalent to (1.1).

The proofs of Corollaries 3.2 and 3.3 are similar to one of Theorem 3.1, it is omitted here.

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REFERENCES

[1] Hardy,G. H., Littlewood, J. E. and Polya, G., *Inequalities*, Cambridge: Cambridge Univ. Press, 1952.

[2] Kuang Jichang, *Applied Inequalities*, 3nd. ed. Shandong Science and Technology Press, 2004, 534-535.

[3] Gao Mingzhe and Hsu Lizhi, *A Survey of Various Refinements And Generalizations of Hilberts Inequalities*, J. Math.Res.& Exp.,Vol. 25, 2(2005), 227-243.

[4] Hu Ke, *On Hilberts Inequality*, Chin.Ann. Math., Ser. B, Vol. 13, 1(1992), 35-39.

[5] He Leping, Gao Mingzhe and Zhou Yu, *On New Extensions of Hilberts Integral Inequality,* Internat. J. Math. & Math. Sci., Vol. 2008 (2008), Article ID 297508, 1-8.

[6] Yang Bicheng, *On a New Inequality Similar to Hardy-Hilberts Inequality*, Vol. 6, 1(2003), 37-44.

[7] Yang Bicheng, *A New Hilbert Type Integral Inequality and Its Generalization*, J. Jilin Univ. (Sci. Ed.), Vol. 43, 5(2005), 580-584.

[8] B. G. Pachpatte, *Inequalities Similar to the Integral Analogue of Hilberts Inequality*, Tamkang J. Math. 30 (1999), 139-146.

[9] Jin Yuming, *Applied Integral Tables*, Hefei: Chinese Science and Technology University Press, 2006, 247, formula: 1373.

[10] Wang Lianxiang and Fang Dexhi, *Mathematical Handbook*, Peoples Education Press, 1979, 231.

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$e^{i\pi}=-1$

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On some new inequalities for the Gamma function

Song-Zhimin, Dou-Xiangkai and Yin Li[2]

ABSTRACT. In this paper,we establish some new inequalities for the Gamma function by using the method of analysis and theory of inequality.

1. INTRODUCTION

The Euler gamma function $\Gamma(x)$ is defined for $x > 0$ by

$$\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt.$$

Alsina and Tomás [3] proved that

$$\frac{1}{n!} \leq \frac{\Gamma(1+x)^n}{\Gamma(1+nx)} \leq 1,$$

for all $x \in [0,1]$ and nonnegative integer n. The inequality can be generalized to

$$\frac{1}{\Gamma(1+a)} \leq \frac{\Gamma(1+x)^a}{\Gamma(1+ax)} \leq 1,$$

for all $x \in [0,1], a > 1$, see[4]. Recently, Shabani [6] using the series representation of the function $\frac{\Gamma'(x)}{\Gamma(x)}$ and the ideas in [5] got some double inequalities involving gamma function.
In particular, Shabani proved following result

$$\frac{\Gamma(a)^c}{\Gamma(b)^d} \leq \frac{\Gamma(a+bx)^c}{\Gamma(b+ax)^d} \leq \frac{\Gamma(a+b)^c}{\Gamma(a+b)^d},$$

for all $x \in [0,1], a > 0$,c, d are positive numbers such that $bc > ad > 0$, and $\frac{\Gamma'(b+ax)}{\Gamma(b+ax)} > 0$. Mansour using $q-$gamma function got some similar inequalities, see [7].
The related results may refer to [8].

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2. MAIN RESULTS

Now we consider the inequality

$$\frac{(m+n)!}{(m+n)^{m+n}} \le \frac{m!}{m^m}\frac{n!}{n^n},$$

which m, n are positive integer. It is a relatively common inequality. Using the form of Gamma function we can get the following inequality:

$$\frac{(m+n)!}{m!n!} \le \frac{(m+n)^{m+n}}{m^m n^n}.$$

Then, the inequality can be rewritten as

$$\frac{\Gamma(m+n+1)}{\Gamma(m+1)\Gamma(n+1)} \le \frac{(m+n)^{m+n}}{m^m n^n},$$

which m, n are positive integer. From the inequality,we can guess the following result:

Theorem 1. Let $x, y \in R^+$, then

$$\frac{(x+y)^{x+y}}{x^x y^y e} \le \frac{\Gamma(x+y+1)}{\Gamma(x+1)\Gamma(y+1)} \le \frac{(x+y)^{x+y}}{x^x y^y}.$$

In order to prove the theorem1, we introduce an important inequality.

Lemma 1. Let G and H are defined by

$$G(u) = e^u u^{-u}\Gamma(1+u), G(0) = 1$$

and

$$H(u,v) = \frac{G(u+v)}{G(u)G(v)} = \frac{u^u v^v}{(u+v)^{u+v}}\frac{\Gamma(1+u+v)}{\Gamma(1+u)\Gamma(1+v)},$$

for $a > 0$ and $b \ge 1$,we have

$$\frac{1}{e} < \frac{1}{G(\frac{b-1}{b})} \le H\left(\frac{1}{a}, \frac{b-1}{b}\right) \le 1,$$

$$\frac{1}{e} < \frac{1}{(1+a)^{\frac{1}{a}}} \le H\left(\frac{1}{a}, \frac{b-1}{b}\right) \le 1.$$

Proof. From Lemma 1, let $x = \frac{1}{a}, y = \frac{b-1}{b}$, when $x, y \in R^+$, we have

$$H\left(x,y\right) = \frac{x^x y^y}{\left(x+v\right)^{x+y}} \frac{\Gamma\left(1+x+y\right)}{\Gamma\left(1+x\right)\Gamma\left(1+y\right)} \leq 1,$$

then

$$\frac{\Gamma\left(1+x+y\right)}{\Gamma\left(1+x\right)\Gamma\left(1+y\right)} \leq \frac{\left(x+y\right)^{x+y}}{x^x y^y}.$$

but

$$\frac{1}{G(y)} \leq H\left(x,y\right),$$

so

$$\frac{y^y}{e^y\Gamma\left(1+y\right)} = \frac{1}{G\left(y\right)} \leq H\left(x,y\right) = \frac{x^x y^y}{\left(x+y\right)^{x+y}} \frac{\Gamma\left(1+x+y\right)}{\Gamma\left(1+x\right)\Gamma\left(1+y\right)}$$

$$\frac{\left(x+y\right)^{x+y}}{x^x e^y\Gamma\left(1+y\right)} \leq \frac{\Gamma\left(1+x+y\right)}{\Gamma\left(1+x\right)\Gamma\left(1+y\right)}.$$

From Lemma 1, we have

$$e^y\Gamma\left(1+y\right) < ey^y,$$

then

$$\frac{\left(x+y\right)^{x+y}}{x^x y^y e} \leq \frac{\Gamma\left(1+x+y\right)}{\Gamma\left(1+x\right)\Gamma\left(1+y\right)}.$$

From the above, we can get the following inequality:

$$\frac{(x+y)^{x+y}}{x^x y^y e} \leq \frac{\Gamma(x+y+1)}{\Gamma(x+1)\Gamma(y+1)} \leq \frac{(x+y)^{x+y}}{x^x y^y},$$

which $x, y \in R^+$.

Theorem 2. Let $x, y \in R^+$, then

$$\frac{(x+y)^{x+y}}{x^x y^y (x+y+1)} \leq \frac{\Gamma(x+y+1)}{\Gamma(x+1)\Gamma(y+1)} \leq \frac{(x+y)^{x+y}}{x^x y^y}.$$

In order to prove the Theorem 2, we introduce following inequality.

Lemma 2. Let $p > 0, q > 0, p > r > 0, q > s > 0$, then

$$B(p,q) \leq (\frac{r}{r+s})^r (\frac{s}{r+s})^s B(p-r, q-s),$$

which $B(p,q)$ is *Beta function.*

Proof. Proof of the inequalities is similar to right of Theorem 1, we'll not go into details. For the left side of the inequality, we can get conclusion from Lemma 2. In fact, let $p = x + 1$, $q = y + 1$, $r = x$, $s = y$, then

$$B(x+1, y+1) \leq (\frac{x}{x+y})^x (\frac{y}{x+y})^y B(1,1).$$

Because

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)},$$

so

$$\frac{\Gamma(x+1)\Gamma(y+1)}{\Gamma(x+y+1)} \leq \frac{x^x y^y (x+y+1)}{(x+y)^{x+y}}.$$

In particular,if $y = n$,we have following inequality.Therefore,we establish a new proof.

Theorem 3. Let $x \in R^+$, $n \in N$, then

$$\frac{(x+n)^{x+n}}{x^x n^n (x+n+1)} \leq \frac{(x+n)(x+n-1)\dots(x+1)}{n!} \leq \frac{(x+n)^{x+n}}{x^x n^n}.$$

Proof. We only consider right side of inequality. Since lnx is monotonous, we only consider following inequality:

$$ln(x+n)+ln(x+n-1)+\dots+ln(x+1)-lnn! \leq (x+n)ln(x+n)-xLnx-nlnn.$$

Let

$$f(x) = (x+n)ln(x+n)-xlnx-nlnn-ln(x+n)-ln(x+n-1)-\dots-ln(x+1)+lnn!.$$

Since

$$f(0) = nlnn - nlnn + lnn! - lnn! = 0,$$

therefore we only prove $f'(x) > 0$.Because of

$$f'(x) = ln(x+n) + 1 - lnx - 1 - \frac{1}{x+n} - \frac{1}{x+n-1} - \dots - \frac{1}{x+1}$$

and $f'(+\infty) = 0$,we only prove $f''(x) < 0$.

Consider

$$\begin{aligned} f''(x) &= \tfrac{x}{x+n}\tfrac{x-(x+n)}{x^2} + \tfrac{1}{(x+1)^2} + \tfrac{1}{(x+2)^2} + \dots + \tfrac{1}{(x+n)^2} \\ &\leq \tfrac{1}{x(x+1)} + \tfrac{1}{(x+1)(x+2)} + \dots + \tfrac{1}{(x+n-1)(x+n)} - \tfrac{n}{x(x+n)} \quad , \\ &= \tfrac{1}{x} - \tfrac{1}{x+n} - \tfrac{n}{x(x+n)} \end{aligned}$$

hence $f'(x) > f'(+\infty) = 0$,then $f(x) > f(0) = 0$.

REFERENCES

[1] Mitrinović, D.S., *Analytic inequalities[M],* Spring Verlag ,1970.
[2] Hardy, G.H., Littlehood, J.E., Polya, G., *Inequalities [M],* Cambridge,1952.
[3] Alsina, A., Tomas, M.S., *A geometric proof of a new inequality for the gamma function*, J.Ineq Pure Appl.Math.6(2005).
[4] *Askey, R.,* The q-gamma and q-beta functions, Applicable Anal.8(1978).
[5] Sándor, J., *A note on certain inequalities for the gamma function,* J.Ineq Pure Appl.Math.6(2005).
[6] Shabani, A. Sh., *Some inequalities for the gamma function*, J.Ineq Pure Appl.Math.8(2007).
[7] Mansour, T., *Inequalities for the q-gamma function,* J.Ineq Pure Appl.Math.9(2008).
[8] Mercer, A.M., *Some inequalities for the Gamma, Beta and Zeta functions,* J.Ineq Pure Appl. Math.7(2006).
[9] Octogon Mathematical Magazine (1993-2009).
[10] Bencze, M., New inequalities for the Gamma function, Creative Math. and Inf., Vol. 18, Nr. 1, 2009, pp. 84-91.

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The integral method in inequalities theory

Mihály Bencze[3]

ABSTRACT. In this paper we present the integral method applying in theory of inequalities. This method offer a linearization equivalence for a lot of inequalities. These offer too the additive and multiplicative version of many catheogory of inequalities.

MAIN RESULTS

Theorem 1. If $f : R_+^n \to R$ such that for all $a_1, a_2, ..., a_n > 0$ holds $f(a_1, a_2, ..., a_n) \geq 0$ then

$$F_u(x_1, x_2, ..., x_n) = \int\limits_0^u f\left(t^{x_1+\alpha}, t^{x_2+\alpha}, ..., t^{x_n+\alpha}\right) dt \geq 0,$$

where $a_k = t^{x_k+\alpha}\ (k = 1, 2, ..., n)$.

Proof. The result is a consequence of the Leibniz theorem.

Corollary 1. If $x, y, z > 0$, then

$$\frac{4}{3}\sum \frac{1}{x} + 12\sum \frac{1}{2x+y} \geq 15\sum \frac{1}{x+2y} + \frac{3}{x+y+z} \tag{1a}$$

Proof. In [1] page 7, problem 16 is proved that

$$4\left(\sum a\right)^3 \geq 27\sum ab^2 + 27abc$$

for all $a, b, c > 0$, which is equivalent with

$$4\sum a^3 + 12\sum a^2b \geq 15\sum ab^2 + 3abc \tag{1m}$$

Let be

$$f(a, b, c) = 4\sum a^3 + 12\sum a^2b - 15\sum ab^2 - 3abc$$

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then $f(a,b,c) \geq 0$ and

$$F_u(x,y,z) = \int_0^u f\left(t^{x-\frac{1}{3}}, t^{y-\frac{1}{3}}, t^{z-\frac{1}{3}}\right) dt =$$

$$= \int_0^u \left(4\sum t^{3x-1} + 12\sum t^{2x+y-1} - 15\sum t^{x+2y-1} - 3t^{x+y+z-1}\right) dt =$$

$$= 4\sum \frac{u^{3x}}{3x} + 12\sum \frac{u^{2x+y}}{2x+y} - 15\sum \frac{u^{x+2y}}{x+2y} - \frac{3u^{x+y+z}}{x+y+z}$$

Because $F_u'(x,y,z) = f\left(t^{x-\frac{1}{3}}, t^{y-\frac{1}{3}}, t^{z-\frac{1}{3}}\right) \geq 0$, therefore $F_u(x,y,z) \geq 0$ for all $u \geq 0$.
If $u = 1$, then

$$F_1(x,y,z) = \frac{4}{3}\sum \frac{1}{x} + 12\sum \frac{1}{2x+y} - 15\sum \frac{1}{x+2y} - \frac{3}{x+y+z} \geq 0$$

which finish the proof. We conclude that the inequalities (1a) and (1m) are equivalent, and we define (1a) the additive version and (1m) the multiplicative version. The equivalence is proved by the integral method. This method can be caracterized like the convexity and log-convexity, or the arithmetic-mean convexity and geometric-mean convexity.
It‘s enough this detailed proof, the next examples are proved short, only indicated the transfer symbols.

Corollary 2. If $x,y,z > 0$, then

$$\frac{1}{4}\sum \frac{1}{x} + \sum \frac{1}{x+y} \geq 3\sum \frac{1}{3x+y} \tag{2a}$$

Proof. In [1] page 67, problem 1 is proved that $\left(\sum a^2\right)^2 \geq 3\sum a^3 b$ or

$$\sum a^4 + 2\sum a^2 b^2 \geq 3\sum a^2 b \tag{2m}$$

In this we take $a = t^{x-\frac{1}{4}}, b = t^{y-\frac{1}{4}}, c = t^{z-\frac{1}{4}}$.
Corollary 3. If $x,y,z > 0$ and $r \in R$, then

$$\frac{1}{4}\sum \frac{1}{x} + \frac{3r^2-1}{2}\sum \frac{1}{x+y} + 3r(1-r)\sum \frac{1}{2x+y+z} \geq 3r\sum \frac{1}{3x+y} \tag{3a}$$

Proof. In [1] page 67, problem 2 is proved that

$$\sum a^4+\left(3r^2-1\right)\sum a^2b^2+3r\left(1-r\right)\sum a^2bc\geq 3r\sum a^3b \tag{3m}$$

for all $a,b,c,r\in R$. In this we take $a=t^{x-\frac{1}{4}},b=t^{y-\frac{1}{4}},c=t^{z-\frac{1}{4}}$.

Corollary 4. If $x,y,z>0$, then

$$\frac{1}{4}\sum\frac{1}{x}+\sum\frac{1}{x+3y}\geq 2\sum\frac{1}{3x+y} \tag{4a}$$

Proof. In [1] page 67, problem 3 is proved that

$$\sum a^4+\sum ab^3\geq 2\sum a^3b \tag{4m}$$

for all $a,b,c>0$. In this we take $a=t^{x-\frac{1}{4}},b=t^{y-\frac{1}{4}},c=t^{z-\frac{1}{4}}$.

Corollary 5. If $x,y,z>0$, then

$$\frac{1}{4}\sum\frac{1}{x}+2\sum\frac{1}{x+3y}\geq\frac{1}{2}\sum\frac{1}{x+y}+2\sum\frac{1}{3x+y} \tag{5a}$$

Proof. In [1] page 67, problem 4 is proved that

$$\sum a^4+2\sum ab^3\geq\sum a^2b^2+2\sum a^3b \tag{5m}$$

In this we take $a=t^{x-\frac{1}{4}},b=t^{y-\frac{1}{4}},c=t^{z-\frac{1}{4}}$.

Corollary 6. If $x,y,z>0$ and $1\leq r\leq 3$, then

$$3\sum\frac{1}{rx+\left(4-r\right)y}\leq\frac{1}{4}\sum\frac{1}{x}+\sum\frac{1}{x+y} \tag{6a}$$

Proof. In [1] page 68, problem 10 is proved that

$$3\sum a^rb^{4-r}\leq\sum a^4+2\sum a^2b^2 \tag{6m}$$

for all $a,b,c>0$ and $1\leq r\leq 3$. In this we take $a=t^{x-\frac{1}{4}},b=t^{y-\frac{1}{4}},c=t^{z-\frac{1}{4}}$.

Corollary 7. If $x,y,z>0$ and $-1\leq r\leq 2$, then

$$\frac{1}{4}\sum\frac{1}{x}+\frac{r}{2}\sum\frac{1}{x+y}\geq\left(r+1\right)\sum\frac{1}{3x+y} \tag{7a}$$

Proof. In [1] page 109, problem 2 is proved that

$$\sum a^4 + r\sum a^2b^2 \geq (r+1)\sum a^3b \tag{7m}$$

for all $a, b, c > 0$ and $-1 \leq r \leq 2$. In this we take
$a = t^{x-\frac{1}{4}}, b = t^{y-\frac{1}{4}}, c = t^{z-\frac{1}{4}}$.

Corollary 8. If $x, y, z > 0$ and $-2 \leq r \leq 2$, then

$$\frac{1}{4}\sum \frac{1}{x} + r\sum \frac{1}{x+3y} \geq \frac{r}{2}\sum \frac{1}{x+y} + \sum \frac{1}{3x+y} \tag{8a}$$

Proof. In [1] page 110, problem 3 is proved that

$$\sum a^4 + r\sum ab^3 \geq r\sum a^2b^2 + \sum a^3b \tag{8m}$$

for all $a, b, c > 0$ and $-2 \leq r \leq 2$. In this we take
$a = t^{x-\frac{1}{4}}, b = t^{y-\frac{1}{4}}, c = t^{z-\frac{1}{4}}$.

Corollary 9. If $x, y, z > 0$, then

$$\frac{7}{4}\sum \frac{1}{x} + 4\sum \frac{1}{3x+y} \geq 3\sum \frac{1}{x+y} + 5\sum \frac{1}{x+3y} \tag{9a}$$

Proof. In [1] page 110, problem 4 is proved that

$$7\sum a^4 + 4\sum a^3b \geq 6\sum a^2b^2 + 5\sum ab^3 \tag{9m}$$

In this we take $a = t^{x-\frac{1}{4}}, b = t^{y-\frac{1}{4}}, c = t^{z-\frac{1}{4}}$.

Corollary 10. If $x_k > 0$ $(k = 1, 2, ..., n)$, then

$$\frac{13}{2}\sum \frac{1}{x_1} \geq 8\sum \frac{1}{x_1+3x_2} + 9\sum \frac{1}{x_1+x_2} \tag{10a}$$

Proof. In [1] page 110, problem 5 is proved that

$$13\sum a_1^4 \geq 4\sum a_1a_2^3 + 9\sum a_1^2a_2^2 \tag{10m}$$

for all $a_k > 0$ and $-2 \leq r \leq 2$. In this we take $a_k = t^{x_k-\frac{1}{4}}$ $(k = 1, 2, ..., n)$.

Corollary 11. If $x, y, z > 0$, then

$$\frac{19}{4}\sum \frac{1}{x} + 27\sum \frac{1}{3x+y} \geq 9\sum \frac{1}{x+y} + 28\sum \frac{1}{x+3y} \tag{11a}$$

Proof. In [1] page 110, problem 6 is proved that

$$19\sum a^4+27\sum a^3b\geq 18\sum a^2b^2+28\sum ab^3 \tag{11m}$$

for all $a,b,c>0$ and $-2\leq r\leq 2$. In this we take
$a=t^{x-\frac{1}{4}},b=t^{y-\frac{1}{4}},c=t^{z-\frac{1}{4}}$.

Corollary 12. If $x_k>0$ $(k=1,2,...,n)$ and $r\geq\frac{1}{\sqrt[3]{4}-1}$, then

$$\frac{r^3-1}{4}\sum_{k=1}^{n}\frac{1}{x_k}+r^2(3-r)\sum\frac{1}{3x_1+x_2}+\frac{3r(1-r)}{2}\sum\frac{1}{x_1+x_2}+$$

$$+(1-3r)\sum\frac{1}{x_1+3x_2}\geq 0 \tag{12a}$$

Proof. In [1] page 110, problem 7 is proved that

$$\left(r^3-1\right)\sum_{k=1}^{n}a_k^4+r^2(3-r)\sum a_1^3a_2+3r(1-r)\sum a_1^2a_2^2+$$

$$+(1-3r)\sum a_1a_2^3\geq 0 \tag{12m}$$

for all $a_k>0$ and $r\geq\frac{1}{\sqrt[3]{4}-1}$. In this we take $a_k=t^{x_k-\frac{1}{4}}$ $(k=1,2,...,n)$.

Corollary 13. If $x,y,z>0$, then

$$\frac{1}{4}\sum\frac{1}{x}+6\sum\frac{1}{x+y}+7\sum\frac{1}{3x+y}\geq 2\sum\frac{1}{x+3y}+$$

$$+6\sum\frac{1}{2x+y+z}+12\sum\frac{1}{x+y+2z} \tag{13a}$$

Proof. In [1] page 110, problem 9 is proved that

$$\sum a^4+12\sum a^2b^2+7\sum a^3b\geq 2\sum ab^3+6\sum a^2bc+12\sum abc^2 \tag{13m}$$

for all $a,b,c>0$. In this we take $a=t^{x-\frac{1}{4}},b=t^{y-\frac{1}{4}},c=t^{z-\frac{1}{4}}$.

Corollary 14. If $x_k>0$ $(k=1,2,...,n)$ and $0\leq r\leq\frac{\sqrt{3}-1}{2}$, then

$$\frac{1}{4}\sum_{k=1}^{n}\frac{1}{x_k}+r\sum\frac{1}{x_1+3x_2}\geq(1+r)\sum\frac{1}{3x_1+x_2} \tag{14a}$$

Proof. In [1] page 110, problem 11 is proved that

$$\sum_{k=1}^{n} a_k^4 + r\sum a_1a_2^3 \geq (1+r)\sum a_1^3a_2 \tag{14m}$$

for all $a_k > 0$ and $0 \leq r \leq \frac{\sqrt{3}-1}{2}$. In this we take $a_k = t^{x_k-\frac{1}{4}}$ $(k = 1, 2, ..., n)$.

Corollary 15. If $x_k > 0$ $(k = 1, 2, ..., n)$, then

$$\frac{1}{4}\sum_{k=1}^{n}\frac{1}{x_k} + \frac{1}{2}\sum\frac{1}{x_1+3x_2} \geq \frac{3}{2}\sum\frac{1}{3x_1+x_2} \tag{15a}$$

Proof. In [1] page 110, problem 12 is proved that

$$\sum_{k=1}^{n} a_k^4 + +\frac{1}{2}\sum a_1a_2^3 \geq \frac{3}{2}\sum a_1^3a_2 \tag{15m}$$

for all $a_k > 0$ $(k = 1, 2, ..., n)$. In this we take $a_k = t^{x_k-\frac{1}{4}}$ $(k = 1, 2, ..., n)$.

Corollary 16. If $x, y, z > 0$, then

$$\frac{3}{4}\sum\frac{1}{x} + 2\sum\frac{1}{2x+y+z} \geq 3\sum\frac{1}{3x+y} + \sum\frac{1}{x+y} \tag{16a}$$

Proof. In [1] page 110, problem 18 is proved that

$$3\sum a^4 + 2\sum a^2bc \geq 3\sum a^3b + 2\sum a^2b^2 \tag{16m}$$

for all $a, b, c > 0$. In this we take $a = t^{x-\frac{1}{4}}, b = t^{y-\frac{1}{4}}, c = t^{z-\frac{1}{4}}$.

Corollary 17. If $x, y, z > 0$, then

$$\frac{1}{4}\sum\frac{1}{x} + 2\sqrt{2}\sum\frac{1}{x+3y} \geq \sum\frac{1}{2x+y+z} + 2\sqrt{2}\sum\frac{1}{3x+y} \tag{17a}$$

Proof. In [1] page 111, problem 19 is proved that

$$\sum a^4 + 2\sqrt{2}\sum ab^3 \geq \sum a^2bc + 2\sqrt{2}\sum a^3b \tag{17m}$$

for all $a, b, c > 0$. In this we take $a = t^{x-\frac{1}{4}}, b = t^{y-\frac{1}{4}}, c = t^{z-\frac{1}{4}}$.

Corollary 18. If $x, y, z > 0$, then

$$\frac{1}{4}\sum\frac{1}{x}+\frac{11}{2}\sum\frac{1}{x+y}\geq 6\sum\frac{1}{3x+y}+6\sum\frac{1}{x+2y+z} \tag{18a}$$

Proof. In [1] page 111, problem 20 is proved that

$$\sum a^4+11\sum a^2b^2\geq 6\sum a^3b+6\sum ab^2c \tag{18m}$$

for all $a, b, c > 0$. In this we take $a=t^{x-\frac{1}{4}}, b=t^{y-\frac{1}{4}}, c=t^{z-\frac{1}{4}}$.

Corollary 19. If $x, y, z > 0$, then

$$\frac{1}{4}\sum\frac{1}{x}+\frac{1}{2}\sum\frac{1}{x+y}\geq\left(\sqrt{6}-2\right)\sum\frac{1}{2x+y+z}+\sqrt{6}\sum\frac{1}{3x+y} \tag{19a}$$

Proof. In [1] page 112, problem 21 is proved that

$$\sum a^4+\sum a^2b^2\geq\left(\sqrt{6}-2\right)\sum a^2bc+\sqrt{6}\sum a^3b \tag{19m}$$

for all $a, b, c > 0$. In this we take $a=t^{x-\frac{1}{4}}, b=t^{y-\frac{1}{4}}, c=t^{z-\frac{1}{4}}$.

Corollary 20. If $x, y, z > 0$, then

$$\frac{1}{4}\sum\frac{1}{x}+5\sum\frac{1}{3x+y}\geq 3\sum\frac{1}{x+y} \tag{20a}$$

Proof. In [1] page 112, problem 22 is proved that

$$\sum a^4+5\sum a^3b\geq 6\sum a^2b^2 \tag{20m}$$

for all $a, b, c > 0$. In this we take $a=t^{x-\frac{1}{4}}, b=t^{y-\frac{1}{4}}, c=t^{z-\frac{1}{4}}$.

Corollary 21. If $x, y, z, t > 0$, then

$$\frac{1}{2}\sum\frac{1}{x+y+2z}+\frac{3}{x+y+z+t}\geq 3\sum\frac{1}{2x+2y+3z+t} \tag{21a}$$

Proof. In [1] page 152, problem 17 is proved that, if $u, v, r, s > 0$ and $uvrs = 1$, then $\sum(u-1)(v-2)\geq 0$. In this we take $u=\frac{a}{b}, v=\frac{b}{c}, r=\frac{c}{d}, s=\frac{d}{a}$ and we obtain:

$$\sum a^2b^2c^4+6a^2b^2c^2d^2\geq 3\sum a^2b^2c^3d \tag{21m}$$

for all $a, b, c, d > 0$. In this we take $a = t^{x-\frac{1}{8}}, b = t^{y-\frac{1}{8}}, c = t^{z-\frac{1}{8}}, d = t^{t-\frac{1}{8}}$.

Corollary 22. If $x_k > 0$ $(k = 1, 2, ..., n)$, $(n \geq 4)$ then

$$\frac{n-1}{2} \sum \frac{1}{2x_1 + x_3 + x_4 + ... + x_n} + \frac{n(n+3)}{2 \sum\limits_{k=1}^{n} x_k} \geq$$

$$\geq (2n+2) \sum \frac{1}{3x_1 + x_2 + 2x_3 + 2x_4 + ... + 2x_n} \qquad (22a)$$

Proof. In [1] page 152, problem 18 is proved that for all $b_k > 0$ $(k = 1, 2, ..., n)$ and $\prod\limits_{k=1}^{n} b_k = 1$ holds

$$(n-1) \sum_{k=1}^{n} b_k^2 + n(n+3) \geq (2n+2) \sum_{k=1}^{n} b_k$$

If in this $b_1 = \frac{a_1}{a_2}, b_2 = \frac{a_2}{a_3}, ..., b_n = \frac{a_n}{a_1}$ then we obtain

$$(n-1) \sum a_1^4 a_3^2 a_4^2 ... a_n^2 + n(n+3) a_1^2 a_2^2 ... a_n^2 \geq$$

$$\geq (2n+2) \sum a_1^3 a_2 a_3^2 a_4^2 ... a_n^2 \qquad (22m)$$

for all $a_k > 0$ $(k = 1, 2, ..., n)$. In this we take $a_k = t^{x_k - \frac{1}{2n}}$ $(k = 1, 2, ..., n)$.

Corollary 23. If $x_k > 0$ $(k = 1, 2, ..., n)$, $(n \geq 4)$ then

$$\frac{1}{n-1} \sum \frac{1}{2x_1 + x_3 + x_4 + ... + x_n} + \frac{n(n-2)}{(n-1) \sum\limits_{k=1}^{n} x_k} \geq$$

$$\geq (n-1) \sum \frac{1}{(n-2) x_1 + n x_2 + (n-1)(x_3 + x_4 + ... + x_n)} \qquad (23a)$$

Proof. In [1] page 152, problem 19 is proved that if $b_k > 0$ $(k = 1, 2, ..., n)$ and $\prod\limits_{k=1}^{n} b_k = 1$, then

$$\sum_{k=1}^{n} b_k^{n-1} + n(n-2) \geq (n-1) \sum_{k=1}^{n} \frac{1}{b_k}$$

If $b_1 = \frac{a_1}{a_2}, b_2 = \frac{a_2}{a_3}, ..., b_n = \frac{a_n}{a_1}$ then we obtain

$$\sum a_1^{2n-2}a_3^{n-1}a_4^{n-1}...a_n^{n-1}+n\left(n-2\right)\prod_{k=1}^{n}a_k^{n-1}\geq$$

$$\geq\left(n-1\right)\sum a_1^{n-2}a_2^{n}a_3^{n-1}a_4^{n-1}...a_n^{n-1}\qquad(23m)$$

for all $a_k>0\ (k=1,2,...,n)$. In this we take $a_k=t^{x_k-\frac{1}{n(n-1)}}\ (k=1,2,...,n)$.

Corollary 24. If $x_k>0\ (k=1,2,...,n)$ and $m\geq n$, then

$$\frac{1}{m}\sum\frac{1}{2x_1+x_3+x_4+...+x_n}+\frac{n}{\sum\limits_{k=1}^{n}x_k}\geq$$

$$\geq\left(m+1\right)\sum\frac{1}{\left(m-1\right)x_1+\left(m+1\right)x_2+mx_3+mx_4+...+mx_n}\qquad(24a)$$

Proof. In [1] page 153, problem 20 is proved that if $b_k>0\ (k=1,2,...,n)$ and $\prod\limits_{k=1}^{n}b_k=1$, then

$$\sum_{k=1}^{n}b_k^m+mn\geq\left(m+1\right)\sum_{k=1}^{n}\frac{1}{b_k}$$

If $b_1=\frac{a_1}{a_2},b_2=\frac{a_2}{a_3},...,b_n=\frac{a_n}{a_1}$ then we obtain

$$\sum a_1^{2m}a_3^{m}a_4^{m}...a_n^{m}+mn\prod_{k=1}^{n}a_k^{m}\geq\left(m+1\right)\sum a_1^{m-1}a_2^{m+1}a_3^{m}a_4^{m}...a_n^{m}\qquad(24m)$$

for all $a_k>0\ (k=1,2,...,n)$. In this we take $a_k=t^{x_k-\frac{1}{mn}}\ (k=1,2,...,n)$.

Corollary 25. If $x_k>0\ (k=1,2,...,n)$, then

$$\frac{1}{n-1}\sum\frac{1}{2x_1+x_3+x_4+...+x_n}+\frac{n\left(n-2\right)}{\left(n-1\right)\sum\limits_{k=1}^{n}x_k}\geq$$

$$\geq\frac{n-1}{2}\sum\frac{1}{nx_1+\left(n-2\right)x_2+\left(n-1\right)\left(x_3+x_4+...+x_n\right)}+$$

$$+\frac{n-1}{2}\sum\frac{1}{\left(n-2\right)x_1+nx_2+\left(n-1\right)\left(x_3+x_4+...+x_n\right)}\qquad(25a)$$

Proof. In [1] page 198, problem 2 is proved that if $b_k > 0$ $(k = 1, 2, ..., n)$ and $\prod\limits_{k=1}^{n} b_k = 1$, then

$$\sum_{k=1}^{n} b_k^{n-1} + n\left(n-2\right) \geq \frac{n-1}{2}\left(\sum_{k=1}^{n} b_k + \sum_{k=1}^{n}\frac{1}{b_k}\right)$$

If $b_1 = \frac{a_1}{a_2}, b_2 = \frac{a_2}{a_3}, ..., b_n = \frac{a_n}{a_1}$ then we obtain

$$\sum a_1^{2n-2} a_3^{n-1} a_4^{n-1} ... a_n^{n-1} + n\left(n-2\right)\prod_{k=1}^{n} a_k^{n-1} \geq$$

$$\geq \frac{n-1}{2}\sum a_1^{n} a_2^{n-2} a_3^{n-1} a_4^{n-1} ... a_n^{n-1} +$$

$$+\frac{n-1}{2}\sum a_1^{n-2} a_2^{n} a_3^{n-1} a_4^{n-1} ... a_n^{n-1} \tag{25m}$$

for all $a_k > 0$ $(k = 1, 2, ..., n)$. In this we take $a_k = t^{x_k - \frac{1}{n(n-1)}}$ $(k = 1, 2, ..., n)$.

Corollary 26. If $x_k > 0$ $(k = 1, 2, ..., n)$, then

$$\frac{n-2}{2}\sum_{k=1}^{n}\frac{1}{x_k} + \frac{n^2}{2\sum\limits_{k=1}^{n} x_k} \geq 2\sum_{1\leq i<j\leq n}\frac{1}{x_i + x_j} \tag{26a}$$

Proof. In [1] page 199, problem 9 is proved that

$$\left(n-2\right)\sum_{k=1}^{n} a_k^2 + n\sqrt[n]{\prod_{k=1}^{n} a_k^2} \geq 2\sum_{1\leq i<j\leq n} a_i a_j \tag{26m}$$

for all $a_k > 0$ $(k = 1, 2, ..., n)$. In this we take $a_k = t^{x_k - \frac{1}{2}}$ $(k = 1, 2, ..., n)$.

Corollary 27. If $x, y, z > 0$, then

$$\frac{1}{3}\sum\frac{1}{x+2y} + \frac{5}{x+y+z} \geq 6\sum\frac{1}{2x+4y+3z} \tag{27a}$$

Proof. In [1] page 219, problem 16 is proved that if $b_1, b_2, b_3 > 0$ and $b_1 b_2 b_3 = 1$, then

$$\sum b_1^3 + 15 \geq 6\sum\frac{1}{b_1}$$

If $b_1 = \frac{a}{b}, b_2 = \frac{b}{c}, b_3 = \frac{c}{a}$ then we obtain

$$\sum a^3b^6 + 15a^3b^3c^3 \geq 6\sum a^2b^4c^3 \tag{27m}$$

for all $a, b, c > 0$. In this we take $a = t^{x-\frac{1}{9}}, b = t^{y-\frac{1}{9}}, c = t^{z-\frac{1}{9}}$.

Corollary 28. If $x_k > 0$ $(k = 1, 2, ..., n)$, then

$$\frac{1}{n}\sum \frac{1}{x_k} + \frac{n(n-1)}{\sum\limits_{k=1}^{n} x_k} \geq \sum_{i,j=1}^{n} \frac{1}{x_i - x_j + \sum\limits_{i=1}^{n} x_k} \tag{28a}$$

Proof. In [1] page 219, problem 21 is proved that

$$\sum_{k=1}^{n} a_k^n + n(n-1)\prod_{k=1}^{n} a_k \geq \prod_{k=1}^{n} a_k \left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} \frac{1}{a_k}\right) \tag{28m}$$

for all $a_k > 0$ $(k = 1, 2, ..., n)$. In this we take $a_k = t^{x_k - \frac{1}{n}}$ $(k = 1, 2, ..., n)$.

Corollary 29. If $x_k > 0$ $(k = 1, 2, ..., n)$ and $m \geq n - 1$, then

$$\frac{1}{m}\sum \frac{1}{2x_1 + x_3 + x_4 + ... + x_n} + \frac{n(m-1)}{m\sum\limits_{k=1}^{n} x_k} \geq$$

$$\geq m\sum \frac{1}{(m-1)x_1 + (m+1)x_2 + m(x_3 + x_4 + ... + x_n)} \tag{29a}$$

Proof. In [1] page 219, problem 17 is proved that if $b_k > 0$ $(k = 1, 2, ..., n)$ and $\prod\limits_{k=1}^{n} b_k = 1$, then

$$\sum_{k=1}^{n} b_k^m + (m-1)n \geq m\sum_{k=1}^{n} \frac{1}{b_k}$$

where $m \geq n - 1$.
If $b_1 = \frac{a_1}{a_2}, b_2 = \frac{a_2}{a_3}, ..., b_n = \frac{a_n}{a_1}$ then we obtain

$$\sum a_1^{2m}a_3^m a_4^m ... a_n^m + (m-1)n\prod_{k=1}^{n} a_k^m \geq m\sum a_1^{m-1}a_2^{m+1}a_3^m a_4^m ... a_n^m \tag{29m}$$

for all $a_k > 0$ $(k = 1, 2, ..., n)$. In this we take $a_k = t^{x_k - \frac{1}{mn}}$ $(k = 1, 2, ..., n)$.

Corollary 30. If $x_k > 0$, $(k = 1, 2, ..., n)$ then

$$\frac{n-1}{n}\sum_{k=1}^{n}\frac{1}{x_k} + \frac{n}{\sum\limits_{k=1}^{n} x_k} \geq \sum_{i,j=1}^{n}\frac{1}{x_i + (n-1)\, x_j} \tag{30a}$$

Proof. In [1] page 220, problem 22 is proved the inequality

$$(n-1)\sum_{k=1}^{n} a_k^n + n\prod_{k=1}^{n} a_k \geq \left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} a_k^{n-1}\right) = \sum_{i,j=1} a_i a_j^{n-1} \tag{30m}$$

for all $a_k > 0$ $(k = 1, 2, ..., n)$. In this we take $a_k = t^{x_k - \frac{1}{n}}$ $(k = 1, 2, ..., n)$.

Corollary 31. If $x_k > 0$ $(k = 1, 2, ..., n)$, then

$$\frac{n-1}{n+1}\sum_{k=1}^{n}\frac{1}{x_k} + \sum\frac{1}{2x_1 + x_2 + x_3 + ... + x_n} \geq \sum_{i,j=1}^{n}\frac{1}{x_i + nx_j} \tag{31a}$$

Proof. In [1] page 220, problem 23 is proved that if $x_k > 0$ $(k = 1, 2, ..., n)$ then

$$(n-1)\sum_{k=1}^{n} a_k^{n+1} + \sum a_1^2 a_2 ... a_n \geq \sum_{i,j=1}^{n} a_i a_j^n \tag{31m}$$

In this we take $a_k = t^{x_k - \frac{1}{n+1}}$ $(k = 1, 2, ..., n)$.

Corollary 32. If $x_k > 0$ $(k = 1, 2, ..., n)$, then

$$\sum_{i,j=1}^{n}\frac{1}{2\sum\limits_{k=1}^{n} x_k + x_i - x_{i+1} + x_{j+1} - x_j} + \frac{n^2}{2\sum\limits_{k=1}^{n} x_k} \geq$$

$$\geq n\sum\frac{1}{3x_1 + x_2 + 2\,(x_3 + x_4 + ... + x_n)} +$$

$$+n\sum\frac{1}{x_1 + 3x_2 + 2\,(x_3 + x_4 + ... + x_n)} \tag{32a}$$

Proof. In [1] page 220, problem 24 is proved that if $b_k > 0$ $(k = 1, 2, ..., n)$, then

$$\left(\sum_{k=1}^{n} b_k - n\right)\left(\sum_{k=1}^{n} \frac{1}{b_k} - n\right) + \prod_{k=1}^{n} b_k + \frac{1}{\prod\limits_{k=1}^{n} b_k} \geq 2.$$

If $b_1 = \frac{a_1}{a_2}, b_2 = \frac{a_2}{a_3}, ..., b_n = \frac{a_n}{a_1}$ then we obtain

$$\sum_{i,j=1}^{n}\left(\prod_{k=1}^{n} a_k^2\right)\frac{a_i}{a_{i+1}} \cdot \frac{a_{j+1}}{a_j} + n^2 \prod_{k=1}^{n} a_k^2 \geq$$

$$\geq n \sum a_1^3 a_2 a_3^2 a_4^2 ... a_n^2 + n \sum a_1 a_2^3 a_3^2 a_4^2 ... a_n^2 \qquad (32m)$$

In this we take $a_k = t^{x_k - \frac{1}{2n}}\ (k = 1, 2, ..., n)$.

Corollary 33. If $x_k > 0$ $(k = 1, 2, ..., n)$, then

$$1).\ \frac{1}{3}\sum_{k=1}^{n} \frac{1}{x_k} + 3\sum \frac{1}{x_1 + x_2 + x_3} \geq \sum \frac{1}{2x_1 + x_2} + \sum \frac{1}{x_1 + 2x_2}$$

$$2).\ \frac{n-1}{6}\sum_{k=1}^{n} \frac{1}{x_k} + \frac{3}{n-2}\sum \frac{1}{x_1 + x_2 + x_3} \geq \sum \frac{1}{2x_1 + x_2} +$$

$$+ \sum \frac{1}{x_1 + 2x_2} \qquad (33a)$$

Proof. In [1] page 271, problem 4 is proved that

$$1).\ \sum_{k=1}^{n} a_k^3 + 3\sum a_1 a_2 a_3 \geq \sum a_1^2 a_2 + \sum a_1 a_2^2$$

$$2).\ \frac{n-1}{2}\sum_{k=1}^{n} a_k^3 + \frac{3}{n-2}\sum a_1 a_2 a_3 \geq \sum a_1^2 a_2 + \sum a_1 a_2^2 \qquad (33m)$$

In this we take $a_k = t^{x_k - \frac{1}{3}}\ (k = 1, 2, ..., n)$.

Corollary 34. If $x_k > 0$ $(k = 1, 2, ..., n)$, then

$$n\sum \frac{1}{x_1 + 3x_2 + 2(x_3 + x_4 + ... + x_n)} + \sum_{i,j=1}^{n} \frac{1}{2\sum\limits_{k=1}^{n} x_k + x_i - x_{i+1} + x_{j+1} - x_j} \geq$$

$$\geq \frac{n(n-2)}{2\sum\limits_{k=1}^{n} x_k} + (n+2)\sum \frac{1}{3x_1 + x_2 + 2(x_3 + x_4 + ... + x_n)} \tag{34a}$$

Proof. In [1] page 374, problem 37 is proved that if $b_k > 0$ $(k = 1, 2, ..., n)$, and $\prod\limits_{k=1}^{n} b_k = 1$, then

$$\sum_{k=1}^{n} \frac{1}{b_k} + \frac{4n}{n + \sum\limits_{k=1}^{n} b_k} \geq n + 2$$

If $b_1 = \frac{a_1}{a_2}, b_2 = \frac{a_2}{a_3}, ..., b_n = \frac{a_n}{a_1}$ then

$$n\sum a_1 a_2^3 a_3^2 a_4^2 ... a_n^2 + \sum_{i,j=1}^{n} \left(\prod_{k=1}^{n} a_k^2\right) \frac{a_i}{a_{i+1}} \cdot \frac{a_{j+1}}{a_j} \geq$$

$$\geq n(n-2)\prod_{k=1}^{n} a_k^2 + (n+2)\sum a_1^3 a_2 a_3^2 a_4^2 ... a_n^2 \tag{34m}$$

In this we take $a_k = t^{x_k - \frac{1}{2n}}$ $(k = 1, 2, ..., n)$.

Corollary 35. If $x_k > 0$ $(k = 1, 2, ..., n)$, then

$$\frac{1}{2}\sum \frac{1}{x+2y} + \frac{3}{x+y+z} \geq \frac{3}{2}\sum \frac{1}{2x+3y+z} + \frac{3}{2}\sum \frac{1}{3x+2y+z} \tag{35a}$$

Proof. In [1] page 378, problem 64 is proved that if $b_1, b_2, b_3 > 0$ and $b_1 b_2 b_3 = 1$, then

$$b_1^2 + b_3^2 + b_3^2 + 6 \geq \frac{3}{2}\left(b_1 + b_2 + b_3 + \frac{1}{b_1} + \frac{1}{b_2} + \frac{1}{b_3}\right)$$

If $b_1 = \frac{a}{b}, b_2 = \frac{b}{c}, b_3 = \frac{c}{a}$, then

$$\sum a^2 b^4 + 6a^2 b^2 c^2 \geq \frac{3}{2}\sum a^2 b^3 c + \frac{3}{2}\sum a^3 b^2 c \tag{35m}$$

In this we take $a = t^{x - \frac{1}{6}}, b = t^{y - \frac{1}{6}}, c = t^{z - \frac{1}{6}}$.
Corollary 36. If $x, y, z > 0$, then

$$\frac{1}{2}\sum \frac{1}{x+2y} + 9\sum \frac{1}{3x+2y+z} \geq 10\sum \frac{1}{2x+3y+z} \tag{36a}$$

Proof. In [1] page 380, problem 80 is proved that if $b_1, b_2, b_3 > 0$ and $b_1 b_2 b_3 = 1$, then

$$\sum b_1^2 + 9\sum b_1 b_2 \geq 10\sum b_1$$

If $b_1 = \frac{a}{b}, b_2 = \frac{b}{c}, b_3 = \frac{c}{a}$, then

$$\sum a^2 b^4 + 9\sum a^3 b^2 c \geq 10\sum a^2 b^3 c \tag{36m}$$

In this we take $a = t^{x-\frac{1}{6}}, b = t^{y-\frac{1}{6}}, c = t^{z-\frac{1}{6}}$.

Corollary 37. If $x, y, z > 0$, then

$$2\sum \frac{1}{2x+2y+z} + \sum \frac{1}{2x+3y} + \sum \frac{1}{x+4y} \geq 4\sum \frac{1}{3x+y+z} \tag{37a}$$

Proof. In [1] page 380, problem 82 is proved that if $a, b, c > 0$, then

$$\sum a + \sum \frac{a^2}{b} \geq \frac{6\sum a^2}{\sum a}$$

or

$$2\sum a^2 b^2 c + \sum a^2 b^3 + \sum a b^4 \geq 4\sum a^3 bc \tag{37m}$$

In this we take $a = t^{x-\frac{1}{5}}, b = t^{y-\frac{1}{5}}, c = t^{z-\frac{1}{5}}$.

Corollary 38. If $x_k > 0$ $(k = 1, 2, ..., n)$, then

$$\frac{n-1}{2n}\sum_{k=1}^{n} x_k + 2\sum_{1\leq i<j\leq n} \frac{1}{x_i + x_j} + \frac{n(n-1)^2}{2\sum\limits_{k=1}^{n} x_k} \geq$$

$$\geq 2(n-1)\sum_{i=1}^{n} \frac{1}{nx_i + \sum\limits_{k=1}^{n} x_k} \tag{38a}$$

Proof. In [1] page 382, problem 91 is proved that if $a_k > 0$ $(k = 1, 2, ..., n)$, then

$$\sum_{k=1}^{n} a_k \geq (n-1)\sqrt[n]{\prod_{k=1}^{n} a_k} + \sqrt{\frac{1}{n}\sum_{k=1}^{n} a_k^2}$$

or

$$\frac{n-1}{n}\sum_{k=1}^{n}a_k^2+2\sum_{1\leq i<j\leq n}a_ia_j+(n-1)^2\sqrt[n]{\prod_{k=1}^{n}a_k^2}\geq$$

$$\geq 2(n-1)\sum\sqrt[n]{a_1^{n+1}a_2a_3...a_n} \tag{38m}$$

In this we take $a_k=t^{x_k-\frac{1}{2}}\ (k=1,2,...,n)$.

Corollary 39. If $x_i>0\ (i=1,2,...,n)$ and $k\in N$, then

$$\frac{n-1}{n+k}\sum_{i=1}^{n}\frac{1}{x_i}+\sum\frac{1}{(k+1)x_1+x_2+x_3+...+x_n}\geq$$

$$\geq\sum_{i,j=1}^{n}\frac{1}{x_i+(n+k-1)x_j} \tag{39a}$$

Proof. In [1] page 382, problem 92 is proved that if $a_i>0\ (i=1,2,...,n)$ and $k\in N$, then

$$(n-1)\sum_{i=1}^{n}a_i^{n+k}+\prod_{i=1}^{n}a_i\sum_{i=1}^{n}a_i^k\geq\left(\sum_{i=1}^{n}a_i\right)\left(\sum_{i=1}^{n}a_i^{n+k-1}\right)$$

or

$$(n-1)\sum_{i=1}^{n}a_i^{n+k}+\sum a_1^{k+1}a_2a_3...a_n\geq\sum_{i,j=1}^{n}a_ia_j^{n+k-1} \tag{39m}$$

In this we take $a_i=t^{x_i-\frac{1}{n+k}}\ (i=1,2,...,n)$.

Corollary 40. If $x,y,z>0$, then

$$\sum\frac{1}{2x+5y}\geq\sum\frac{1}{2x+4y+z} \tag{40a}$$

Proof. In [2] page 9, problem 17 is proved that

$$\sum\frac{a^3}{b^2}\geq\sum\frac{a^2}{b}$$

or

$$\sum a^2b^5 \geq \sum a^2b^4c \tag{40m}$$

In this we take $a = t^{x-\frac{1}{7}}, b = t^{y-\frac{1}{7}}, c = t^{z-\frac{1}{7}}$.

Corollary 41. If $x, y > 0$ and $m, n \in N^*$ then

$$\frac{(n-1)(m-1)}{m+n}\left(\frac{1}{x}+\frac{1}{y}\right)+(m+n-1)\left(\frac{1}{mx+ny}+\frac{1}{nx+my}\right) \geq$$

$$\geq mn\left(\frac{1}{(m+n-1)x+y}+\frac{1}{x+(m+n-1)y}\right) \tag{41a}$$

Proof. In [2] page 17, problem 77 is proved that if $a, b > 0$ and $m, n \in N^*$ then

$$(n-1)(m-1)\left(a^{m+n}+b^{m+n}\right)+(m+n-1)\left(a^mb^n+a^nb^m\right) \geq$$

$$\geq mn\left(a^{m+n-1}b+ab^{m+n-1}\right) \tag{41m}$$

In this we take $a = t^{x-\frac{1}{m+n}}, b = t^{y-\frac{1}{m+n}}$.

Corollary 42. If $x, y, z, t > 0$, then

$$\frac{1}{4}\sum\frac{1}{x}+\frac{2}{\sum x} \geq \frac{1}{2}\left(\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+t}+\frac{1}{t+x}+\frac{1}{x+z}+\frac{1}{y+t}\right) \tag{42a}$$

Proof. In [2] page 21, problem 104 is proved that if $a, b, c, d > 0$, then

$$\sum a^4 + 2abcd \geq a^2b^2+b^2c^2+c^2d^2+d^2a^2+a^2c^2+b^2d^2 \tag{42m}$$

In this we take $a = t^{x-\frac{1}{4}}, b = t^{y-\frac{1}{4}}, c = t^{z-\frac{1}{4}}, d = t^{t-\frac{1}{4}}$.

Corollary 43. If $x_i > 0$ $(i = 1, 2, ..., n)$, then

$$\frac{1}{2}\sum_{i=1}^{n}\frac{1}{x_i}+2\sum_{1\leq i<j\leq n}\frac{1}{x_i+x_j} \leq \sum_{i,j=1}^{n}\frac{ij}{(i+j-1)(x_i+x_j)} \tag{43a}$$

Proof. In [2] page 21, problem 105 is proved that if $a_i > 0$ $(i = 1, 2, ..., n)$, then

$$\left(\sum_{i=1}^{n} a_i\right)^2 \leq \sum_{i,j=1}^{n} \frac{ija_ia_j}{i+j-1}$$

or

$$\sum_{i=1}^{n} a_i^2 + 2\sum_{1\leq i<j\leq n} a_ia_j \leq \sum_{i,j=1}^{n} \frac{ija_ia_j}{i+j-1} \tag{43m}$$

In this we take $a_i = t^{x_i - \frac{1}{2}}$ $(i = 1, 2, ..., n)$.

Corollary 44. If $x_k > 0$ $(k = 1, 2, ..., n)$, then

$$\frac{1}{2}\sum \frac{1}{2x_1 + x_3 + x_4 + ... + x_n} \geq \frac{2n\sqrt[n]{n-1}}{n-1}\cdot$$

$$\cdot\sum \frac{1}{3x_1 + x_2 + 2\left(x_3 + x_4 + ... + x_n\right)} + \frac{n\left(1 - \frac{2\sqrt[n]{n-1}}{n-1}\right)}{2\sum\limits_{k=1}^{n} x_k} \tag{44a}$$

Proof. In [2] page 22, problem 112 is proved that if $b_k > 0$ $(k = 1, 2, ..., n)$, $\prod\limits_{k=1}^{n} b_k = 1$ then

$$\sum_{k=1}^{n} b_k^2 - n \geq \frac{2n\sqrt[n]{n-1}}{n-1}\left(\sum_{k=1}^{n} b_k - n\right)$$

If $b_1 = \frac{a_1}{a_2}, b_2 = \frac{a_2}{a_3}, ..., b_n = \frac{a_n}{a_1}$ then

$$\sum a_1^4 a_3^2 a_4^2 ... a_n^2 \geq$$

$$\geq \frac{2n\sqrt[n]{n-1}}{n-1}\sum a_1^3 a_2 a_3^2 a_4^2 ... a_n^2 + n\left(1 - \frac{2\sqrt[n]{n-1}}{n-1}\right)\prod_{k=1}^{n} a_k^2 \tag{44m}$$

In this we take $a_k = t^{x_k - \frac{1}{2n}}$ $(k = 1, 2, ..., n)$.

Corollary 45. If $x_k > 0$ $(k = 1, 2, ..., n)$, then

$$\frac{n-2}{n}\sum \frac{1}{2x_1 + x_3 + x_4 + ... + x_n} + \frac{n}{2\sum\limits_{k=1}^{n} x_k} \geq$$

$$\geq 2\sum \frac{1}{2\sum\limits_{k=1}^{n} x_k + x_i + x_j - x_{i+1} - x_{j+1}} \tag{45a}$$

Proof. In [2] page 22, problem 17 is proved that, if $b_k > 0$ $(k = 1, 2, ..., n)$, $\prod\limits_{k=1}^{n} b_k = 1$ then

$$\sum_{1\leq i<j\leq n} (b_i - b_j)^2 \geq \sum_{k=1}^{n} b_k^2 - n.$$

If $b_1 = \frac{a_1}{a_2}, b_2 = \frac{a_2}{a_3}, ..., b_n = \frac{a_n}{a_1}$ then

$$(n-2)\sum a_1^4 a_3^2 a_4^2 ... a_n^2 + n\prod_{k=1}^{n} a_k^2 \geq 2\sum_{1\leq i<j\leq n} \frac{a_i}{a_{i+1}} \cdot \frac{a_j}{a_{j+1}} \left(\prod_{k=1}^{n} a_k^2\right) \tag{45m}$$

In this we take $a_k = t^{x_k - \frac{1}{2n}}$ $(k = 1, 2, ..., n)$.

Corollary 46. If $x, y, z > 0$, then

$$\frac{1}{2}\sum \frac{1}{x+2y} + \frac{1}{2}\sum \frac{1}{2x+y} \geq \sum \frac{1}{3x+y+2z} + \sum \frac{1}{x+3y+2z} \tag{46a}$$

Proof. In [3] page 25, problem 10 is proved that, if $a, b, c > 0$, then

$$\sum \left(\frac{a^2}{b^2} + \frac{b^2}{a^2}\right)^2 \geq \sum \left(\frac{a}{b} + \frac{b}{a}\right)^2$$

or

$$\sum a^4 b^8 + \sum a^8 b^4 \geq \sum a^6 b^2 c^4 + \sum a^2 b^6 c^4 \tag{46m}$$

In this we take $a = t^{x - \frac{1}{12}}, b = t^{y - \frac{1}{12}}, c = t^{z - \frac{1}{12}}$.

Corollary 47. If $x_k > 0$ $(k = 1, 2, ..., n)$, then

$$\frac{1}{2}\sum_{k=1}^{n} \frac{1}{x_k} - \sum \frac{1}{x_1 + x_2} \geq$$

$$\geq \frac{n}{4(n-1)} \max\left\{\frac{(x_1 - x_2)^2}{x_1 x_2 (x_1 + x_2)}; \frac{(x_2 - x_3)^2}{x_2 x_3 (x_2 + x_3)}; ...; \frac{(x_n - x_1)^2}{x_n x_1 (x_n + x_1)}\right\} \tag{47a}$$

Proof. In [3] page 25, problem 14 is proved that, if $a_k > 0$ $(k = 1, 2, ..., n)$, then

$$\sum_{k=1}^{n} a_k^2 - \sum a_1 a_2 \geq \frac{n}{2(n-1)} \max \left\{ (a_1 - a_2)^2, (a_2 - a_3)^2, ..., (a_n - a_1)^2 \right\}$$

In this we take $a_k = t^{x_k - \frac{1}{2}}$ $(k = 1, 2, ..., n)$.

Corollary 48. If $x, y, z > 0$, then

$$\frac{1}{3} \sum \frac{1}{x} + \frac{3}{\sum x} \geq \sum \frac{1}{2x + y} + \sum \frac{1}{x + 2y} \tag{48a}$$

Proof. In [3] page 26, problem 2.30 is proved that, if $a, b, c > 0$, then

$$\sum a^3 + 3abc \geq \sum a^2 b + \sum ab^2 \tag{48m}$$

In this we take $a = t^{x - \frac{1}{3}}, b = t^{y - \frac{1}{3}}, c = t^{z - \frac{1}{3}}$.

Corollary 49. If $x, y, z > 0$, then

$$\frac{2}{3} \sum \frac{1}{x} \geq \sum \frac{1}{2x + y} + \sum \frac{1}{x + 2y} \tag{49a}$$

Proof. In [3] page 26, problem 2.31 is proved that

$$2 \sum a^3 \geq \sum a^2 b + \sum ab^2 \tag{49m}$$

for all $a, b, c > 0$. In this we take $a = t^{x - \frac{1}{3}}, b = t^{y - \frac{1}{3}}, c = t^{z - \frac{1}{3}}$.

Corollary 50. If $x, y, z > 0$ and $n \in N^*$ then

$$\sum \frac{1}{(n+1)x + y} + \sum \frac{1}{x + (n+1)y} \geq 2 \sum \frac{1}{nx + y + z} \tag{50a}$$

Proof. In [3] page 27, problem 2.47 is proved that

$$\sum a^{n+1} b + \sum ab^{n+1} \geq 2 \sum a^n bc \tag{50m}$$

In this we take $a = t^{x - \frac{1}{n+2}}, b = t^{y - \frac{1}{n+2}}, c = t^{z - \frac{1}{n+2}}$.

Corollary 51. If $x_i > 0$ $(i = 1, 2, ..., n)$ and $k \in N^*$ then

$$\frac{1}{k} \sum \frac{1}{2x_1 + x_3 + x_4 + ... + x_n} \geq$$

$$\geq \frac{1}{(k+1)\, x_1 + (k-1)\, x_2 + k\,(x_3 + x_4 + ... + x_n)} \qquad (51a)$$

Proof. In Wildt József MC 2009 Mihály Bencze proved that if $a_i > 0$ $(i = 1, 2, ..., n)$ and $k \in N^*$, then

$$\sum \left(\frac{a_1}{a_2}\right)^k \geq \sum \frac{a_1}{a_2}$$

or

$$\sum a_1^{2k} a_3^k a_4^k ... a_n^k \geq \sum a_1^{k+1} a_2^{k-1} a_3^k a_4^k ... a_n^k \qquad (51m)$$

In this we take $a_i = t^{x_i - \frac{1}{nk}}$ $(i = 1, 2, ..., n)$.

Corollary 52. If $x, y, z > 0$, then

$$\frac{1}{3} \sum \frac{1}{x+y} \geq \sum \frac{1}{2x + 3y + z} \qquad (52a)$$

Proof. In [3] page 29, problem 2.72 is proved that if $b_1, b_2, b_3 > 0$ and $b_1 b_2 b_3 = 1$, then

$$\sum \frac{b_1}{b_2} \geq \sum b_1.$$

If $b_1 = \frac{a}{b}, b_2 = \frac{b}{c}, b_3 = \frac{c}{a}$, then we obtain

$$\sum a^3 b^3 \geq \sum a^2 b^3 c \qquad (52m)$$

In this we take $a = t^{x - \frac{1}{6}}, b = t^{y - \frac{1}{6}}, c = t^{z - \frac{1}{6}}$.

Corollary 53. If $x_k > 0$ $(k = 1, 2, ..., n)$, then

$$\sum_{k=1}^{n} \frac{1}{x_k} \geq \frac{n\left(n^2 - 1\right)}{12} \min_{1 \leq i < j \leq n} \frac{(x_i - x_j)^2}{x_i x_j (x_i + x_j)} \qquad (53a)$$

Proof. In [3] page 121, problem 9.11 is proved that, if $a_k > 0$ $(k = 1, 2, ..., n)$, then

$$\sum_{k=1}^{n} a_k^2 \geq \frac{n\left(n^2-1\right)}{12} \min_{1\leq i<j\leq n} \left(a_i - a_j\right)^2 \tag{53b}$$

In this we take $a_k = t^{x_k - \frac{1}{2}}\ (k = 1, 2, ..., n)$.

Corollary 54. If $x, y, z, t > 0$, then

$$\sum \frac{1}{4x+y} \geq \sum \frac{1}{2x+y+z+t} \tag{54a}$$

Proof. In [3] page 121, problem 9.23 is proved that, if $a, b, c, d > 0$, then

$$\sum a^4 b \geq \sum a^2 bcd \tag{54m}$$

In this we take $a = t^{x-\frac{1}{5}}, b = t^{y-\frac{1}{5}}, c = t^{z-\frac{1}{5}}, d = t^{t-\frac{1}{5}}$.

Corollary 55. If $x, y, z > 0$, then

$$\frac{1}{2}\sum \frac{1}{x+2y} + 15\sum \frac{1}{3x+2y+z} \geq 16\sum \frac{1}{2x+3y+z} \tag{55a}$$

Proof. In [1] page 456, is proved that if $b_1, b_2, b_3 > 0$ and $b_1 b_2 b_3 = 1$, then

$$\sum b_1^2 + 15\sum b_1 b_2 \geq 16\sum b_1$$

If $b_1 = \frac{a}{b}, b_2 = \frac{b}{c}, b_3 = \frac{c}{a}$, then we get

$$\sum a^2 b^4 + 15\sum a^3 b^2 c \geq 16\sum a^2 b^3 c \tag{55m}$$

In this we take $a = t^{x-\frac{1}{5}}, b = t^{y-\frac{1}{5}}, c = t^{z-\frac{1}{5}}$.

Corollary 56. If $x, y, z > 0$, then

$$\frac{1}{5}\sum \frac{1}{x} + 10\left(\sum \frac{1}{2x+y} + \sum \frac{1}{x+2y}\right) + 5\left(\sum \frac{1}{4x+y} + \sum \frac{1}{x+4y}\right) +$$

$$+30\sum \frac{1}{2x+2y+z} \geq 61\sum \frac{1}{3x+y+z} \tag{56a}$$

Proof. In [1] page 16, problem 3 is proved that

$$\left(\sum a\right)^5 \geq 81abc\sum a^2$$

for all $a, b, c > 0$ which can be written in following form:

$$\sum a^5 + 10\left(\sum a^2b^3 + \sum a^3b^2\right) + 5\left(\sum a^4b + \sum ab^4\right) +$$

$$+30\sum a^2b^2c \geq 61\sum a^3bc \tag{56m}$$

In this we take $a = t^{x-\frac{1}{5}}, b = t^{y-\frac{1}{5}}, c = t^{z-\frac{1}{5}}$.

Corollary 57. If $x, y, z > 0$, then

$$1).\ \sum \frac{1}{x+4y} + \sum \frac{1}{2x+2y+z} \geq 2\sum \frac{1}{3x+y+z}$$

$$2).\ \frac{1}{2}\sum \frac{1}{2x+y} + \frac{3}{2\sum x} \geq 2\sum \frac{1}{2x+3y+z} \tag{57a}$$

Proof. In [5] page 2, problem 2 is proved that

$$1).\ \sum \frac{a^3}{b} + \sum ab \geq 2\sum a^2 \text{ and}$$

$$2).\ \sum \frac{a^3c}{b} + 3abc \geq 2\sum a^2c, \text{ which can written}$$

$$1).\ \sum ab^4 + \sum a^2b^2c \geq 2\sum a^3bc$$

$$2).\ \sum a^2b^4 + 3a^2b^2c^2 \geq 2\sum a^2b^3c \tag{57m}$$

In these we take:
1). $a = t^{x-\frac{1}{5}}, b = t^{y-\frac{1}{5}}, c = t^{z-\frac{1}{5}}$ and
2). $a = t^{x-\frac{1}{6}}, b = t^{y-\frac{1}{6}}, c = t^{z-\frac{1}{6}}$.

Corollary 58. If $\alpha, x_k > 0$ $(k = 1, 2, ..., n)$, then

$$1).\ \sum \frac{1}{(\alpha+2)x_1 + x_3 + x_4 + ... + x_n} + \sum \frac{1}{\alpha x_1 + 2x_2 + x_3 + x_4 + ... + x_n} \geq$$

$$\geq 2\sum \frac{1}{(\alpha+1)x_1 + x_2 + x_3 + ... + x_n}$$

$$2).\ \frac{1}{2}\sum\frac{1}{2x_1+x_3+x_4+...+x_n}+\frac{n}{2\sum\limits_{k=1}^{n}x_k}\geq$$

$$\geq 2\sum\frac{1}{3x_1+x_2+2\left(x_3+x_4+...+x_n\right)} \tag{58a}$$

Proof. In [5] page 3 is proved that

$$1).\ \sum a_1^{\alpha+2}a_3a_4...a_n+\sum a_1^{\alpha}a_2^2a_3a_4...a_n\geq 2\sum a_1^{\alpha+1}a_2a_3...a_n$$

$$2).\ \sum a_1^4a_3^2a_4^2...a_n^2+n\prod_{k=1}^{n}a_k^2\geq 2\sum a_1^3a_2a_3^2a_4^2...a_n^2 \tag{58m}$$

In these we take:
1). $a_k=t^{x_k-\frac{1}{\alpha+n}}$ $(k=1,2,...,n)$ and
2). $a_k=t^{x_k-\frac{1}{2n}}$ $(k=1,2,...,n)$

Corollary 59. If $x,y,z>0$, then

$$\frac{1}{9}\sum\frac{1}{x}+2\sum\frac{1}{x+2y}\geq\frac{4}{x+y+z}+\sum\frac{1}{2x+y} \tag{59a}$$

Proof. In [5] page 6, problem 5 is proved that
(59m)

$$\frac{1}{3}\sum a^3+2\sum ab^2\geq 4abc+\sum a^2b \tag{59m}$$

In this we take $a=t^{x-\frac{1}{3}},b=t^{y-\frac{1}{3}},c=t^{z-\frac{1}{3}}$.

Corollary 60. If $x,y,z,t>0$, then

$$1).\ \frac{1}{4}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{3}{t}\right)\geq 2\left(\frac{1}{x+y+2t}+\frac{1}{y+z+2t}+\frac{1}{z+x+2t}\right)$$

$$2).\ \frac{3}{4}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{t}\right)\geq$$

$$\geq\sum\frac{1}{x+y+2t}+\sum\frac{1}{y+z+2t}+\sum\frac{1}{z+x+2t} \tag{60a}$$

Proof. In [5] page 53, problem 58 is proved that

$$1).\ a^4+b^4+c^4+3d^4\geq 2\left(ab+bc+ca\right)d^2$$

$$2).\ 3\left(a^4+b^4+c^4+d^4\right)\geq \sum abd^2+\sum bcd^2+\sum cad^2 \qquad (60m)$$

In these we take $a=t^{x-\frac{1}{4}}, b=t^{y-\frac{1}{4}}, c=t^{z-\frac{1}{4}}$.

Corollary 61. If $x_k>0\ (k=1,2,...,n)$, then

$$\frac{n^2\left(n^2+n-1\right)}{3}\sum_{k=1}^{n}\frac{1}{x_k}+3n^2\left(\sum\frac{1}{2x_1+x_2}+\sum\frac{1}{x_1+2x_2}\right)+$$

$$+6n^2\sum\frac{1}{x_1+x_2+x_3}\geq\left(2n-1\right)n^2\sum_{i,j=1}^{n}\frac{1}{2x_i+x_j} \qquad (61a)$$

Proof. In [1] page 150, problem 1 is proved that, if $b_k>0\ (k=1,2,...,n)$ and $\sum\limits_{k=1}^{n}b_k=n$, then

$$(n-1)\sum_{k=1}^{n}b_k^3+n^2\geq(2n-1)\sum_{k=1}^{n}b_k^2$$

If $b_k=\frac{na_k}{\sum\limits_{i=1}^{n}a_i}\ (k=1,2,...,n)$, then we obtain:

$$n^2\left(n^2-n+1\right)\sum_{k=1}^{n}a_k^3+3n^2\left(\sum a_1^2a_2+\sum a_1a_2^2\right)+6n^2\sum a_1a_2a_3\geq$$

$$\geq n^2\left(2n-1\right)\sum_{i,j=1}^{n}a_i^2a_j \qquad (61m)$$

In this we take $a_k=t^{x_k-\frac{1}{3}}\ (k=1,2,...,n)$.

Corollary 62. If $x_k>0\ (k=1,2,...,n)$, then

$$\frac{n^2\left(n+1\right)}{3}\sum_{k=1}^{n}\frac{1}{x_k}+3n^2\left(\sum\frac{1}{2x_1+x_2}+\sum\frac{1}{x_1+2x_2}\right)+$$

$$+6n^2\sum\frac{1}{x_1+x_2+x_3}\leq\left(n+1\right)\sum_{i,j=1}^{n}\frac{1}{x_i+2x_j} \qquad (62a)$$

Proof. In [1] page 150, problem 2 is proved that, if $b_k > 0$ $(k = 1, 2, ..., n)$ and $\sum\limits_{k=1}^{n} b_k = n$, then

$$\sum_{k=1}^{n} b_k^3 + n^2 \leq (n+1) \sum_{k=1}^{n} b_k^2$$

If $b_k = \frac{na_k}{\sum\limits_{i=1}^{n} a_i}$ $(k = 1, 2, ..., n)$, then we get:

$$n^2 (n+1) \sum_{k=1}^{n} a_k^3 + 3n^2 \left(\sum a_1^2 a_2 + \sum a_1 a_2^2 \right) + 6n^2 \sum a_1 a_2 a_3 \leq$$

$$\leq (n+1) \sum_{i,j=1}^{n} a_i a_j^2 \qquad (62m)$$

In this we take $a_k = t^{x_k - \frac{1}{3}}$ $(k = 1, 2, ..., n)$.

Corollary 63. If $x_k > 0$ $(k = 1, 2, ..., n)$, then

$$\sum_{i,j=1}^{n} \frac{1}{3x_i + \sum\limits_{k=1}^{n} x_k - x_j} \geq \left(5n^2 - 8n + 4\right) \sum \frac{1}{3x_1 + x_2 + x_3 + ... + x_n} +$$

$$+2(n-2)^2 \sum \frac{1}{2(x_1 + x_2) + x_3 + x_4 + ... + x_n} \qquad (63a)$$

Proof. In [1] page 150, problem 5 is proved that, if $b_k > 0$ $(k = 1, 2, ..., n)$ and $\sum\limits_{k=1}^{n} b_k = 1$, then

$$\sum_{k=1}^{n} \frac{1}{b_k} \geq (n-2)^2 + 4n(n-1) \sum_{k=1}^{n} b_k^2$$

If $b_k = \frac{na_k}{\sum\limits_{i=1}^{n} a_i}$ $(k = 1, 2, ..., n)$, then we obtain:

$$\sum_{i,j=1}^{n} \frac{a_i^3 a_1 a_2 ... a_n}{a_j} \geq$$

$$\geq \left(5n^2 - 8n + 4\right) \sum a_1^3 a_2 a_3 ... a_n + 2(n-2)^2 \sum a_1^2 a_2^2 a_3 a_4 ... a_n \qquad (63m)$$

In this we take $a_k = t^{x_k - \frac{1}{n+2}}$ $(k = 1, 2, ..., n)$.

Corollary 64. If $x_k > 0$ $(k = 1, 2, ..., n)$ and $m \in N^*$ then

$$\frac{n^{m-1}}{m} \sum_{k=1}^{n} \frac{1}{x_k} \geq \sum_{\substack{0 \leq i_1 < ... < i_m \leq m \\ i_1 + ... + i_n = m}} \frac{m!}{i_1!...i_n! \left(i_1 x_1 + i_2 x_2 + ... + i_n x_n\right)} \tag{64a}$$

Proof. We have for all $a_k > 0$ $(k = 1, 2, ..., n)$ the following inequality

$$n^{m-1} \sum_{k=1}^{n} a_k^m \geq \left(\sum_{k=1}^{n} a_k\right)^m = \sum_{\substack{0 \leq i_1 < ... < i_m \leq m \\ i_1 + ... + i_n = m}} \frac{m!}{i_1!...i_n!} a_1^{i_1} a_2^{i_2} ... a_n^{i_n} \tag{64m}$$

In this we take $a_k = t^{x_k - \frac{1}{n}}$ $(k = 1, 2, ..., n)$.

Corollary 65. If $x, y, z > 0$, then

$$\frac{13}{4} \sum \frac{1}{x} \geq 2 \left(\sum \frac{1}{3x + y} + \frac{1}{x + 3y}\right) + \frac{3}{2} \sum \frac{1}{x + y} + 6 \sum \frac{1}{2x + y + z} \tag{65a}$$

Proof. If in Corrolary 64 we take $m = 4$ and $n = 3$, then we have:

$$13 \sum a^4 \geq 2 \left(\sum a^3 b + \sum a b^3\right) + 3 \sum a^2 b^2 + 6 \sum a^2 bc \tag{65m}$$

In this we take $a = t^{x - \frac{1}{4}}, b = t^{y - \frac{1}{4}}, c = t^{z - \frac{1}{4}}$.

Corollary 66. If $x_k > 0$ $(k = 1, 2, ..., n)$ then

$$\frac{2n + 1}{3} \sum_{k=1}^{n} \frac{1}{x_k} + 3 \left(\sum \frac{1}{2x_1 + x_2} + \sum \frac{1}{x_1 + 2x_2}\right) + 6 \sum \frac{1}{x_1 + x_2 + x_3} \leq$$

$$\leq (2n + 1) \sum_{i,j=1}^{n} \frac{1}{x_i + 2x_j} \tag{66a}$$

Proof. In [1] page 154, problem 29 is proved that, if $b_k > 0$ $(k = 1, 2, ..., n)$ and $\sum\limits_{k=1}^{n} b_k = n$, then

$$2 \sum_{k=1}^{n} b_k^3 + n^2 \leq (2n + 1) \sum_{k=1}^{n} b_k^2$$

If $b_k = \frac{na_k}{\sum\limits_{i=1}^{n} a_i}$ $(k = 1, 2, ..., n)$, then we get

$$(2n+1)\sum_{k=1}^{n} a_k^3 + 3\left(\sum a_1^2 a_2 + \sum a_1 a_2^2\right) + 6\sum a_1 a_2 a_3 \leq (2n+1)\sum_{i,j=1}^{n} a_i a_j^2 \tag{66m}$$

In this we take $a_k = t^{x_k - \frac{1}{3}}$ $(k = 1, 2, ..., n)$.

Corollary 67. If $x, y, z, t > 0$, then

$$1).\ \frac{1}{3}\sum \frac{1}{x} + 3\sum \frac{1}{x+y+z} \geq \sum \frac{1}{2x+y} + \sum \frac{1}{x+2y} \tag{67a}$$

$$2).\ \sum \frac{1}{x} + 3\sum \frac{1}{x+y+z} \geq 2\left(\sum \frac{1}{2x+y} + \sum \frac{1}{x+2y}\right)$$

Proof. In [1] page 271, problem 3 is proved that, if $b_1, b_2, b_3, b_4 > 0$ and $b_1 + b_2 + b_3 + b_4 = 1$, then
1). $4\sum b_1^3 + 15\sum b_1 b_2 b_3 \geq 1$
2). $11\sum b_1^3 + 21\sum b_1 b_2 b_3 \geq 2$
If $b_1 = \frac{a}{s}, b_2 = \frac{b}{s}, b_3 = \frac{c}{s}, b_4 = \frac{c}{s}$ where $s = a + b + c + d$, then we obtain

$$1).\ \sum a^3 + 3\sum abc \geq \sum a^2 b + \sum ab^2$$

$$2).\ 3\sum a^3 + 3\sum abc \geq 2\left(\sum a^2 b + \sum ab^2\right) \tag{67m}$$

In these we take $a = t^{x-\frac{1}{3}}, b = t^{y-\frac{1}{3}}, c = t^{z-\frac{1}{3}}, d = t^{t-\frac{1}{3}}$.

Corollary 68. (The additive version of Cauchy-Schwarz inequality) If $x_k, y_k > 0$ $(k = 1, 2, ..., n)$, then

$$\frac{1}{2}\sum_{i,j=1} \frac{1}{x_i + y_j} \geq \frac{1}{2}\sum_{k=1}^{n} \frac{1}{x_k + y_k} + 2\sum_{1 \leq i < j \leq n} \frac{1}{x_i + y_i + x_j + y_j} \tag{68a}$$

Proof. Using the Cauchy-Schwarz inequality we have

$$\sum_{i,j=1}^{n} a_i^2 b_j^2 = \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) \geq \left(\sum_{k=1}^{n} a_k b_k\right)^2 =$$

$$=\sum_{k=1}^{n}a_k^2b_k^2+2\sum_{1\leq i<j\leq n}a_ib_ia_jb_j \tag{68m}$$

In this we take $a_k=t^{x_k-\frac{1}{4}}, b_k=t^{y_k-\frac{1}{4}}$ $(k=1,2,...,n)$.

Corollary 69. (The additive version of D.K. Callebaut‘s inequality) If $x_k, y_k>0$ $(k=1,2,...,n)$ and $0<\alpha<\beta<1$, then

$$\frac{1}{2}\sum_{k=1}^{n}\frac{1}{x_k+y_k}+2\sum_{1\leq i<j\leq n}\frac{1}{x_i+y_i+x_j+y_j}\leq$$

$$\leq\sum_{i,j=1}^{n}\frac{1}{(1+\alpha)\,x_i+(1-\alpha)\,y_i+(1-\alpha)\,x_j+(1+\alpha)\,y_j}\leq$$

$$\leq\sum_{i,j=1}^{n}\frac{1}{(1+\beta)\,x_i+(1-\beta)\,y_i+(1-\beta)\,x_j+(1+\beta)\,y_j}\leq$$

$$\leq\frac{1}{2}\sum_{i,j=1}^{n}\frac{1}{x_i+x_j} \tag{69a}$$

Proof. The D.K. Callebaut‘s inequality say if $a_k, b_k>0$ $(k=1,2,...,n)$ and $0<\alpha<\beta<1$, then

$$\sum_{k=1}^{n}a_k^2b_k^2+2\sum_{1\leq i<j\leq n}a_ib_ia_jb_j=\left(\sum_{k=1}^{n}a_kb_k\right)^2\leq$$

$$\leq\left(\sum_{k=1}^{n}a_k^{1+\alpha}b_k^{1-\alpha}\right)\left(\sum_{k=1}^{n}a_k^{1-\alpha}b_k^{1+\alpha}\right)=\sum_{i,j=1}^{n}a_i^{1+\alpha}b_i^{1-\alpha}a_j^{1-\alpha}b_j^{1+\alpha}\leq$$

$$\leq\left(\sum_{k=1}^{n}a_k^{1+\beta}b_k^{1-\beta}\right)\left(\sum_{k=1}^{n}a_k^{1-\beta}b_k^{1+\beta}\right)=\sum_{i,j=1}^{n}a_i^{1+\beta}b_i^{1-\beta}a_j^{1-\beta}b_j^{1+\beta}\leq$$

$$\leq\left(\sum_{k=1}^{n}a_k^2\right)\left(\sum_{k=1}^{n}b_k^2\right) \tag{69m}$$

In this we take $a_k=t^{x_k-\frac{1}{4}}, b_k=t^{y_k-\frac{1}{4}}$ $(k=1,2,...,n)$.

Corollary 70. If $x_i > 0$ $(i = 1, 2, ..., n)$ and $k \in N$, then

$$\frac{1}{k+1}\sum_{i,j=1}^{n}\frac{1}{x_i - x_j + \sum\limits_{p=1}^{n} x_p} \geq \sum_{i,j=1}^{n}\frac{1}{kx_i - kx_j + (k+1)\sum\limits_{p=1}^{n} x_p} \tag{70a}$$

Proof. In [4] M. Bencze proved that, if $a_i > 0$ $(i = 1, 2, ..., n)$ then

$$\left(\sum_{i=1}^{n} a_i^{k+1}\right)\left(\sum_{i=1}^{n}\frac{1}{a_i^{k+1}}\right) \geq \left(\sum_{i=1}^{n} a_i^{k}\right)\left(\sum_{i=1}^{n}\frac{1}{a_i^{k}}\right) \tag{70m}$$

or

$$\sum_{i,j=1}^{n}\frac{a_i^{k+1}\prod\limits_{p=1}^{n} a_p^{k+1}}{a_j^{k+1}} \geq \sum_{i,j=1}^{n}\frac{a_i^{k}\prod\limits_{p=1}^{n} a_p^{k+1}}{a_j^{k}}$$

In this we take $a_i = t^{x_i - \frac{1}{(k+1)n}}$ $(i = 1, 2, ..., n)$.

Corollary 71. If $x_k > 0$ $(k = 1, 2, ..., n)$ then

$$\frac{1}{n}\sum_{1 \leq i < j \leq n}\left(\frac{x_i - x_j}{\sqrt{x_i x_j}}\right)^2 \geq \frac{1}{n}\sum_{k=1}^{n}\frac{1}{x_k} - \frac{n}{\sum\limits_{k=1}^{n} x_k} \geq \frac{1}{n^2}\sum_{1 \leq i < j \leq n}\left(\frac{x_i - x_j}{\sqrt{x_i x_j}}\right)^2$$

Proof. In [7] M. Bencze proved the following inequality, if $a_k > 0$ $(k = 1, 2, ..., n)$ then

$$\frac{1}{n}\sum_{1 \leq i < j \leq n}\left(\sqrt{a_i} - \sqrt{a_j}\right)^2 \geq \frac{1}{n}\sum_{k=1}^{n} a_k - \sqrt[n]{\prod_{k=1}^{n} a_k} \geq \frac{1}{n^2}\sum_{1 \leq i < j \leq n}\left(\sqrt{a_i} - \sqrt{a_j}\right)^2$$

In this we take $a_k = t^{x_k - 1}$ $(k = 1, 2, ..., n)$.

Corollary 72. If $x, y, z > 0$ then

$$\frac{1}{2}\sum\frac{1}{2x+y} + \frac{1}{2}\sum\frac{1}{x+2y} + \sum\frac{1}{4x+y+z} \geq \sum\frac{1}{x+3y+2z} +$$

$$+\sum\frac{1}{x+2y+3z} + \frac{3}{2(x+y+z)} \tag{72a}$$

Proof. In [6] page 116, problem 258 is proved that, if $a, b, c > 0$, then

$$\left(a^2+ab+b^2\right)\left(b^2+bc+c^2\right)\left(c^2+ca+a^2\right) \geq (ab+bc+ca)^3 \text{ or}$$

$$\sum a^4b^2 + \sum a^2b^4 + \sum a^4bc \geq \sum ab^3c^2 + \sum ab^2c^3 + 3a^2b^2c^2 \qquad (72m)$$

In this we take $a = t^{x-\frac{1}{6}}, b = t^{y-\frac{1}{6}}, c = t^{z-\frac{1}{6}}$.

Corollary 73. If $x, y, z > 0$ then

$$\frac{3}{4}\min\left\{\frac{(x-y)^2}{xy}, \frac{(y-z)^2}{yz}, \frac{(z-x)^2}{zx}\right\} \leq \frac{1}{2}\sum\frac{1}{x} - \sum\frac{1}{x+y} \leq$$

$$\leq \frac{3}{4}\max\left\{\frac{(x-y)^2}{xy}, \frac{(y-z)^2}{yz}, \frac{(z-x)^2}{zx}\right\} \qquad (73a)$$

Proof. In [6] page 187, problem 426 is proved that, if $a, b, c > 0$, then

$$\frac{3}{2}\min\left\{(a-b)^2; (b-c)^2; (c-a)^2\right\} \leq a^2+b^2+c^2-ab-bc-ca \leq$$

$$\leq \frac{3}{2}\left\{(a-b)^2; (b-c)^2; (c-a)^2\right\} \qquad (73m)$$

In this we take $a = t^{x-\frac{1}{2}}, b = t^{y-\frac{1}{2}}, c = t^{z-\frac{1}{2}}$.

Corollary 74. If $x, y > 0$ then

$$\frac{1}{2^n}\sum_{k=0}^{n}\frac{\binom{n}{k}}{(n-k)x+ky} \leq \frac{1}{n+1}\sum_{k=0}^{n}\frac{1}{(n-k)x+ky} \leq \frac{1}{2n}\left(\frac{1}{x}+\frac{1}{y}\right) \qquad (74a)$$

Proof. In [6] page 223, problem 466 is proved that, for all $a, b > 0$ holds

$$\left(\frac{a+b}{2}\right)^n \leq \frac{1}{n+1}\sum_{k=0}^{n}a^{n-k}b^k \leq \frac{a^n+b^n}{2} \qquad (74m)$$

In this we take $a = t^{x-\frac{1}{n}}, b = t^{y-\frac{1}{n}}$.

Corollary 75. If $x_k > 0$ $(k = 1, 2, ..., n)$ and $\alpha \in [0, 1]$, then

$$\frac{n}{\sum\limits_{k=1}^{n} x_k} \leq \frac{1}{n} \sum \frac{1}{(1-\alpha) x_1 + \alpha x_2} \leq \frac{1}{n} \sum_{k=1}^{n} \frac{1}{x_k} \tag{75a}$$

Proof. In [6] page 255, problem 491 is proved that,

$$\sqrt[m]{\prod_{k=1}^{n} a_k} \leq \frac{1}{n} \sum a_1 \left(\frac{a_2}{a_1}\right)^{\alpha} \leq \frac{1}{n} \sum_{k=1}^{n} a_k \tag{75m}$$

In this we take $a_k = t^{x_k - 1}$ $(k = 1, 2, ..., n)$.

Corollary 76. (The additive version of Chebishev‘s inequality). If $x_{i1}, x_{i2}, ..., x_{im}$ $(i = 1, 2, ..., n)$ is positive increasing (or decreasing) sequence, $p_k > 0$ $(k = 1, 2, ..., n)$, then

$$\sum_{i_1, i_2, ..., i_m = 1}^{n} \frac{p_{i_1} p_{i_2} ... p_{i_m}}{x_{i_1 1} + x_{i_2 2} + ... + x_{i_m m}} \leq$$

$$\leq \left(\sum_{k=1}^{n} p_k\right)^{m-1} \sum_{i=1}^{n} \frac{p_i}{x_{i1} + x_{i2} + ... + x_{im}} \tag{76a}$$

Proof. Using the generalized Chebishev‘s inequality (see [8], page 370 we can written, if $a_{i1}, a_{i2}, ..., a_{im}$ $(i = 1, 2, ..., n)$ is increasing (or decreasing) positive sequence, then

$$\sum_{i_1, ..., i_m = 1}^{n} p_{i_1} ... p_{i_m} a_{i_1 1} a_{i_2 2} ... a_{i_m m} \leq \left(\sum_{k=1}^{n} p_k\right)^{m-1} \sum_{i=1}^{n} p_i a_{i1} a_{i2} ... a_{im} \tag{76m}$$

In this we take $a_{ij} = t^{x_{ij} - \frac{1}{m}}$ $(i = 1, 2, ..., n; j = 1, 2, ..., m)$.

Corollary 77. If $x, y, z > 0$ then

$$\sum \frac{1}{2x + 3y + z} + \sum \frac{1}{x + 3y + 2z} \leq \frac{1}{2 \sum x} + \frac{1}{3} \sum \frac{1}{x + y} +$$

$$+ \sum \frac{1}{4x + y + z} \tag{77a}$$

Proof. In inequality

$$\sum a^2b^3c+\sum ab^3c^2\leq a^2b^2c^2+\sum a^3b^3+\sum a^4bc \tag{77m}$$

where $a,b,c>0$ we take $a=t^{x-\frac{1}{6}},b=t^{y-\frac{1}{6}},c=t^{z-\frac{1}{6}}$.

Corollary 78. If $x,y,z>0$ then

$$\frac{1}{6}\sum\frac{1}{x}+\frac{2}{3}\sum\frac{1}{x+y}\geq 2\sum\frac{1}{5x+y}+\sum\frac{1}{4x+y+z} \tag{78a}$$

Proof. In inequality

$$\sum a^6+2\sum a^3b^3\geq 2\sum a^5b+\sum a^4bc \tag{78m}$$

for all $a,b,c>0$ we take $a=t^{x-\frac{1}{6}},b=t^{y-\frac{1}{6}},c=t^{z-\frac{1}{6}}$.

Corollary 79. If $p_k,x_k>0$ $(k=1,2,...,n)$, then

$$\left(\sum_{k=1}^{n}\frac{p_k}{x_k}\right)\left(\sum_{k=1}^{n}p_kx_k\right)\geq\left(\sum_{k=1}^{n}p_k\right)^2 \tag{79a}$$

Proof. If $p_k,x_k>0$ $(k=1,2,...,n)$, then from weighted AM-GM inequality we have

$$\frac{\sum\limits_{k=1}^{n}p_ka_k}{\sum\limits_{k=1}^{n}p_k}\geq\left(\prod_{k=1}^{n}a_k^{p_k}\right)^{\frac{1}{\sum\limits_{k=1}^{n}p_k}} \tag{79m}$$

If $a_k=t^{x_k-1}$ $(k=1,2,...,n)$, then we obtain the result, which is the weighted AM-HM inequality.

Remark. We conclude that the weighted AM-HM inequality is the additive version of the weighted AM-GM inequality.

REFERENCES

[1] Cîrtoaje, V., *Algebraic inequalities,* Gil, 2006.
[2] Andreescu, T., Cîrtoaje, V., Dospinescu, G., Lascu, M., *Old and new inequalities,* Gil, 2004.
[3] Panaitopol, L., Bandila, V., Lascu, M., *Egyenlőtlenségek, Gil, 1996*

[4] Octogon Mathematical Magazine (1993-2009)
[5] Bencze, M. and Arslanagic, S., *A mathematical Ptroblem Book,* Sarajevo, 2008.
[6] Drimbe, M.O., *Inegualitati,* Gil, 2003
[7] Bencze, M., *New method for generating new inequalities,* Octogon Mathematical Magazine, vol. 16, No. 2, October 2008, pp. 1051-1057.
[8] Mitrinovic, D.S., *Analytic Inequalities,* Springer Verlag, 1970.

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Hermite-Hadamard and Fejér Inequalities for Wright-Convex Functions

Vlad Ciobotariu-Boer [4]

ABSTRACT. In this paper, we establish several inequalities of Hermite-Hadamard and Fejér type for Wright-convex functions.

1. INTRODUCTION

Throughout this paper we will consider a real-valued convex function f, defined on a nonempty interval $I \subset \mathbb{R}$, and $a, b \in I$, with $a < b$.
In the conditions above, we have:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \cdot \int\limits_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

The inequalities (1.1) are known as the Hermite-Hadamard inequalities (see [10], [17], [20]). In [9], Fejér established the following weighted generalization of the inequalities (1.1):

$$f\left(\frac{a+b}{2}\right) \cdot \int\limits_a^b p(x)\mathrm{d}x \leq \int\limits_a^b f(x)p(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2} \cdot \int_a^b p(x)\mathrm{d}x, \tag{1.2}$$

where $p : [a, b] \to \mathbb{R}$ is a nonnegative, integrable, and symmetric about $x = \frac{a+b}{2}$. The last inequalities are known as the Fejér inequalities.

In recent years, many extensions, generalizations, applications and similar results of the inequalities (1.1) and (1.2) were deduced (see [1]-[4], [6]-[8], [12]-[16], [18], [21], [22]).
In [5], Dragomir established the following theorem, which is a refinement of the first inequality of (1.1):

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Theorem 1.1. If H is defined on $[0,1]$ by

$$H(t)=\frac{1}{b-a}\cdot\int\limits_a^b f\left(tx+(1-t)\cdot\frac{a+b}{2}\right)\mathrm{d}x,$$

where the function f is convex on $[a,b]$, then H is convex, nondecreasing on $[0,1]$, and for all $t\in[0,1]$, we have

$$f\left(\frac{a+b}{2}\right)=H(0)\leq H(t)\leq H(1)=\frac{1}{b-a}\cdot\int\limits_a^b f(x)\mathrm{d}x. \tag{1.3}$$

In [19], Yang and Hong established the following theorem which is a refinement of the second inequality of (1.1):

Theorem 1.2. If F is defined on $[0,1]$ by

$$F(t)=\frac{1}{2(b-a)}\cdot\int\limits_a^b\left[f\left(\frac{1+t}{2}\cdot a+\frac{1-t}{2}\cdot x\right)+f\left(\frac{1+t}{2}\cdot b+\frac{1-t}{2}\cdot x\right)\right]\mathrm{d}x,$$

where the function f is convex on $[a,b]$, then F is convex, nondecreasing on $[0,1]$, and for all $t\in[0,1]$, we have

$$\frac{1}{b-a}\cdot\int_a^b f(x)\mathrm{d}x=F(0)\leq F(t)\leq F(1)=\frac{f(a)+f(b)}{2}. \tag{1.4}$$

In [20], Yang and Tseng established the following theorem, which refines the inequality (1.2):

Theorem 1.3. If P,Q are defined on $[0,1]$ by

$$P(t)=\int\limits_a^b f\left(tx+(1-t)\cdot\frac{a+b}{2}\right)\cdot p(x)\mathrm{d}x$$

and

$$Q(t)=\frac{1}{2}\cdot\int\limits_a^b\left[f\left(\frac{1+t}{2}\cdot a+\frac{1-t}{2}\cdot x\right)\cdot p\left(\frac{x+a}{2}\right)+\right.$$

$$+f\left(\frac{1+t}{2}\cdot b+\frac{1-t}{2}\cdot x\right)\cdot p\left(\frac{x+b}{2}\right)\Bigg]\,\mathrm{d}x,$$

where the function f is convex on $[a,b]$, then P,Q are convex and increasing on $[0,1]$, and for all $t\in[0,1]$:

$$f\left(\frac{a+b}{2}\right)\cdot\int_a^b p(x)\mathrm{d}x=P(0)\le P(t)\le P(1)=\int_a^b f(x)p(x)\mathrm{d}x \tag{1.5}$$

and

$$\int_a^b f(x)\cdot p(x)\mathrm{d}x=Q(0)\le Q(t)\le Q(1)=\frac{f(a)+f(b)}{2}\cdot\int_a^b p(x)\mathrm{d}x, \tag{1.6}$$

where $p:[a,b]\to\mathbb{R}$ is nonnegative, integrable and symmetric about $x=\frac{a+b}{2}$. In the following, we recall the definition of a Wright-convex function:

Definition 1.1. (see [15]) We say that $f:[a,b]\to\mathbb{R}$ is a *Wright-convex function* if for all $x,y\in[a,b]$ with $x<y$ and $\delta\ge 0$, so that $x+\delta\in[a,b]$, we have:

$$f(x+\delta)+f(y)\le f(y+\delta)+f(x).$$

Denoting the set of all convex functions on $[a,b]$ by $K([a,b])$ and the set of all Wright-convex functions on $[a,b]$ by $W([a,b])$, then $K([a,b])\subset W([a,b])$, the inclusion being strict (see [14], [15]).
Next, we give a theorem that characterizes Wright-convex functions (see [18]):

Theorem 1.4. If $f:[a,b]\to\mathbb{R}$, then the following statements are equivalent:
(i) $f\in W([a,b])$;
(ii) for all $s,t,u,v\in[a,b]$ with $s\le t\le u\le v$ and $t+u=s+v$, we have $f(t)+f(u)\le f(s)+f(v)$.

In [18], Tseng, Yang and Dragomir established the following theorems for Wright-convex functions, related to the inequalities (1.1):

Theorem 1.5. Let $f \in W([a,b]) \cap L_1[a,b]$. Then, the inequalities (1.1) hold.

Theorem 1.6. Let $f \in W([a,b]) \cap L_1[a,b]$ and let H be defined as in Theorem 1.1. Then $H \in W([0,1])$ is nondecreasing on $[0,1]$, and the inequality (1.3) holds for all $t \in [0,1]$.

Theorem 1.7. Let $f \in W([a,b]) \cap L_1[a,b]$ and let F be defined as in Theorem 1.2. Then $F \in W([0,1])$ is nondecreasing on $[0,1]$, and the inequality (1.4) holds for all $t \in [0,1]$.
In [12], Ming-In-Ho established the following theorems for Wright-convex functions related to the inequalities (1.2):

Theorem 1.8. Let $f : W([a,b]) \cap L_1[a,b]$ and let p defined as in Theorem 1.3. Then the inequalities (1.2) hold.

Theorem 1.9. Let p, P, Q be defined as in Theorem 1.3. Then $P, Q \in W([0,1])$ are nondecreasing on $[0,1]$, and the inequalities (1.5) and (1.6) hold for all $t \in [0,1]$.
In [3], we established the following theorems for convex functions related to inequalitites (1.2) and (1.2):

Theorem 1.10. If R, S are defined on $[0,1]$ by

$$R(t) = \frac{1}{b-a} \cdot \int_a^b f\left(\frac{1+t}{2} \cdot \frac{a+b}{2} + \frac{1-t}{2} \cdot x\right) \mathrm{d}x$$

and

$$S(t) = \frac{1}{2(b-a)} \cdot \int_a^b \left[f\left(\frac{a+b}{2} - t \cdot \frac{b-x}{2}\right) + f\left(\frac{a+b}{2} + t \cdot \frac{x-a}{2}\right)\right] \mathrm{d}x,$$

where the function f is convex on $[a,b]$, then R is convex, nonincreasing on $[0,1]$ and S is convex, nondecreasing on $[0,1]$, and for all $t \in [0,1]$, we have:

$$f\left(\frac{a+b}{2}\right) = R(1) \le R(t) \le R(0) = \frac{1}{b-a} \cdot \int_a^b f\left(\frac{a+b}{4} + \frac{x}{2}\right) \mathrm{d}x \le$$

$$\leq \frac{1}{2} \cdot f\left(\frac{a+b}{2}\right) + \frac{1}{2(b-a)} \cdot \int\limits_a^b f(x)\mathrm{d}x \leq \frac{1}{b-a} \cdot \int\limits_a^b f(x)\mathrm{d}x \tag{1.7}$$

and

$$f\left(\frac{a+b}{2}\right) = S(0) \leq S(t) \leq S(1) = \frac{1}{b-a} \cdot \int\limits_a^b f(x)\mathrm{d}x. \tag{1.8}$$

Theorem 1.11. If T, U are defined on $[0,1]$ by

$$T(t) = \int\limits_a^b f\left(\frac{1+t}{2} \cdot \frac{a+b}{2} + \frac{1-t}{2} \cdot x\right) \cdot p(x)\mathrm{d}x$$

and

$$U(t) = \frac{1}{2} \cdot \int\limits_a^b \left[f\left(\frac{a+b}{2} - t \cdot \frac{b-x}{2}\right) \cdot p\left(\frac{x+a}{2}\right) + \right.$$
$$\left. + f\left(\frac{a+b}{2} + t \cdot \frac{x-a}{2}\right) \cdot p\left(\frac{x+b}{2}\right)\right] \mathrm{d}x,$$

where the function f is convex on $[a,b]$ and p is defined on $[a,b]$ as in Theorem 1.3, then T is convex, nonincreasing on $[0,1]$ and U is convex, nondecreasing on $[0,1]$, and for all $t \in [0,1]$ we have:

$$f\left(\frac{a+b}{2}\right) \cdot \int\limits_a^b p(x)\mathrm{d}x = T(1) \leq T(t) \leq T(0) = \int\limits_a^b f\left(\frac{a+b}{4} + \frac{x}{2}\right) \cdot p(x)\mathrm{d}x \leq$$

$$\leq \frac{1}{2} \cdot f\left(\frac{a+b}{2}\right) \cdot \int\limits_a^b p(x)\mathrm{d}x + \frac{1}{2} \cdot \int\limits_a^b f(x)p(x)\mathrm{d}x \leq \int\limits_a^b f(x)p(x)\mathrm{d}x \tag{1.9}$$

and

$$f\left(\frac{a+b}{2}\right) \cdot \int\limits_a^b p(x)\mathrm{d}x = U(0) \leq U(t) \leq U(1) = \int\limits_a^b f(x)p(x)\mathrm{d}x. \tag{1.10}$$

In this paper, we establish some results related to Theorem 1.10 and Theorem 1.11 for Wright-convex functions.

MAIN RESULTS

Theorem 2.1. Let $f \in W([a,b]) \cap L_1[a,b]$ and let R be defined as in Theorem 1.10. Then, $R \in W([0,1])$ is nonincreasing on $[0,1]$, and the inequalities (1.7) hold for all $t \in [0,1]$

Proof. If $s,t,u,v \in [0,1]$ with $s \le t \le u \le v$ and $t+u = s+v$, then, for all $x \in \left[a, \frac{a+b}{2}\right]$, we have

$$a \le \frac{1+s}{2} \cdot \frac{a+b}{2} + \frac{1-s}{2} \cdot x \le \frac{1+t}{2} \cdot \frac{a+b}{2} + \frac{1-t}{2} \cdot x \le$$

$$\le \frac{1+u}{2} \cdot \frac{a+b}{2} + \frac{1-u}{2} \cdot x \le \frac{1+v}{2} \cdot \frac{a+b}{2} + \frac{1-v}{2} \cdot x \le \frac{a+b}{2}$$

and, for all $x \in \left[\frac{a+b}{2}, b\right]$, we have

$$\frac{a+b}{2} \le \frac{1+v}{2} \cdot \frac{a+b}{2} + \frac{1-v}{2} \cdot x \le \frac{1+u}{2} \cdot \frac{a+b}{2} + \frac{1-u}{2} \cdot x \le$$

$$\le \frac{1+t}{2} \cdot \frac{a+b}{2} + \frac{1-t}{2} \cdot x \le \frac{1+s}{2} \cdot \frac{a+b}{2} + \frac{1-s}{2} \cdot x \le b.$$

Denoting

$$s_1 := \frac{1+s}{2} \cdot \frac{a+b}{2} + \frac{1-s}{2} \cdot x,$$

$$t_1 := \frac{1+t}{2} \cdot \frac{a+b}{2} + \frac{1-t}{2} \cdot x,$$

$$u_1 := \frac{1+u}{2} \cdot \frac{a+b}{2} + \frac{1-u}{2} \cdot x,$$

$$v_1 := \frac{1+v}{2} \cdot \frac{a+b}{2} + \frac{1-v}{2} \cdot x,$$

we note that for $x \in \left[a, \frac{a+b}{2}\right]$, $s_1, t_1, u_1, v_1 \in \left[a, \frac{a+b}{2}\right]$ with $s_1 \le t_1 \le u_1 \le v_1$ and $t_1 + u_1 = s_1 + v_1$. Since $f \in W([a,b])$, taking into account the Theorem 1.4, we deduce:

$$f(t_1) + f(u_1) \le f(s_1) + f(v_1) \quad \text{for all } x \in \left[a, \frac{a+b}{2}\right]. \tag{2.1}$$

Denoting

$$s_2 := \frac{1+v}{2} \cdot \frac{a+b}{2} + \frac{1-v}{2} \cdot x,$$

$$t_2 := \frac{1+u}{2} \cdot \frac{a+b}{2} + \frac{1-u}{2} \cdot x,$$

$$u_2 := \frac{1+t}{2} \cdot \frac{a+b}{2} + \frac{1-t}{2} \cdot x,$$

$$v_2 := \frac{1+s}{2} \cdot \frac{a+b}{2} + \frac{1-s}{2} \cdot x,$$

for $x \in \left[\frac{a+b}{2}, b\right]$, we note that $s_2, t_2, u_2, v_2 \in \left[\frac{a+b}{2}, b\right]$ with $s_2 \leq t_2 \leq u_2 \leq v_2$ and $t_2 + u_2 = s_2 + v_2$. Since $f \in W([a, b])$, taking into account the Theorem 1.4, we obtain:

$$f(t_2) + f(u_2) \leq f(s_2) + f(v_2) \quad \text{for all } x \in \left[\frac{a+b}{2}, b\right]. \tag{2.2}$$

Integrating the inequality (2.1) over x on $\left[a, \frac{a+b}{2}\right]$, the inequality (2.2) over x on $\left[\frac{a+b}{2}, b\right]$ and adding the obtained inequalities and multiplying the result by $\frac{1}{b-a}$, we find:

$$R(t) + R(u) \leq R(s) + R(v),$$

namely $R \in W([0, 1])$.
In order to prove the monotonicity of $R \in W([0, 1])$, we consider $0 \leq t_1 < t_2 \leq 1$. Then, we have:

$$a \leq \frac{1+t_1}{2} \cdot \frac{a+b}{2} + \frac{1-t_1}{2} \cdot x \leq \frac{1+t_2}{2} \cdot \frac{a+b}{2} + \frac{1-t_2}{2} \cdot x \leq$$

$$\leq \frac{1+t_2}{2} \cdot \frac{a+b}{2} + \frac{1-t_2}{2} \cdot (a+b-x) \leq \frac{1+t_1}{2} \cdot \frac{a+b}{2} + \frac{1-t_1}{2} \cdot (a+b-x) \leq \frac{a+b}{2}$$

for all $x \in \left[a, \frac{a+b}{2}\right]$ and

$$\frac{a+b}{2} \leq \frac{1+t_1}{2} \cdot \frac{a+b}{2} + \frac{1-t_1}{2} \cdot (a+b-x) \leq \frac{1+t_2}{2} \cdot \frac{a+b}{2} + \frac{1-t_2}{2} \cdot (a+b-x) \leq$$

$$\leq \frac{1+t_2}{2} \cdot \frac{a+b}{2} + \frac{1-t_2}{2} \cdot x \leq \frac{1+t_1}{2} \cdot \frac{a+b}{2} + \frac{1-t_1}{2} \cdot x \leq b$$

for all $x \in \left[\frac{a+b}{2}, b\right]$.
Considering

$$s_3 := \frac{1+t_1}{2} \cdot \frac{a+b}{2} + \frac{1-t_1}{2} \cdot x,$$

$$t_3 := \frac{1+t_2}{2} \cdot \frac{a+b}{2} + \frac{1-t_2}{2} \cdot x,$$

$$u_3 := \frac{1+t_2}{2} \cdot \frac{a+b}{2} + \frac{1-t_2}{2} \cdot (a+b-x),$$

$$v_3 := \frac{1+t_1}{2} \cdot \frac{a+b}{2} + \frac{1-t_1}{2} \cdot (a+b-x),$$

for all $x \in \left[a, \frac{a+b}{2}\right]$, we note that $s_3, t_3, u_3, v_3 \in \left[a, \frac{a+b}{2}\right]$ with $s_3 \leq t_3 \leq u_3 \leq v_3$ and $t_3 + u_3 = s_3 + v_3$. Applying Theorem 1.4, we find:

$$f(t_3) + f(u_3) \leq f(s_3) + f(v_3) \quad \text{for all } x \in \left[a, \frac{a+b}{2}\right]. \tag{2.3}$$

Denoting

$$s_4 := \frac{1+t_1}{2} \cdot \frac{a+b}{2} + \frac{1-t_1}{2} \cdot (a+b-x),$$

$$t_4 := \frac{1+t_2}{2} \cdot \frac{a+b}{2} + \frac{1-t_2}{2} \cdot (a+b-x),$$

$$u_4 := \frac{1+t_2}{2} \cdot \frac{a+b}{2} + \frac{1-t_2}{2} \cdot x,$$

$$v_4 := \frac{1+t_1}{2} \cdot \frac{a+b}{2} + \frac{1-t_1}{2} \cdot x,$$

for all $x \in \left[\frac{a+b}{2}, b\right]$, we note that $s_4, t_4, u_4, v_4 \in \left[\frac{a+b}{2}, b\right]$ with $s_4 \leq t_4 \leq u_4 \leq v_4$ and $t_4 + u_4 = s_4 + v_4$. Since $f \in W([a,b])$, taking into account Theorem 1.4, we obtain:

$$f(t_4) + f(u_4) \leq f(s_4) + f(v_4) \quad \text{for all } x \in \left[\frac{a+b}{2}, b\right]. \tag{2.4}$$

Integrating the inequality (2.3) over x on $\left[a, \frac{a+b}{2}\right]$, the inequality (2.4) over x on $\left[\frac{a+b}{2}, b\right]$, adding the obtained inequalities and multiplying the result by $\frac{1}{b-a}$, we deduce

$$2R(t_2) \leq 2R(t_1),$$

namely R is nonincreasing on the interval $[0,1]$.
Now, we note that

$$x \leq \frac{a+b}{4} + \frac{x}{2} \leq \frac{a+b}{4} + \frac{x}{2} \leq \frac{a+b}{2} \quad \text{for all } x \in \left[a, \frac{a+b}{2}\right] \tag{2.5}$$

and

$$\frac{a+b}{2} \leq \frac{a+b}{4} + \frac{x}{2} \leq \frac{a+b}{4} + \frac{x}{2} \leq x \quad \text{for all } x \in \left[\frac{a+b}{2}, b\right]. \tag{2.6}$$

Since $f \in W([a,b])$, taking into account the Theorem 1.4, we have, from (2.5) and (2.6):

$$2 \cdot f\left(\frac{a+b}{4} + \frac{x}{2}\right) \leq f\left(\frac{a+b}{2}\right) + f(x), \quad \text{for all } x \in [a,b]. \tag{2.7}$$

Multiplying the inequality (2.7) by $\frac{1}{2(b-a)}$ and integrating the obtained result over x on $[a,b]$, we deduce:

$$\frac{1}{b-a} \cdot \int_a^b f\left(\frac{a+b}{4} + \frac{x}{2}\right) \mathrm{d}x \leq \frac{1}{2} \cdot f\left(\frac{a+b}{2}\right) + \frac{1}{2(b-a)} \int_a^b f(x) \mathrm{d}x. \tag{2.8}$$

The monotonicity of R on $[0,1]$, the inequality (2.8) and the first inequality of (1.1) for Wright-convex functions, imply the inequalities (1.7) for Wright-convex functions.

Remark 2.1. *The inequalities (1.7) refine the first inequality of (1.1) for Wright-convex functions.*

Theorem 2.2. Let $f \in W([a,b]) \cap L_1[a,b]$ and let S be defined as in Theorem 1.10. Then $S \in W([0,1])$ is nondecreasing on $[0,1]$, and the inequalities (1.8) hold for all $t \in [0,1]$.

Proof. If $s,t,u,v \in [0,1]$ with $s \leq t \leq u \leq v$ and $t+u = s+v$, then, for all $x \in [a,b]$, we have

$$a \leq \frac{a+b}{2} - v \cdot \frac{b-x}{2} \leq \frac{a+b}{2} - u \cdot \frac{b-x}{2} \leq$$

$$\leq \frac{a+b}{2} - t \cdot \frac{b-x}{2} \leq \frac{a+b}{2} - s \cdot \frac{b-x}{2} \leq \frac{a+b}{2} \tag{2.9}$$

and

$$\frac{a+b}{2} \leq \frac{a+b}{2} + s \cdot \frac{x-a}{2} \leq \frac{a+b}{2} + t \cdot \frac{x-a}{2} \leq$$

$$\leq \frac{a+b}{2} + u \cdot \frac{x-a}{2} \leq \frac{a+b}{2} + v \cdot \frac{x-a}{2} \leq b. \tag{2.10}$$

Considering

$$s_5 := \frac{a+b}{2} - v \cdot \frac{b-x}{2},\ t_5 := \frac{a+b}{2} - u \cdot \frac{b-x}{2},\ u_5 := \frac{a+b}{2} - t \cdot \frac{b-x}{2},$$

$$v_5 := \frac{a+b}{2} - s \cdot \frac{b-x}{2}$$

in (2.9), we note that $s_5, t_5, u_5, v_5 \in \left[a, \frac{a+b}{2}\right]$, with $s_5 \leq t_5 \leq u_5 \leq v_5$ and $t_5 + u_5 = s_5 + v_5$. Since $f \in W([a,b])$, taking into account the Theorem 1.4, we find

$$f(t_5) + f(u_5) \leq f(s_5) + f(v_5) \quad \text{for all } x \in [a,b]. \tag{2.11}$$

Putting

$$s_6 := \frac{a+b}{2} + s \cdot \frac{x-a}{2},\ t_6 := \frac{a+b}{2} + t \cdot \frac{x-a}{2},\ u_6 := \frac{a+b}{2} + u \cdot \frac{x-a}{2},$$

$$v_6 := \frac{a+b}{2} + v \cdot \frac{x-a}{2}$$

in (2.10), we note that $s_6, t_6, u_6, v_6 \in \left[\frac{a+b}{2}, b\right]$, with $s_6 \leq t_6 \leq u_6 \leq v_6$ and $t_6 + u_6 = s_6 + v_6$. Since $f \in W([a,b])$, taking into account the Theorem 1.4, we have

$$f(t_6) + f(u_6) \leq f(s_6) + f(v_6) \quad \text{for all } x \in [a,b]. \tag{2.12}$$

Adding the inequalities (2.11) and (2.12), integrating over x on $[a,b]$ and multiplying by $\frac{1}{2(b-a)}$ the obtained result, we deduce

$$S(t) + S(u) \leq S(s) + S(v),$$

namely $S \in W([0,1])$.
In order to prove the monotonicity of S on the interval $[0,1]$, we take $0 \leq t_1 < t_2 \leq 1$. Then, for al $x \in [a,b]$, we have

$$a \leq \frac{a+b}{2} - t_2 \cdot \frac{b-x}{2} \leq \frac{a+b}{2} - t_1 \cdot \frac{b-x}{2} \leq \frac{a+b}{2} + t_1 \cdot \frac{b-x}{2} \leq$$

$$\leq \frac{a+b}{2} + t_2 \cdot \frac{b-x}{2} \leq b. \tag{2.13}$$

Considering

$$s_7 := \frac{a+b}{2} - t_2 \cdot \frac{b-x}{2},\ t_7 := \frac{a+b}{2} - t_1 \cdot \frac{b-x}{2},\ u_7 := \frac{a+b}{2} + t_1 \cdot \frac{b-x}{2},$$

$$v_7 := \frac{a+b}{2} + t_2 \cdot \frac{b-x}{2}$$

in (2.13), we note that $s_7, t_7, u_7, v_7 \in [a,b]$, with $s_7 \leq t_7 \leq u_7 \leq v_7$ and $t_7 + u_7 = s_7 + v_7$. Since $f \in W([a,b])$, taking into account the Theorem 1.4, we deduce

$$f(t_7) + f(u_7) \leq f(s_7) + f(v_7) \quad \text{for all } x \in [a,b]. \tag{2.14}$$

Integrating the last inequality over x on $[a,b]$, we obtain

$$\int_a^b f\left(\frac{a+b}{2} - t_1 \cdot \frac{b-x}{2}\right) \mathrm{d}x + \int_a^b f\left(\frac{a+b}{2} + t_1 \cdot \frac{b-x}{2}\right) \mathrm{d}x \leq$$

$$\leq \int_a^b f\left(\frac{a+b}{2} - t_2 \cdot \frac{b-x}{2}\right) \mathrm{d}x + \int_a^b f\left(\frac{a+b}{2} + t_2 \cdot \frac{b-x}{2}\right) \mathrm{d}x$$

or

$$\int_a^b f\left(\frac{a+b}{2} - t_1 \cdot \frac{b-x}{2}\right) \mathrm{d}x + \int_a^b f\left(\frac{a+b}{2} + t_1 \cdot \frac{x-a}{2}\right) \mathrm{d}x \leq$$

$$\leq \int_a^b f\left(\frac{a+b}{2} - t_2 \cdot \frac{b-x}{2}\right) \mathrm{d}x + \int_a^b f\left(\frac{a+b}{2} + t_2 \cdot \frac{x-a}{2}\right) \mathrm{d}x. \tag{2.15}$$

The inequality (2.15) is equivalent to $S(t_1) \leq S(t_2)$, namely S is nondecreasing on the interval $[0, 1]$.
The monotonicity of S implies the inequalities (1.8) for Wright-convex functions.

Remark 2.2. The inequalities (1.8) refine the first inequality of (1.1) for Wright-convex functions.

Theorem 2.3. Let $f \in W([a, b]) \cap L_1[a, b]$ and let T be defined as in Theorem 1.11. Then $T \in W([0, 1])$ is nonincreasing on $[0, 1]$ and the inequalities (1.9) hold for all $t \in [0, 1]$.
Proof. If $s, t, u, v \in [0, 1]$ with $s \leq t \leq u \leq v$ and $t + u = s + v$, then the inequalities (2.1) and (2.2) hold. Multiplying those inequalities by $p(x)$, integrating the obtained results: the first one over x on $\left[a, \frac{a+b}{2}\right]$ and the second one over x on $\left[\frac{a+b}{2}, b\right]$, multiplying by $\frac{1}{2}$ and adding the found inequalities, we deduce

$$T(t) + T(u) \leq T(s) + T(v),$$

namely $T \in W([0, 1])$.
In order to prove the monotonicity of T, we take $0 \leq t_1 < t_2 \leq 1$. Then, the inequalities (2.3) and (2.4) hold. Multiplying those relations by $p(x)$ and integrating the obtained results, we may write:

$$\int_a^{\frac{a+b}{2}} f\left(\frac{1+t_2}{2} \cdot \frac{a+b}{2} + \frac{1-t_2}{2} \cdot x\right) p(x)\mathrm{d}x +$$

$$+ \int_a^{\frac{a+b}{2}} f\left(\frac{1+t_2}{2} \cdot \frac{a+b}{2} + \frac{1-t_2}{2} \cdot (a+b-x)\right) p(a+b-x)\mathrm{d}x \leq$$

$$\leq \int_a^{\frac{a+b}{2}} f\left(\frac{1+t_1}{2} \cdot \frac{a+b}{2} + \frac{1-t_1}{2} \cdot x\right) p(x)\mathrm{d}x +$$

$$+ \int_a^{\frac{a+b}{2}} f\left(\frac{1+t_1}{2} \cdot \frac{a+b}{2} + \frac{1-t_1}{2} \cdot (a+b-x)\right) p(a+b-x)\mathrm{d}x \tag{2.16}$$

and

$$\int_{\frac{a+b}{2}}^{b} f\left(\frac{1+t_2}{2} \cdot \frac{a+b}{2} + \frac{1-t_2}{2} \cdot x\right) p(x)\mathrm{d}x +$$

$$+\int_{\frac{a+b}{2}}^{b} f\left(\frac{1+t_2}{2}\cdot\frac{a+b}{2}+\frac{1-t_2}{2}\cdot(a+b-x)\right)p(a+b-x)\mathrm{d}x \le$$

$$\le \int_{\frac{a+b}{2}}^{b} f\left(\frac{1+t_1}{2}\cdot\frac{a+b}{2}+\frac{1-t_1}{2}\cdot x\right)p(x)\mathrm{d}x+$$

$$+\int_{\frac{a+b}{2}}^{b} f\left(\frac{1+t_1}{2}\cdot\frac{a+b}{2}+\frac{1-t_1}{2}\cdot(a+b-x)\right)p(a+b-x)\mathrm{d}x. \qquad (2.17)$$

Adding the inequalities (2.16) and (2.17), we find $2\cdot T(t_2)\le 2\cdot T(t_1)$, namely T is nonincreasing on the interval $[0,1]$.
Since $f\in W([a,b])$, taking into account the inequality (2.7), we deduce

$$f\left(\frac{a+b}{4}+\frac{x}{2}\right)\cdot p(x)\le\frac{1}{2}\cdot f\left(\frac{a+b}{2}\right)\cdot p(x)+\frac{1}{2}\cdot f(x)\cdot p(x)$$

$$\text{for all } x\in[a,b]. \qquad (2.18)$$

Integrating the inequality (2.18) over x on $[a,b]$, we find

$$\int_a^b f\left(\frac{a+b}{4}+\frac{x}{2}\right)\cdot p(x)\mathrm{d}x\le\frac{1}{2}\cdot f\left(\frac{a+b}{2}\right)\int_a^b p(x)\mathrm{d}x+$$

$$+\frac{1}{2}\cdot\int_a^b f(x)\cdot p(x)\mathrm{d}x. \qquad (2.19)$$

The monotonicity of T on $[0,1]$, the inequality (2.19) and the first inequality of (1.2) for Wright-convex functions imply the inequalities (1.9) for Wright-convex functions.

Remark 2.3. *If we set* $p(x)\equiv 1(x\in[a,b])$ *in Theorem 2.3, then we find Theorem 2.1.*

Theorem 2.4, Let $f\in W([a,b])\cap L_1[a,b]$ and let U be defined as in Theorem 1.11. Then $U\in W([0,1])$ is nondecreasing on $[0,1]$, and the inequalities (1.10) hold for all $t\in[0,1]$.
Proof. If $0\le s\le t\le u\le v\le 1$ and $t+u=s+v$, then, for all $x\in[a,b]$, the inequalities (2.11) and (2.12) hold.

Multiplying (2.11) by $p\left(\frac{x+a}{2}\right)$ and integrating the obtained result over x on $[a,b]$, we have

$$\int_a^b f\left(\frac{a+b}{2}-u\cdot\frac{b-x}{2}\right)\cdot p\left(\frac{x+a}{2}\right)\mathrm{d}x+\int_a^b f\left(\frac{a+b}{2}-t\cdot\frac{b-x}{2}\right)\cdot$$

$$\cdot p\left(\frac{x+a}{2}\right)\mathrm{d}x\le\int_a^b f\left(\frac{a+b}{2}-v\cdot\frac{b-x}{2}\right)\cdot p\left(\frac{x+a}{2}\right)\mathrm{d}x+$$

$$+\int_a^b f\left(\frac{a+b}{2}-s\cdot\frac{b-x}{2}\right)\cdot p\left(\frac{x+a}{2}\right)\mathrm{d}x. \tag{2.20}$$

Multiplying (2.12) by $p\left(\frac{x+b}{2}\right)$ and integrating the obtained result over x on $[a,b]$, we have

$$\int_a^b f\left(\frac{a+b}{2}+t\cdot\frac{x-a}{2}\right)\cdot p\left(\frac{x+b}{2}\right)\mathrm{d}x+\int_a^b f\left(\frac{a+b}{2}+u\cdot\frac{x-a}{2}\right)\cdot p\left(\frac{x+b}{2}\right)\mathrm{d}x\le$$

$$\le\int_a^b f\left(\frac{a+b}{2}+s\cdot\frac{x-a}{2}\right)\cdot p\left(\frac{x+b}{2}\right)\mathrm{d}x+$$

$$+\int_a^b f\left(\frac{a+b}{2}+v\cdot\frac{x-a}{2}\right)\cdot p\left(\frac{x+b}{2}\right)\mathrm{d}x. \tag{2.21}$$

Adding the inequalities (2.20) and (2.21) and multiplying the result by $\frac{1}{2}$, we find $U(t)+U(u)\le U(s)+U(v)$, namely $U\in W([0,1])$.
Next, we take $0\le t_1<t_2\le 1$. Then, the inequality (2.14) holds.
Multiplying the inequality (2.14) by $p\left(\frac{x+a}{2}\right)$ and integrating the obtained result over x on $[a,b]$, we have

$$\int_a^b f\left(\frac{a+b}{2}-t_1\cdot\frac{b-x}{2}\right)\cdot p\left(\frac{x+a}{2}\right)\mathrm{d}x+\int_a^b f\left(\frac{a+b}{2}+t_1\cdot\frac{b-x}{2}\right)\cdot$$

$$\cdot p\left(\frac{x+a}{2}\right)\mathrm{d}x \le \int_a^b f\left(\frac{a+b}{2} - t_2\cdot\frac{b-x}{2}\right)\cdot p\left(\frac{x+a}{2}\right)\mathrm{d}x +$$

$$+\int_a^b f\left(\frac{a+b}{2} + t_2\cdot\frac{b-x}{2}\right)\cdot p\left(\frac{x+a}{2}\right)\mathrm{d}x$$

or

$$\int_a^b f\left(\frac{a+b}{2} - t_1\cdot\frac{b-x}{2}\right)\cdot p\left(\frac{x+a}{2}\right)\mathrm{d}x + \int_a^b f\left(\frac{a+b}{2} + t_1\cdot\frac{x-a}{2}\right)\cdot$$

$$\cdot p\left(\frac{2a+b-x}{2}\right)\mathrm{d}x \le \int_a^b f\left(\frac{a+b}{2} - t_2\cdot\frac{b-x}{2}\right)\cdot p\left(\frac{x+a}{2}\right)\mathrm{d}x +$$

$$+\int_a^b f\left(\frac{a+b}{2} + t_2\cdot\frac{x-a}{2}\right)\cdot p\left(\frac{2a+b-x}{2}\right)\mathrm{d}x.$$

Using the symmetry of p about $x=\frac{a+b}{2}$ in the last inequality, we deduce

$$\int_a^b f\left(\frac{a+b}{2} - t_1\cdot\frac{b-x}{2}\right)\cdot p\left(\frac{x+a}{2}\right)\mathrm{d}x + \int_a^b f\left(\frac{a+b}{2} + t_1\cdot\frac{x-a}{2}\right)\cdot$$

$$\cdot p\left(\frac{x+b}{2}\right)\mathrm{d}x \le \int_a^b f\left(\frac{a+b}{2} - t_2\cdot\frac{b-x}{2}\right)\cdot p\left(\frac{x+a}{2}\right)\mathrm{d}x +$$

$$+\int_a^b f\left(\frac{a+b}{2} + t_2\cdot\frac{x-a}{2}\right)\cdot p\left(\frac{x+b}{2}\right)\mathrm{d}x$$

which, multiplied by $\frac{1}{2}$, gives $U(t_1)\le U(t_2)$, namely U is nondecreasing on the interval $[0,1]$.
From the monotonicity of U, we deduce the inequalities (1.10) for Wright-convex functions.
Remark 2.4. *The Theorem 2.4 is a weighted generalization of Theorem 2.2.*

REFERENCES

[1] M. Akkouchi, *A result on the mapping H of S.S. Dragomir with Applications*, Facta Universitatis (Niš), Ser. Math. Inform. 17 (2002), 5–12.

[2] M. Akkouchi, *On the mapping H of S.S. Dragomir*, Facta Universitatis (Niš), Ser. Math. Inform. 20 (2005), 21–31.

[3] V. Ciobotariu-Boer, *Refinements of some Hermite-Hadamard and Fejér inequalitites for convex functions*, Octogon Mathematical Magazine, 16(1), 2008, 147–156.

[4] P. Czinder and Z. Páles, *An extension of the Hermite-Hadamard inequality and an application for Gini and Stolarsky means*, J.I.P.A.M., 5(2) (2004), Art. 2, 8pp.

[5] S.S. Dragomir, *Two mappings in connection to Hadamard's inequalities*, J. Math. Anal. Appl., 167 (1992), 49–56.

[6] S.S. Dragomir, *New refinements of the Hermite-Hadamard integral inequality for convex functions and applications*, Soochow Journal of Mathematics, 28(4) (2002), 357–374.

[7] S.S. Dragomir, Y.J. Cho and S.S. Kim, *Inequalities of Hadamard's type for Lipschitzian mappings and their applications*, J. Math. Anal. Appl., 245 (2000), 489–501.

[8] S.S. Dragomir and A. Mcandrew, *Refinements of the Hermite-Hadamard inequality for convex functions*, J.I.P.A.M., 6(5) (2005), Art. 140, 6 pp.

[9] L. Fejér, *Uber die Fourierreihen*, II, Math. Naturwiss. Anz. Ungar. Akad. Wiss., 24 (1906), 369–390 (Hungarian).

[10] J. Hadamard, *Étude sur les propriétés des fonctions entiéres en particulier d'une fonction considérée par Riemann*, J. Math. Pures Appl., 58 (1893), 171–215.

[11] G.H. Hardy, J.E. Littlewood and G. Pólya, *Inequalities*, 1st ed. and 2nd ed., Cambridge University Press, Cambridge, England (1934, 1952).

[12] Minh-In-Ho, *Fejér inequalities for Wright-convex functions*, J.I.P.A.M., 8(1) (2007), Art. 9, 9 pp.

[13] S. Hussain and M. Anwar, *On certain inequalities improving the Hermite-Hadamard inequality*, J.I.P.A.M., 8(2) (2007), Art. 60, 5 pp.

[14] H. Kenyon, *Note on convex functions*, Amer. Math. Monthly, 63 (1956), 107.

[15] V.L. Klee, *Solution of a problem of E.M. Wright on convex functions*, Amer. Math. Monthly, 63 (1956), 106–107.

[16] M. Matić and J. Pečarić, *On inequalitites of Hadamard's type for Lipschitzian mappings*, Tamkang J. Math., 32(2) (2001), 127–130.

[17] D.S. Mitrinović and I.B. Lacković, *Hermite and convexity*, Aequationes Math., 25 (1985), 229–232.
[18] K.L. Tseng, G.S. Yang and S.S. Dragomir, *Hadamard inequalitites for Wright-convex functions*, Demonstratio Math.., 37(3) (2004), 525–532.
[19] G.S. Yang and M.C. Hong, *A note o Hadamard's inequality*, Tamkang J. Math., 28(1) (1997), 33–37.
[20] G.S. Yang and K.L. Tseng, *On certain integral inequalities related to Hermite-Hadamard inequalities*, J. Math. Anal. Appl., 239 (1999), 180–187.
[21] G.S. Yang and K.L. Tseng, *Inequalitites of Hadamard's Type for Lipschitzian Mappings*, J. Math. Anal. Appl., 260 (2001), 230–238.
[22] L.-C. Wang, *Some refinements of Hermite-Hadamard inequalitites for convex functions*, Univ. Beograd Publ. Elek. Fak., Ser. Mat., 15 (2004), 39–44.

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New inequalities for the triangle

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ABSTRACT. In this paper we will prove some new inequalities for the triangle. Among these, we will improve Euler's Inequality, Mitrinović's Inequality and Weitzenböck's Inequality, thus:

$$R \geq \frac{4}{\sum\limits_{cyclic} \sqrt{F_\lambda\left(\frac{1}{h_a}, \frac{1}{h_b}\right)(n)}} \geq 2r;\ \ s \geq \frac{1}{2}\sum_{cyclic} \sqrt{F_\lambda\left(s-a, s-b\right)(n)} \geq 3\sqrt{3}r;$$

and

$$a^{2\alpha} + b^{2\alpha} + c^{2\alpha} \geq \frac{1}{2}\sum_{cyclic} \sqrt{F_\lambda\left(a^{2\alpha}, b^{2\alpha}\right)(n)} \geq 3\left(\frac{4\Delta}{\sqrt{3}}\right)^{\alpha},$$

where $F_\lambda(x,y)(n) = \left[\left(1+(1-2\lambda)^n\right)x + \left(1-(1-2\lambda)^n\right)y\right] \cdot$
$\cdot \left[\left(1-(1-2\lambda)^n\right) + \left(1+(1-2\lambda)^n\right)y\right]$, with $\lambda \in [0,1]$, for any $x, y \geq 0$ and for all integers $n \geq 0$.

1. INTRODUCTION

Among well known the geometric inequalities, we recall the famous inequality of Euler, $R \geq 2r$, the inquality of Mitrinović, $s \geq 3\sqrt{3}r$, and in the year 1919 Weitzenböck published in Mathematische Zeitschrift the following inequality,

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}\Delta.$$

This inequality later, in 1961, was given at the International Mathematical Olympiad. In 1927, this inequality appeared as the generalization

$$\Delta \leq \frac{\sqrt{3}}{4}\left(\frac{a^k + b^k + c^k}{3}\right)^{\frac{2}{k}},$$

in one of the issues of the *American Mathematical Monthly.* For $k = 2$, we obtain the Weitzenböck Inequality.

In this paper we will prove several improvements for these inqualities.

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2. MAIN RESULTS

In the following, we will use the notations: $a, b, c-$ the lengths of the sides, h_a, h_b, h_c- the lengths of the altitudes, r_a, r_b, r_c- the radii of the excircles, s is the semi-perimeter; R is the circumradius, $r-$ the inradius, and $\Delta-$ the area of the triangle ABC.

Lemma 2.1 If $x, y \geq 0$ and $\lambda \in [0,1]$, then the inequality

$$\left(\frac{x+y}{2}\right)^2 \geq [(1-\lambda)\,x+\lambda y]\cdot[\lambda x+(1-\lambda)\,y] \geq xy \tag{2.1}$$

holds.
Proof. The inequality

$$\left(\frac{x+y}{2}\right)^2 \geq [(1-\lambda)\,x+\lambda y]\cdot[\lambda x+(1-\lambda)\,y]$$

is equivalent to

$$(1-2\lambda)^2\,x^2-2\,(1-2\lambda)^2\,xy+(1-2\lambda)^2\,y^2 \geq 0,$$

which means that

$$(1-2\lambda)^2\,(x-y)^2 \geq 0,$$

which is true. The equality holds if and only if $\lambda=\frac{1}{2}$ or $x=y$.

The inequality

$$[(1-\lambda)\,x+\lambda y]\cdot[\lambda x+(1-\lambda)\,y] \geq xy$$

becomes

$$\lambda\,(1-\lambda)\,x^2-2\lambda\,(1-\lambda)\,xy+\lambda\,(1-\lambda)\,y^2 \geq 0$$

Therefore, we obtain

$$\lambda\,(1-\lambda)\,(x-y)^2 \geq 0,$$

which is true, because $\lambda \in [0,1]$. The equality holds if and only if $\lambda \in \{0,1\}$ or $x=y$.
We consider the expression

$$F_\lambda\,(x,y)\,(n)=[(1+(1-2\lambda)^n)\,x+(1-(1-2\lambda)^n)\,y]\cdot$$

$$\cdot\left[\left(1-(1-2\lambda)^{n}\right)x+\left(1+(1-2\lambda)^{n}\right)y\right],$$

with $\lambda \in [0,1]$, for any $x, y \geq 0$, and for all integers $n \geq 0$.

Theorem 2.2 There are the following relations:

$$F_{\lambda}\left((1-\lambda)x+\lambda y, \lambda x+(1-\lambda)y\right)(n)=F_{\lambda}(x,y)(n+1); \tag{2.2}$$

$$F_{\lambda}(x,y)(n+1) \geq F_{\lambda}(x,y)(n) \tag{2.3}$$

and

$$(x+y)^{2} \geq F_{\lambda}(x,y)(n) \geq 4xy, \tag{2.4}$$

for any $\lambda \in [0,1]$,for any $x, y \geq 0$ and all integers $n \geq 0$.
Proof. We make the following calculation:

$$F_{\lambda}\left((1-\lambda)x+\lambda y, \lambda x+(1-\lambda)y\right)(n)=$$

$$=\left[\left(1+(1-2\lambda)^{n}\right)\left((1-\lambda)x+\lambda y\right)+\left(1-(1-2\lambda)^{n}\right)\left(\lambda x+(1-\lambda)y\right)\right]\cdot$$

$$\cdot\left[\left(1-(1-2\lambda)^{n}\right)\left((1-\lambda)x+\lambda y\right)+\left(1+(1-2\lambda)^{n}\right)\left(\lambda x+(1-\lambda)y\right)\right]=$$

$$=\{\left[1-\lambda+(1-\lambda)(1-2\lambda)^{n}+\lambda-\lambda(1-2\lambda)^{n}\right]x+$$

$$+\left[\lambda+\lambda(1-2\lambda)^{n}+1-\lambda-(1-\lambda)(1-2\lambda)^{n}\right]y\}$$

$$\cdot\{\left[1-\lambda-(1-\lambda)(1-2\lambda)^{n}+\lambda+\lambda(1-2\lambda)^{n}\right]x+$$

$$+\left[\lambda-(1-2\lambda)^{n}+1-\lambda+(1-\lambda)(1-2\lambda)^{n}\right]y\}=$$

$$=\left[\left(1+(1-2\lambda)^{n+1}\right)x+\left(1-(1-2\lambda)^{n+1}\right)y\right]$$

$$\left[\left(1-(1-2\lambda)^{n+1}\right)x+\left(1+(1-2\lambda)^{n+1}\right)y\right]=$$

$$= F_{\lambda}(x, y)(n+1),$$

so $F_{\lambda}((1-\lambda)x + \lambda y, \lambda x + (1-\lambda)y)(n) = F_{\lambda}(x, y)(n+1)$.
We use the induction on n. For $n = 0$, we obtain the inequality

$$F_{\lambda}(x, y)(1) \geq F_{\lambda}(x, y)(0).$$

Therefore, we deduce the following inequality:

$$4[(1-\lambda)x + \lambda y] \cdot [\lambda x + (1-\lambda)y] \geq 4xy,$$

which is true, from Lemma 2.1.
We assume it is true for every integer $\leq n$, so

$$F_{\lambda}(x, y)(n+1) \geq F_{\lambda}(x, y)(n)$$

We will prove that

$$F_{\lambda}(x, y)(n+2) \geq F_{\lambda}(x, y)(n+1). \tag{2.5}$$

Using the substitutions $x \to (1-\lambda)x + \lambda y$ and $y \to \lambda x + (1-\lambda)y$ in the inequality (2.3), we deduce

$$F_{\lambda}((1-\lambda)x + \lambda y, \lambda x + (1-\lambda)y)(n+1) \geq$$

$$\geq F_{\lambda}((1-\lambda)x + \lambda y, \lambda x + (1-\lambda)y)(n),$$

so, from equality (2.2), we have

$$F_{\lambda}(x, y)(n+2) \geq F_{\lambda}(x, y)(n+1).$$

so we obtain (2.6).
According to inequality (2.3), we can write the sequence of inequalities

$$F_{\lambda}(x, y)(n) \geq F_{\lambda}(x, y)(n-1) \geq ... \geq F_{\lambda}(x, y)(1) \geq F_{\lambda}(x, y)(0) = 4xy.$$

Therefore, we have

$$F_{\lambda}(x, y)(n) \geq 4xy, \text{ for any } \lambda \in [0, 1], x, y \geq 0 \text{ and for all integers } n \geq 0.$$

If $\lambda \in (0,1)$, then $1-2\lambda \in (-1,1)$ and passing to limit when $n \to \infty$, we obtain

$$\lim_{n\to\infty} F_\lambda(x,y)(n) = (x+y)^2 .$$

Since the sequence $(F_\lambda(x,y)(n))_{n\geq 0}$ is increasing, we deduce

$$(x+y)^2 \geq F_\lambda(x,y)(n), \text{ for any } \lambda \in (0,1), x,y \geq 0 \text{ and for all integers } n \geq 0.$$

From the inequalities above we have that

$$(x+y)^2 \geq F_\lambda(x,y)(n) \geq 4xy, \text{ for any } \lambda \in (0,1), x,y \geq 0 \text{ and for all integers } n \geq 0.$$

If $\lambda = 0$ and $\lambda = 1$, then $F_\lambda(x,y)(n) = 4xy$, so

$$(x+y)^2 \geq F_\lambda(x,y)(n) \geq 4xy.$$

It follows that

$$(x+y)^2 \geq F_\lambda(x,y)(n) \geq 4xy, \text{ for any } \lambda \in [0,1], x,y \geq 0 \text{ and for all integers } n \geq 0.$$

Thus, the proof of Theorem 2.2 is complete.

Remark 1. *It is easy to see that there is the sequence of inequalities*

$$(x+y)^2 \geq ... \geq F_\lambda(x,y)(n) \geq F_\lambda(x,y)(n-1) \geq ...$$

$$... \geq F_\lambda(x,y)(1) \geq F_\lambda(x,y)(0) = 4xy. \tag{2.6}$$

Corollary 2.3. There are the following inequalities:

$$x+y \geq \sqrt{F_\lambda(x,y)(n)} \geq 2\sqrt{xy}; \tag{2.7}$$

$$x^2+y^2 \geq \sqrt{F_\lambda(x^2,y^2)(n)} \geq 2xy; \tag{2.8}$$

$$x+y+z \geq \frac{1}{2}\sum_{cyclic} \sqrt{F_\lambda(x,y)(n)} \geq \sqrt{xy}+\sqrt{yz}+\sqrt{zx}; \tag{2.9}$$

$$x^2+y^2+z^2 \geq \frac{1}{2}\sum_{cyclic}\sqrt{F_\lambda\left(x^2,y^2\right)(n)} \geq xy+yz+zx; \tag{2.10}$$

$$x^2+y^2+z^2+xy+yz+zx \geq \frac{1}{2}\sum_{cyclic}F_\lambda\left(x,y\right)(n) \geq 2\left(xy+yz+zx\right) \tag{2.11}$$

and

$$\left(x+y\right)\left(y+z\right)\left(z+x\right) \geq \sqrt{\prod_{cyclic}F_\lambda\left(x,y\right)(n)} \geq 8xyz, \tag{2.12}$$

for any $\lambda \in [0,1]$, for any $x, y \geq 0$, and for all integers $n \geq 0$.

Proof. From Theorem 2.2, we easily deduce inequality (2.7). Using the substitutions $x \to x^2$ and $y \to y^2$ in inequality (2.7), we obtain inequality (2.8). Similarly to inequality (2.7), $x+y \geq \sqrt{F_\lambda\left(x,y\right)(n)} \geq 2\sqrt{xy}$, we can write the following inequalities:

$$y+z \geq \sqrt{F_\lambda\left(y,z\right)(n)} \geq 2\sqrt{yz} \text{ and } z+x \geq \sqrt{F_\lambda\left(z,x\right)(n)} \geq 2\sqrt{zx},$$

which means, by adding, that

$$x+y+z \geq \frac{1}{2}\sum_{cyclic}\sqrt{F_\lambda\left(x,y\right)(n)} \geq \sqrt{xy}+\sqrt{yz}+\sqrt{zx}.$$

It is easy to see that, by making the substitutions $x \to x^2$ and $y \to y^2$ in inequality (2.9), we obtain inequality (2.10). Similar to inequality (2.4), $\left(x+y\right)^2 \geq F_\lambda\left(x,y\right)(n) \geq 4xy$, we obtain the following inequalities:

$$\left(y+z\right)^2 \geq F_\lambda\left(y,z\right)(n) \geq 4yz \text{ and } \left(z+x\right)^2 \geq F_\lambda\left(z,x\right)(n) \geq 4zx.$$

By adding them, we have inequality (2.11) and by multiplying them, we obtain inequality (2.12).

Lemma 2.4 For any triangle ABC, the following inequality,

$$\sqrt{ab}+\sqrt{bc}+\sqrt{ca} \geq \frac{4\Delta}{R}, \tag{2.13}$$

holds.

Proof. We apply the arithmetic-geometric mean inequality and we find that

$$\sqrt{ab}+\sqrt{bc}+\sqrt{ca} \geq 3\sqrt[3]{abc}.$$

It is sufficient to show that

$$3\sqrt[3]{abc} \geq \frac{4\Delta}{R}. \tag{2.14}$$

Inequality (2.14) is equivalent to

$$27abc \geq \frac{64\Delta^3}{R^3},$$

so

$$27 \cdot 4R\Delta \geq \frac{64\Delta^3}{R^3},$$

which means that

$$27R^4 \geq 16\Delta^2. \tag{2.15}$$

Using Mitrinović's Inequality, $3\sqrt{3}R \geq 2s$, and Euler's Inequality $R \geq 2r$, we deduce, by multiplication, that

$$3\sqrt{3}R^2 \geq 4\Delta.$$

It follows (2.15).

Corollary 2.5. In any triangle ABC, there are the following inequalities:

$$R \geq \frac{4}{\sum\limits_{cyclic} \sqrt{F_\lambda\left(\frac{1}{h_a}, \frac{1}{h_b}\right)(n)}} \geq 2r; \tag{2.16}$$

$$s \geq \frac{1}{2}\sum_{cyclic} \sqrt{F_\lambda\left(s-a, s-b\right)(n)} \geq 3\sqrt{3}r \tag{2.17}$$

and

$$a^{2\alpha} + b^{2\alpha} + c^{2\alpha} \geq \frac{1}{2}\sum_{cyclic} \sqrt{F_\lambda\left(a^{2\alpha}, b^{2\alpha}\right)(n)} \geq 3\left(\frac{4\Delta}{\sqrt{3}}\right)^\alpha, \tag{2.18}$$

for any $\lambda \in [0,1], x, y \geq 0$, $n \geq 0$ and α is a real numbers.

Proof. Making the substitutions $x = \frac{1}{h_a}, y = \frac{1}{h_b}$ and $z = \frac{1}{h_c}$ in inequality (2.9), we obtain

$$\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c} \geq \frac{1}{2}\sum_{cyclic} \sqrt{F_\lambda\left(\frac{1}{h_a}, \frac{1}{h_b}\right)(n)} \geq \frac{1}{\sqrt{h_a h_b}}+\frac{1}{\sqrt{h_b h_c}}+\frac{1}{\sqrt{h_c h_a}}. \tag{2.19}$$

According to the equalities

$$h_a = \frac{2\Delta}{a}, h_b = \frac{2\Delta}{b} \text{ and } h_c = \frac{2\Delta}{c},$$

we have

$$\frac{1}{\sqrt{h_a h_b}} + \frac{1}{\sqrt{h_b h_c}} + \frac{1}{\sqrt{h_c h_a}} = \frac{1}{2\Delta}\left(\sqrt{ab} + \sqrt{bc} + \sqrt{ca}\right).$$

From Lemma 2.4, we deduce

$$\frac{1}{\sqrt{h_a h_b}} + \frac{1}{\sqrt{h_b h_c}} + \frac{1}{\sqrt{h_c h_a}} \geq \frac{2}{R}.$$

If we use the identity

$$\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} = \frac{1}{r}$$

and inequality from above then inequality (2.18) becomes

$$\frac{1}{r} \geq \frac{1}{2} \sum_{cyclic} \sqrt{F_\lambda\left(\frac{1}{h_a}, \frac{1}{h_b}\right)(n)} \geq \frac{2}{R}. \tag{2.20}$$

Consequently the inequalities (2.16) follows.
If in inequality (2.9) we take $x = s - a, y = s - b$ and $z = s - c$, then we deduce the inequality

$$s \geq \frac{1}{2} \sum_{cyclic} \sqrt{F_\lambda (s-a, s-b)(n)} \geq$$

$$\geq \sqrt{(s-a)(s-b)} + \sqrt{(s-b)(s-c)} + \sqrt{(s-c)(s-a)}. \tag{2.21}$$

But, we know the identity $\sum\limits_{cyclic} \sqrt{(s-a, s-b)} = \sum\limits_{cyclic} \sqrt{bc} \sin \frac{A}{2}$.
Using the arithmetic-geometric mean inequality, we obtain

$$\begin{aligned} \sum_{cyclic} \sqrt{bc} \sin \frac{A}{2} &\geq 3\sqrt[3]{abc \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}} = 3\sqrt[3]{4R\Delta \cdot \frac{r}{4R}} = \\ = 3\sqrt[3]{\Delta r} &= 3\sqrt[3]{sr^2} \geq 3\sqrt[3]{3\sqrt{3}r^3} = 3\sqrt{3}r. \end{aligned}$$

Hence,

$$\sqrt{(s-a)(s-b)}+\sqrt{(s-b)(s-c)}+\sqrt{(s-c)(s-a)}\geq 3\sqrt{3}r, \tag{2.22}$$

which means, according to inequalities (2.21) and (2.22), that

$$s\geq \frac{1}{2}\sum_{cyclic}\sqrt{F_{\lambda}(s-a,s-b)(n)}\geq 3\sqrt{3}r.$$

Making the substitutions $x=a^{\alpha}, y=b^{\alpha}$, and $z=c^{\alpha}$ in inequality (2.9), we obtain the following inequality:

$$a^{2\alpha}+b^{2\alpha}+c^{2\alpha}\geq \frac{1}{2}\sum_{cyclic}\sqrt{F_{\lambda}(a^{2\alpha},b^{2\alpha})(n)}\geq a^{\alpha}b^{\alpha}+b^{\alpha}c^{\alpha}+c^{\alpha}a^{\alpha}. \tag{2.23}$$

Applying the arithmetic- geometric mean inequality and Pólya-Szegő's Inequality, $\sqrt[3]{a^2b^2c^2}\geq \frac{4\Delta}{\sqrt{3}}$, we deduce

$$a^{\alpha}b^{\alpha}+b^{\alpha}c^{\alpha}+c^{\alpha}a^{\alpha}\geq \sqrt[3]{(a^2b^2c^2)^{\alpha}}=3\left(\sqrt[3]{a^2b^2c^2}\right)^{\alpha}\geq 3\left(\frac{4\Delta}{\sqrt{3}}\right)^{\alpha},$$

so

$$a^{\alpha}b^{\alpha}+b^{\alpha}c^{\alpha}+c^{\alpha}a^{\alpha}\geq 3\left(\frac{4\Delta}{\sqrt{3}}\right)^{\alpha}. \tag{2.24}$$

According to inequalities (2.23) and (2.24), we obtain the inequality

$$a^{2\alpha}+b^{2\alpha}+c^{2\alpha}\geq \frac{1}{2}\sum_{cyclic}\sqrt{F_{\lambda}(a^{2\alpha},b^{2\alpha})(n)}\geq 3\left(\frac{4\Delta}{\sqrt{3}}\right)^{\alpha}.$$

Thus, the statement is true.

Remark 2. *a) Inequality (2.16) implies the sequence of inequalities*

$$R\geq \frac{4}{\sum\limits_{cyclic}\sqrt{F_{\lambda}\left(\frac{1}{h_a},\frac{1}{h_b}\right)(0)}}\geq ...\geq \frac{4}{\sum\limits_{cyclic}\sqrt{F_{\lambda}\left(\frac{1}{h_a},\frac{1}{h_b}\right)(n-1)}}\geq$$

$$\geq \frac{4}{\sum\limits_{cyclic}\sqrt{F_{\lambda}\left(\frac{1}{h_a},\frac{1}{h_b}\right)(n)}}\geq ...\geq 2r \tag{2.25}$$

b) For $\alpha=1$ *in inequality (2.18), we obtain*

$$a^2+b^2+c^2 \geq \frac{1}{2}\sum_{cyclic}\sqrt{F_\lambda\left(a^2,b^2\right)(n)} \geq 4\sqrt{3}\Delta, \tag{2.26}$$

which proves Weitzenböck's Inequality, namely

$$a^2+b^2+c^2 \geq 4\sqrt{3}\Delta.$$

Corollary 2.6. For any triangle ABC, there are the following inequalities:

$$2\left(s^2-r^2-4Rr\right) \geq \frac{1}{2}\sum_{cyclic}\sqrt{F_\lambda\left(a^2,b^2\right)(n)} \geq s^2+r^2+4Rr, \tag{2.27}$$

$$s^2-2r^2-8Rr \geq \frac{1}{2}\sum_{cyclic}\sqrt{F_\lambda\left((s-a)^2,(s-b)^2\right)(n)} \geq r\left(4R+r\right), \tag{2.28}$$

$$\frac{\left(s^2+r^2+4Rr\right)^2-8s^2Rr}{4R^2} \geq \frac{1}{2}\sum_{cyclic}\sqrt{F_\lambda\left(h_a^2,h_b^2\right)(n)} \geq \frac{2s^2r}{R}, \tag{2.29}$$

$$(4R+r)^2-2s^2 \geq \frac{1}{2}\sum_{cyclic}\sqrt{F_\lambda\left(r_a^2,r_b^2\right)(n)} \geq s^2, \tag{2.30}$$

$$\frac{8R^2+r^2-s^2}{8R^2} \geq \frac{1}{2}\sum_{cyclic}\sqrt{F_\lambda\left(\sin^4\frac{A}{2},\sin^4\frac{B}{2}\right)(n)} \geq \frac{s^2+r^2-8Rr}{16R^2} \tag{2.31}$$

and

$$\frac{(4R+r)^2-s^2}{4R^2} \geq \frac{1}{2}\sum_{cyclic}\sqrt{F_\lambda\left(\cos^4\frac{A}{2},\cos^4\frac{B}{2}\right)(n)} \geq$$

$$\geq \frac{s^2+(4R+r)^2}{8R^2}. \tag{2.32}$$

Proof. According to Corollary 2.3 we have the inequality

$$x^2+y^2+z^2 \geq \frac{1}{2}\sum_{cyclic}\sqrt{F_\lambda\left(x^2,y^2\right)(n)} \geq xy+yz+zx.$$

Using the substitutions

$$\begin{aligned}(x,y,z) \quad &\in \quad \{(a,b,c),(s-a,s-b,s-c),(h_a,h_b,h_c),(r_a,r_b,r_c),\\ &\left(\sin^2\frac{A}{2},\sin^2\frac{B}{2},\sin^2\frac{C}{2}\right),\left(\cos^2\frac{A}{2},\cos^2\frac{B}{2},\cos^2\frac{C}{2}\right)\Bigg\},\end{aligned}$$

we deduce the inequalities required.

Corollary 2.7. In any triangle ABC, there are the following inequalities:

$$3s^2-r^2-4Rr \geq \frac{1}{2}\sum_{cyclic} F_\lambda(a,b)(n) \geq 2\left(s^2+r^2+4Rr\right), \tag{2.33}$$

$$s^2-r^2-4Rr \geq \frac{1}{2}\sum_{cyclic} F_\lambda(s-a,s-b)(n) \geq 2r(4R+r), \tag{2.34}$$

$$\frac{\left(s^2+r^2+4Rr\right)^2-8s^2Rr}{4R^2} \geq \frac{1}{2}\sum_{cyclic} F_\lambda(h_a,h_b)(n) \geq \frac{4s^2r}{R} \tag{2.35}$$

and

$$(4R+r)^2-s^2 \geq \frac{1}{2}\sum_{cyclic} F_\lambda(r_a,r_b)(n) \geq 2s^2 \tag{2.36}$$

Proof. According to Corollary 2.3, we have the inequality

$$x^2+y^2+z^2+xy+yz+zx \geq \frac{1}{2}\sum_{cyclic} F_\lambda(x,y)(n) \geq 2(xy+yz+zx).$$

Using the substitutions

$$(x,y,z) \in \{(a,b,c),(s-a,s-b,s-c),(h_a,h_b,h_c),(r_a,r_b,r_c)\}$$

we deduce the inequalities from the statement.

Corollary 2.8. For any triangle ABC there are the following inequalities:

$$2s\left(s^2+r^2+2Rr\right) \geq \prod_{cuclic} \sqrt{F_\lambda(a,b)(n)} \geq 32sRr, \tag{2.37}$$

$$4sRr \geq \prod_{cyclic} \sqrt{F_\lambda((s-a),(s-b))(n)} \geq 8sr^2, \tag{2.38}$$

$$\frac{s^2r\left(s^2+r^2+4Rr\right)}{R^2}\geq\prod_{cyclic}\sqrt{F_\lambda\left(h_a,h_b\right)(n)}\geq\frac{16s^2r^2}{R},\tag{2.39}$$

$$4s^2R\geq\prod_{cyclic}\sqrt{F_\lambda\left(r_a,r_b\right)(n)}\geq 8s^2r,\tag{2.40}$$

$$\frac{(2R-r)\left(s^2+r^2-8Rr\right)-2Rr^2}{32R^3}\geq\prod_{cyclic}\sqrt{F_\lambda\left(\sin^2\frac{A}{2},\sin^2\frac{B}{2}\right)(n)}\geq$$

$$\geq\frac{r^2}{2R^2}\tag{2.41}$$

and

$$\frac{(4R+r)^3+s^2(2R+r)}{32R^3}\geq\prod_{cyclic}\sqrt{F_\lambda\left(\cos^2\frac{A}{2},\cos^2\frac{B}{2}\right)(n)}\geq\frac{s^2}{2R^2}\tag{2.42}$$

Proof. According to Corollary 2.3, we have the inequality

$$(x+y)(y+z)(z+x)\geq\sqrt{\prod_{cyclic}F_\lambda(x,y)(n)}\geq 8xyz.$$

Using the substitutions

$$\begin{aligned}(x,y,z)\ \in\ &\{(a,b,c),(s-a,s-b,s-c),(h_a,h_b,h_c),(r_a,r_b,r_c),\\&\left(\sin^2\frac{A}{2},\sin^2\frac{B}{2},\sin^2\frac{C}{2}\right),\left(\cos^2\frac{A}{2},\cos^2\frac{B}{2},\cos^2\frac{C}{2}\right)\right\},\end{aligned}$$

we deduce the inequalities required.

Remark 3. *From Corollary 2.7, we obtain the inequality*

$$\begin{aligned}2s\left(s^2+r^2+2Rr\right)\ &\geq\ \prod_{cyclic}\sqrt{F_\lambda(a,b)(n)}\geq 32sRr\geq\\&\geq\ 8\prod_{cyclic}\sqrt{F_\lambda((s-a),(s-b))(n)}\geq 64sr^2.\end{aligned}$$

We consider the expression

$$G\left(x,y\right)\left(n\right)=\frac{xy\left(x^{n-1}+y^{n-1}\right)\left(x^{n+1}+y^{n+1}\right)}{\left(x^{n}+y^{n}\right)^{2}}, \tag{2.43}$$

where $x,y>0$ *and for all integers* $n\geq 0$*.*

Theorem 2.9. For any $x,y>0$ and for all integers $n\geq 0$, there are the following relations:

$$a)\ \left(\frac{x+y}{2}\right)^{2}\geq G\left(x,y\right)\left(n\right)\geq xy \tag{2.44}$$

and

$$b)\ G\left(x,y\right)\left(n+1\right)\leq G\left(x,y\right)\left(n\right). \tag{2.45}$$

Proof. We take $\lambda=\frac{x^{n}}{x^{n}+y^{n}}$,for all integers $n\geq 0$, in inequality (2.1), because $\lambda\in(0,1)$, and we deduce

$$\left(\frac{x+y}{2}\right)^{2}\geq\frac{xy\left(x^{n-1}+y^{n-1}\right)\left(x^{n+1}+y^{n+1}\right)}{\left(x^{n}+y^{n}\right)^{2}}\geq xy,$$

so,

$$\left(\frac{x+y}{2}\right)^{2}\geq G\left(x,y\right)\left(n\right)\geq xy.$$

To prove inequality (2.45), we can write

$$\frac{G\left(x,y\right)\left(n+1\right)}{G\left(x,y\right)\left(n\right)}-1=$$
$$=-\frac{\left(xy\right)^{n-1}\left(x-y\right)^{2}\left(x^{2}+xy+y^{2}\right)\left(x^{2n}+x^{2n-1}y+x^{2n}y^{2}+...+y^{2n}\right)}{\left(x^{n+1}+y^{n+1}\right)^{3}\left(x^{n-1}+y^{n-1}\right)}\leq 0.$$

Consequently, we have

$$G\left(x,y\right)\left(n+1\right)\leq G\left(x,y\right)\left(n\right).$$

Remark 4. *It is easy to see that there is the sequence of inequalities*

$$\frac{\left(x+y\right)^{2}}{4}=G\left(x,y\right)\left(0\right)\geq G\left(x,y\right)\left(1\right)\geq ...$$

$$\geq G(x,y)(n-1) \geq G(x,y)(n) \geq ... \geq xy. \tag{2.46}$$

Corollary 2.10
There are the following inequalities:

$$\frac{x+y}{2} \geq \sqrt{G(x,y)(n)} \geq \sqrt{xy}; \tag{2.47}$$

$$\frac{x^2+y^2}{2} \geq \sqrt{G(x^2,y^2)(n)} \geq \sqrt{xy}; \tag{2.48}$$

$$x+y+z \geq \sum_{cyclic} \sqrt{G(x,y)(n)} \geq \sqrt{xy}+\sqrt{yz}+\sqrt{zx}; \tag{2.49}$$

$$x^2+y^2+z^2 \geq \sum_{cyclic} \sqrt{G(x^2,y^2)(n)} \geq xy+yz+zx; \tag{2.50}$$

$$\frac{1}{2}\left(x^2+y^2+z^2+xy+yz+zx\right) \geq \sum_{cyclic} G(x,y)(n) \geq xy+yz+zx \tag{2.51}$$

and

$$\frac{1}{8}(x+y)(y+z)(z+x) \geq \sqrt{\prod_{cyclic} G(x,y)(n)} \geq xyz, \tag{2.52}$$

for any $x, y > 0$ and for all integers $n \geq 0$.
Proof. From Theorem 2.9, we easily deduce inequality (2.47). Using the substitutions $x \to x^2$ and $y \to y^2$ in inequality (2.47), we obtain inequality (2.48). Similarly to inequality (2.47), $\frac{x+y}{2} \geq \sqrt{G(x,y)(n)} \geq \sqrt{xy}$, we can write the following inequalities:

$$\frac{y+z}{2} \geq \sqrt{G(y,z)(n)} \geq \sqrt{yz} \text{ and } \frac{z+x}{2} \geq \sqrt{G(z,x)(n)} \geq \sqrt{zx},$$

which means, by adding, that

$$x+y+z \geq \sum_{cyclic} \sqrt{G(x,y)(n)} \geq \sqrt{xy}+\sqrt{yz}+\sqrt{zx}.$$

It is easy to see that by making the substitutions $x \to x^2$ and $y \to y^2$ in inequality (2.49), we obtain inequality (2.50). Similarly to inequality (2.44), $\left(\frac{x+y}{2}\right)^2 \geq G(x,y)(n) \geq xy$, we obtain the following inequalities:

$$\left(\frac{y+z}{2}\right)^2 \geq G\left(y,z\right)(n) \geq yz \text{ and } \left(\frac{z+x}{2}\right)^2 \geq G\left(z,x\right)(n) \geq zx.$$

By adding them, we have inequality (2.51) and by multiplying them, we obtain inequality (2.52).

Corollary 2.11. In any triangle ABC , there are the following inequalities:

$$R \geq \frac{2}{\sum\limits_{cyclic} \sqrt{G\left(\frac{1}{h_a}, \frac{1}{h_b}\right)(n)}} \geq 2r; \tag{2.53}$$

$$s \geq \sum_{cyclic} \sqrt{G\left(s-a, s-b\right)(n)} \geq 3\sqrt{3}r \tag{2.54}$$

and

$$a^{2\alpha} + b^{2\alpha} + c^{2\alpha} \geq \sum_{cyclic} \sqrt{G\left(a^{2\alpha}, b^{2\alpha}\right)(n)} \geq 3\left(\frac{4\Delta}{\sqrt{3}}\right)^{\alpha}, \tag{2.55}$$

for any $n \geq 0$ and for every real numbers α.
Proof. Making the substitutions $x = \frac{1}{h_a}, y = \frac{1}{h_b}$ and $z = \frac{1}{h_c}$ in inequality (2.49), we obtain

$$\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} \geq \sum_{cyclic} \sqrt{G\left(\frac{1}{h_a}, \frac{1}{h_b}\right)(n)} \geq \frac{1}{\sqrt{h_a h_b}} + \frac{1}{\sqrt{h_b h_c}} + \frac{1}{\sqrt{h_c h_a}}. \tag{2.56}$$

From inequality (2.13), we have

$$\frac{1}{\sqrt{h_a h_b}} + \frac{1}{\sqrt{h_b h_c}} + \frac{1}{\sqrt{h_c h_a}} \geq \frac{2}{R},$$

and from the identity

$$\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} = \frac{1}{r}$$

we deduce

$$\frac{1}{r} \geq \sum_{cyclic} \sqrt{G\left(\frac{1}{h_a}, \frac{1}{h_b}\right)(n)} \geq \frac{2}{R}. \tag{2.57}$$

Consequently

$$R \geq \frac{2}{\sum\limits_{cyclic} \sqrt{G\left(\frac{1}{h_a}, \frac{1}{h_b}\right)(n)}} \geq 2r.$$

If in inequality (2.49) we take $x = s - a, y = s - b$ and $z = s - c$, then we deduce the inequality

$$s \geq \sum_{cyclic} \sqrt{G(s-a, s-b)(n)} \geq \sqrt{(s-a)(s-b)}+$$

$$+\sqrt{(s-b)(s-c)} + \sqrt{(s-c)(s-a)}. \tag{2.58}$$

But

$$\sqrt{(s-a)(s-b)} + \sqrt{(s-b)(s-c)} + \sqrt{(s-c)(s-a)} \geq 3\sqrt{3}r,$$

which means that

$$s \geq \sum_{cyclic} \sqrt{G(s-a, s-b)(n)} \geq 3\sqrt{3}r.$$

Making the substitutions , and in inequality (2.49), we obtain the following inequality:

$$a^{2\alpha} + b^{2\alpha} + c^{2\alpha} \geq \sum_{cyclic} \sqrt{G\left(a^{2\alpha}, b^{2\alpha}\right)(n)} \geq a^\alpha b^\alpha + b^\alpha c^\alpha + c^\alpha a^\alpha. \tag{2.59}$$

Applying the arithmetic-geometric mean inequality and Pólya-Szegö's Inequality, $\sqrt[3]{a^2b^2c^2} \geq \frac{4\Delta}{\sqrt{3}}$, we deduce

$$a^\alpha b^\alpha + b^\alpha c^\alpha + c^\alpha a^\alpha \geq \left(\frac{4\Delta}{\sqrt{3}}\right)^\alpha.$$

Therefore

$$s \geq \sum_{cyclic} \sqrt{G(s-a, s-b)(n)} \geq 3\sqrt{3}r.$$

Remark 5. *a) Inequality (2.53) implies the sequence of inequalities*

$$R \geq ... \frac{2}{\sum\limits_{cyclic} \sqrt{G\left(\frac{1}{h_a}, \frac{1}{h_b}\right)(n)}} \geq \frac{2}{\sum\limits_{cyclic} \sqrt{G\left(\frac{1}{h_a}, \frac{1}{h_b}\right)(n-1)}} \geq ...$$

$$\geq \frac{2}{\sum\limits_{cyclic} \sqrt{\left(\frac{1}{h_a}, \frac{1}{h_b}\right)(0)}} \geq 2r \tag{2.60}$$

b) For $\alpha = 1$ in inequality (2.55), we obtain

$$a^2 + b^2 + c^2 \geq \sum_{cyclic} \sqrt{G\left(a^2, b^2\right)(n)} \geq 4\sqrt{3}\Delta, \tag{2.61}$$

which proves Weitzenböck's Inequality, namely

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}\Delta.$$

Corollary 2.12. For any triangle ABC , there are the following inequalities:

$$2\left(s^2 - r^2 - 4Rr\right) \geq \sum_{cyclic} \sqrt{G\left(a^2, b^2\right)(n)} \geq s^2 + r^2 + 4Rr, \tag{2.62}$$

$$s^2 - 2r^2 - 8Rr \geq \sum_{cyclic} \sqrt{G\left((s-a)^2, (s-b)^2\right)(n)} \geq r\left(4R + r\right), \tag{2.63}$$

$$\frac{\left(s^2 + r^2 + 4Rr\right)^2 - 8s^2Rr}{4R^2} \geq \sum_{cyclic} \sqrt{G\left(h_a^2, h_b^2\right)(n)} \geq \frac{2s^2r}{R}, \tag{2.64}$$

$$(4R + r)^2 - 2s^2 \geq \sum_{cyclic} \sqrt{G\left(r_a^2, r_b^2\right)(n)} \geq s^2 \tag{2.65}$$

$$\frac{8R^2 + r^2 - s^2}{8R^2} \geq \sum_{cyclic} \sqrt{G\left(\sin^4 \frac{A}{2}, \sin^4 \frac{B}{2}\right)(n)} \geq \frac{s^2 + r^2 - 8Rr}{16R^2} \tag{2.66}$$

and

$$\frac{4\left(R + r\right)^2 - s^2}{4R^2} \geq \sum_{cyclic} \sqrt{G\left(\cos^4 \frac{A}{2}, \cos^4 \frac{B}{2}\right)(n)} \geq \frac{s^2 + (4R + r)^2}{8R^2}. \tag{2.67}$$

Proof. According to Corollary 2.10, we have the inequality

$$x^2 + y^2 + z^2 \geq \sum_{cyclic} \sqrt{G\left(x^2, y^2\right)(n)} \geq xy + yz + zx.$$

Using the substitutions

$$(x,y,z) \in \left\{(a,b,c),(s-a,s-b,s-c),(h_a,h_b,h_c),(r_a,r_b,r_c), \left(\sin^2\frac{A}{2},\sin^2\frac{B}{2},\sin^2\frac{C}{2}\right),\left(\cos^2\frac{A}{2},\cos^2\frac{B}{2},\cos^2\frac{C}{2}\right)\right\},$$

we deduce the inequalities required.

Corollary 2.13. In any triangle ABC there are the following inequalities:

$$\frac{1}{2}\left(3s^2-r^2-4Rr\right) \geq \sum_{cyclic} G(a,b)(n) \geq s^2+r^2+4Rr, \tag{2.68}$$

$$\frac{1}{2}\left(s^2-r^2-4Rr\right) \geq \sum_{cyclic} G(s-a,s-b)(n) \geq r(4R+r), \tag{2.69}$$

$$\frac{\left(s^2+r^2+4Rr\right)^2}{8R^2} \geq \sum_{cyclic} G(h_a,h_b)(n) \geq \frac{2s^2r}{R} \tag{2.70}$$

and

$$\frac{1}{2}\left[(4R+r)^2-s^2\right] \geq \sum_{cyclic} G(r_a,r_b)(n) \geq s^2. \tag{2.71}$$

Proof. According to Corollary 2.10, we have the inequality

$$\frac{1}{2}\left(x^2+y^2+z^2+xy+yz+zx\right) \geq \sum_{cyclic} G(x,y)(n) \geq xy+yz+zx.$$

Using the substitutions

$$(x,y,z) \in \{(a,b,c),(s-a,s-b,s-c),(h_a,h_b,h_c),(r_a,r_b,r_c)\},$$

we deduce the inequalities from the statement.

Corollary 2.14. For any triangle ABC there are the following inequalities:

$$\frac{1}{4}s\left(s^2+r^2+2Rr\right) \geq \prod_{cyclic} \sqrt{G(a,b)(n)} \geq 4sRr, \tag{2.72}$$

$$\frac{1}{2}sRr \geq \prod_{cyclic} \sqrt{G((s-a),(s-b))(n)} \geq sr^2, \tag{2.73}$$

$$\frac{s^2r\left(s^2+r^2+4Rr\right)}{8R^2}\geq\prod_{cyclic}\sqrt{G\left(h_a,h_b\right)(n)}\geq\frac{2s^2r^2}{R},\tag{2.74}$$

$$\frac{1}{2}s^2R\geq\prod_{cyclic}\sqrt{G\left(r_a,r_b\right)(n)}\geq s^2r,\tag{2.75}$$

$$\frac{(2R-r)\left(s^2+r^2-8Rr\right)-2Rr^2}{256R^3}\geq$$

$$\geq\prod_{cyclic}\sqrt{G\left(\sin^2\frac{A}{2},\sin^2\frac{B}{2}\right)(n)}\geq\frac{r^2}{16R^2}\tag{2.76}$$

and

$$\frac{(4R+r)^3+s^2\left(2R+r\right)}{256R^3}\geq\prod_{cyclic}\sqrt{G\left(\cos^2\frac{A}{2},\cos^2\frac{B}{2}\right)(n)}\geq\frac{s^2}{16R^2}.\tag{2.77}$$

Proof. According to Corollary 2.10, we have the inequality

$$\frac{1}{8}\left(x+y\right)\left(y+z\right)\left(z+x\right)\geq\sqrt{\prod_{cyclic}G\left(x,y\right)(n)}\geq xyz.$$

Using the substitutions

$$\begin{aligned}(x,y,z)\ \in\ &\{(a,b,c),(s-a,s-b,s-c),(h_a,h_b,h_c),(r_a,r_b,r_c),\\&\left(\sin^2\frac{A}{2},\sin^2\frac{B}{2},\sin^2\frac{C}{2}\right),\left(\cos^2\frac{A}{2},\cos^2\frac{B}{2},\cos^2\frac{C}{2}\right)\Bigg\},\end{aligned}$$

we deduce the inequalities required.

Remark 6. From Corollary 2.13, we obtain the inequality

$$\frac{1}{4}s\left(s^2+r^2+2Rr\right)\geq\geq\prod_{cyclic}\sqrt{G\left(a,b\right)(n)}\geq 4sRr\geq$$

$$\geq 8\prod_{cyclic}\sqrt{G\left(\left(s-a\right),\left(s-b\right)\right)(n)}\geq 8sr^2.\tag{2.78}$$

REFERENCES

[1] Bencze M. and Minculete N., *New refinements of some geometrical inequalities*, Octogon Mathematical Magazine, Vol. 16, no.2, 2008.
[2] Botema, O., Djordjević R. Z., Janić, R. R., Mitrinović, D. S. and Vasić, P. M., *Geometric Inequalities*, Gröningen,1969.
[3] Mitrinović, D. S., *Analytic Inequalities*, Springer Verlag Berlin, Heidelberg, New York, 1970.
[4] Minculete, N. and Bencze, M., *A Generalization of Weitzenböck's Inequality*, Octogon Mathematical Magazine Vol. 16, no.2, 2008.
[5] Minculete, N., *Teoreme şi probleme specifice de geometrie*, Editura Eurocarpatica, Sfântu Gheorghe, 2007 (in Romanian).

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Helly‘s theorem on fuzzy valued functions

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ABSTRACT. Some recently developed notions of fuzzy valued functions, fuzzy distribution functions proposed by H.C.Wu [9] are employed to establish Hellys theorem on fuzzy valued functions. To establish Hellys theorem, and for a convenient discussion of fuzzy random variables a more strong sense of measurability for fuzzy valued functions is introduced.

1. INTRODUCTION

The notion of fuzzy random variables, with necessary theoretical framework was introduced by Kwakernaak [5] and Puri and Ralescu [8]. The notion of normality of fuzzy random variables was also discussed by M.L.Puri et.al [7]. To make fuzzy random variables amenable to statistical analysis for imprecise data where dimness of perception is prevalent, H.C.Wu[9-12] has contributed a variety of research papers on fuzzy random variables which expose various rudiments of fuzzy random variables such as, weak and strong law of large numbers, weak and strong convergence with probability, fuzzy distribution functions, fuzzy probability density functions, fuzzy expectation, fuzzy variance and fuzzy valued functions governed by strong measurability conditions. In [9] H.C.Wu has introduced the notion of fuzzy distribution functions for fuzzy random variables.

Since the α-level set of a closed fuzzy number is a compact interval, in order to make the end points of the α-level set of a fuzzy random variables to be the usual random variables H.C.Wu [9-12] has introduced the concept of strong measurability for fuzzy random variables. In this paper, Hellys theorem, and Helly Bray theorem for fuzzy valued functions and fuzzy probability distribution function are introduced. These results are based on the concept of strong measurability for fuzzy random variables.

In section 2, we introduce some preliminaries related to fuzzy numbers, such as strong and weak convergence of sequence of fuzzy numbers to a fuzzy

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numbers. Section 3 is devoted to fuzzy random variables and its fuzzy distribution functions. The theoretical settings of fuzzy random variables are derived from H.C. Wu [9-12].
In section 3 we introduce the definition of a fuzzy valued function and other notions related to the measurability of a fuzzy valued function. The notion of fuzzy distribution function is also introduced in this section. This section conclude with the definitions of strong and weak convergence in distribution of fuzzy random variables. In section 4 we present the Hellys theorem and Helly- Bray theorem for fuzzy valued functions and fuzzy distribution functions. Throughout this paper we denote the indicator function of the set A by 1_A.

2. FUZZY NUMBERS

In this section we provide some limit properties of fuzzy numbers by applying the Hausdorff metric. We also introduce the notions of fuzzy real numbers and fuzzy random variables.

Definition 2.1.
(i) Let f be a real valued function on a topological space. If $\{x; f(x) \geq \alpha\}$ is closed for each α, then f is said to be upper semi continuous.
(ii) A real valued function f is said to be upper semi continuous at y if and only if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $|x - y| < \delta$ implies $f(x) < f(y) + \varepsilon$
(iii) $f(x)$ is said to be lower semicontinuous if $-f(x)$ is upper semicontinuous.

Definition 2.2. Let $F : X \to R^n$ be a set valued mapping. F is said to be continuous at $x_0 \in X$ if F is both upper semi continuous and lower semi continuous at x_0.

Theorem 2.1. (Bazarra and Shetty [2]) Let S be a compact set in R^n. If f is upper semicontinuous on S then f assumes maximum over S and if f is lower semi-continuous on S then f assumes minimum over S.

Definition 2.3. Let X be a universal set. Then a fuzzy subset A of X is defined by its membership function $\mu_A : X \to [0, 1]$. We denote $A_\alpha = \{x; \mu(x) \geq \alpha\}$ as the α-level set of A where A_0 is the closure of the set $\{x; \mu_A(x) \neq 0\}$.

Definition 2.4.
(i) A is called a normal fuzzy set if their exist x such that $\mu_A(x) = 1$.

(ii) A is called a convex fuzzy set if
$\mu_A(\lambda x + (1-\lambda)y) \geq min\{\mu_A(x), \mu_A(y)\}$ for $\lambda \in [0,1]$.
Theorem 2.2. [13] A is a convex fuzzy set if and only if $\{x; \mu_A(x) \geq \alpha\}$ is a convex set for all α.

Definition 2.5. Let $X = R$
(i) $\widetilde{m}$ is called a fuzzy number if $\widetilde{m}$ is a normal convex fuzzy set and the α-level set $\widetilde{m}_\alpha$ is bounded $\forall \alpha \neq 0$.
(ii) $\widetilde{m}$ is called a closed fuzzy number if $\widetilde{m}$ is a fuzzy number and its membership function $\mu_{\widetilde{m}}$ is upper semicontinuous.
(iii) $\widetilde{m}$ is called a bounded fuzzy number if $\widetilde{m}$ is a fuzzy number and its membership function $\mu_{\widetilde{m}}$ has compact support.

Theorem 2.3. [10] If $\widetilde{m}$ is a closed fuzzy number then the α-level set of $\widetilde{m}$ is a closed interval, which is denoted by

$$\widetilde{m}_\alpha = \left[\widetilde{m}^L_\alpha, \widetilde{m}^U_\alpha\right]$$

Definition 2.6. $\widetilde{m}$ is called a canonical fuzzy number, if it is a closed and bounded fuzzy number and its membership function is strictly increasing on the interval $\left[\widetilde{m}^L_0, \widetilde{m}^L_1\right]$ and strictly decreasing on the interval $\left[\widetilde{m}^U_1, \widetilde{m}^U_0\right]$.

Theorem 2.4. [10] Suppose that $\widetilde{a}$ is a canonical fuzzy number. Let $g(\alpha) = \widetilde{a}^L_\alpha$ and $h(\alpha) = \widetilde{a}^U_\alpha$. Then $g(\alpha)$ and $h(\alpha)$ are continuous functions of α.

Theorem 2.5. (Zadeh [14] resolution identity)
(i) Let A be a fuzzy set with membership function μ_A and $A_\alpha = \{x; \mu_A(x) \geq \alpha\}$. Then

$$\mu_A(x) = \sup_{\alpha \in [0,1]} \alpha 1_{A_\alpha}(x)$$

(ii) (Negoita and Relescu [6]) Let A be a set and $\{A_\alpha : 0 \leq \alpha \leq 1\}$ be a family of subsets of A such that the following conditions are satisfied.
a. $A_0 = A$
b. $A_\alpha \subseteq A_\beta$ for $\alpha > \beta$
c. $A_\alpha = \bigcap_{n=1}^\infty A_{\alpha_n}$ for $\alpha_n \uparrow \alpha$
Then the function $\mu : A \rightarrow [0,1]$ defined by $\mu(x) = \sup\limits_{\alpha \in [0,1]} \alpha 1_{A_\alpha}(x)$ has the property that

$$A_\alpha = \{x; \mu(x) \geq \alpha\} \ \forall \alpha \in [0,1]$$

With the help of α-level sets of a fuzzy set A we can construct closed fuzzy number. Let g and h be two functions from $[0,1]$ into R.
Let $\{A_\alpha = [g(\alpha), h(\alpha)] : 0 \leq \alpha \leq 1\}$ be a family of closed intervals. Then we can induce a fuzzy set A with the membership function

$$\mu_A(r) = \sup_{0 \leq \alpha \leq 1} \alpha 1_{A_\alpha}(r)$$

via the form of resolution identity.

Theorem 2.6. [11] Let $\{A_\alpha : 0 \leq \alpha \leq 1\}$ be a set of decreasing closed intervals. i.e., $A_\alpha \subseteq A_\beta$ for $\alpha > \beta$. Then $f(\alpha) = \alpha 1_{A_\alpha}$ is upper semicontinuous for any fixed r.

Theorem 2.7. [11] Let $\widetilde{a}$ and $\widetilde{b}$ be two canonical fuzzy numbers. Then $\widetilde{a} \oplus \widetilde{b}$, $\widetilde{a} \ominus \widetilde{b}$ and $\widetilde{a} \otimes \widetilde{b}$ are also canonical fuzzy numbers. Further more we have for $\alpha \in [0,1]$

$$\left(\widetilde{a} \oplus \widetilde{b}\right)_\alpha = \left[\widetilde{a}_\alpha^L + \widetilde{b}_\alpha^L, \widetilde{a}_\alpha^U + \widetilde{b}_\alpha^U\right]$$
$$\left(\widetilde{a} \ominus \widetilde{b}\right)_\alpha = \left[\widetilde{a}_\alpha^L - \widetilde{b}_\alpha^L, \widetilde{a}_\alpha^U - \widetilde{b}_\alpha^U\right]$$
$$\left(\widetilde{a} \otimes \widetilde{b}\right)_\alpha = \left[\min\left\{\widetilde{a}_\alpha^L\widetilde{b}_\alpha^L, \widetilde{a}_\alpha^L\widetilde{b}_\alpha^U, \widetilde{a}_\alpha^U\widetilde{b}_\alpha^L, \widetilde{a}_\alpha^U\widetilde{b}_\alpha^U\right\}; \max\left\{\widetilde{a}_\alpha^L\widetilde{b}_\alpha^L, \widetilde{a}_\alpha^L\widetilde{b}_\alpha^U, \widetilde{a}_\alpha^U\widetilde{b}_\alpha^L, \widetilde{a}_\alpha^U\widetilde{b}_\alpha^U\right\}\right]$$

Let $A \subseteq R^n$ and $B \subseteq R^n$. The Hausdorff metric is defined by

$$d_H(A,B) = \max\left\{\sup_{a \in A} \inf_{b \in B} \|a - b\|, \sup_{b \in B} \inf_{a \in A} \|a - b\|\right\}$$

Let $\Im$ be the set of all fuzzy numbers and $\Im_b$ be the set of all bounded fuzzy numbers. Puri and Ralescu [8] have defined the metric $d_\Im$ in $\Im$ as

$$d_\Im\left(\widetilde{a}, \widetilde{b}\right) = \sup_{0 < \alpha \leq 1} dH\left(\widetilde{a}_\alpha, \widetilde{b}_\alpha\right)$$

and $d_{\Im_b}$ in $\Im_b \left(\widetilde{a}, \widetilde{b}\right) = \sup\limits_{0 < \alpha \leq 1} d_H\left(\widetilde{a_\alpha}, \widetilde{b_\alpha}\right)$.

Lemma 2.1. [9] Let $\widetilde{a}$ and $\widetilde{b}$ be two closed fuzzy numbers.
Then we have

$$d_H\left(\widetilde{a}_\alpha,\widetilde{b}_\alpha\right)=\max\left\{\left|\widetilde{a}_\alpha^L-\widetilde{b}_\alpha^L\right|,\left|\widetilde{a}_\alpha^U-\widetilde{b}_\alpha^U\right|\right\}.$$

Definition 2.7. [9]
(i) Let $\{\widetilde{a}_n\}$ be a sequence of closed (canonical) fuzzy numbers. $\{\widetilde{a}_n\}$ is said to converge strongly if there is a closed (canonical) fuzzy number $\widetilde{a}$ with the following property. $\forall\varepsilon>0$, $\exists N>0$ such that for $n>N$, we have
$d_{\Im}\left(\widetilde{a}_n,\widetilde{a}\right)<a$ $\left(d_{\Im_b}\left(\widetilde{a}_n,\widetilde{a}\right)<\varepsilon\right)$. We say that the sequence $\{\widetilde{a}_n\}$ converges to $\widetilde{a}_n$ strongly and it is denoted as $\underset{s\widetilde{a}}{\lim\limits_{n\to\infty}\widetilde{a}_n=}$.
(ii) Let $\{\widetilde{a}_n\}$ be a sequence of closed (canonical) fuzzy numbers. $\{\widetilde{a}_n\}$ is said to converge weakly if there is a closed (canonical) fuzzy number with $\widetilde{a}_n$ the following property:

$$\lim_{n\to\infty}\left(\widetilde{a}_n\right)_\alpha^L=\widetilde{a}_\alpha^L \text{ and } \lim_{n\to\infty}\left(\widetilde{a}_n\right)_\alpha^U=\widetilde{a}_\alpha^U \text{ for all } \alpha\in[0,1]$$

we say that the sequence $\{\widetilde{a}_n\}$ converges to $\widetilde{a}_n$ weakly and it is denoted as $\underset{w\widetilde{a}}{\lim\limits_{n\to\infty}\widetilde{a}_n=}$. We note that $\underset{s\widetilde{a}}{\lim\limits_{n\to\infty}\widetilde{a}_n=}$ is equivalent to $\lim\limits_{n\to\infty}d_{\Im_b}\left(\widetilde{a}_n,\widetilde{a}\right)=0$.

3. FUZZY RANDOM VARIABLE AND ITS DISTRIBUTION FUNCTION

In this section we provide the theoretical framework of fuzzy random variables and its distribution functions proposed by H.C. Wu [9]. Given a real number x one can induce a fuzzy number $\widetilde{x}$ with membership function $\mu_{\widetilde{x}}(r)$ such that $\mu_{\widetilde{x}}(x)=1$ and $\mu_{\widetilde{x}}(r)<1$ for $r\neq x$. We call $\widetilde{x}$ as a fuzzy real number induced by the real number x. Let $\Im_R$ be a set of all fuzzy real numbers induced by the real number system R. We define the relation $\tilde{}$ on $\Im_R$ as $\widetilde{x}^1\sim\widetilde{x}^2$ if and only if $\widetilde{x}^1$ and $\widetilde{x}^2$ are induced by the same real number x. Then $\tilde{}$ is an equivalence relation. This equivalence relation induces the equivalence classes $[\widetilde{x}]=\left\{\begin{array}{c}\widetilde{a}|\widetilde{a}\sim\\ x\end{array}\right\}$. The quotient set $\Im_{R/N}$ is the set of all equivalence classes. Then the cardinality of $\Im_{R/N}$ is equal to the cardinality of the real number system R, since the map $R\to\Im_{R/N}$ specified by $x\to[\widetilde{x}]$ is a bijection. We call $\Im_{R/N}$ as the fuzzy real number system. For practical purposes we take only one element $\widetilde{x}$ from each equivalence class $[\widetilde{x}]$ to form the fuzzy real number system $\left(\Im_{R/N}\right)_R$.
i.e. $\left(\Im_{R/N}\right)_R=\left\{\widetilde{x}|\widetilde{x}\in[\widetilde{x}],\widetilde{x}\text{ is the only element from }[\widetilde{x}]\right\}$.
If the fuzzy real number system $\left(\Im_{R/N}\right)_R$ consists of canonical fuzzy real numbers, then we call $\left(\Im_{R/N}\right)_R$ as the canonical fuzzy real number system.

Let $(X, \mathcal{M})$ be a measurable space and $(R, \mathcal{B})$ be a Borel measurable space. Let $f : X \to \mathcal{P}(\mathcal{R})$ (power set of R) be a set valued function. According to Aumann [1] the set valued function f is called measurable if and only if $\{(x, y)\, ; y \in f\,(x)\}$ is $calM \times \mathcal{B}$ measurable. The function $\widetilde{f}\,(x)$ is called a fuzzy valued function if $\widetilde{f} : x \to \Im$ (the set of all fuzzy numbers).

Definition 3.1. [9] Let $\left(\Im_{R/N}\right)_R$ be a canonical fuzzy real number system and $\widetilde{X} : \Omega \to \left(\Im_{R/N}\right)_R$ be a closed-fuzzy valued function. $\widetilde{X}$ is called a fuzzy random variable if X is measurable (or equivalently strongly measurable).

Theorem 3.1. [9] Let $\left(\Im_{R/N}\right)_R$ be a canonical fuzzy real number system and $\widetilde{X} : \Omega \to \left(\Im_{R/N}\right)_R$ be a closed-fuzzy valued function. $\widetilde{X}$ is a fuzzy random variable if and only if $\widetilde{X}^L_\alpha$ and $\widetilde{X}^U_\alpha$ are random variables for all $\alpha \in [0, 1]$.

If $\widetilde{x}$ is a canonical fuzzy real number, then $\widetilde{x}^L_1 = \widetilde{x}^U_1$. Let $\widetilde{XZ}$ be a fuzzy random variable. By theorem 3.1 $\widetilde{X}^L_\alpha$ and $\widetilde{X}^U_\alpha$ are random variables for all α, and $\widetilde{X}^L_1 = \widetilde{X}^U_1$. Let $F(x)$ be a continuous distribution function of a random variable X. Let $\widetilde{X}^L_\alpha$ and $\widetilde{X}^U_\alpha$ have the same distribution function $F(x)$ for all $\alpha \in [0, 1]$. If $\widetilde{x}$ is any fuzzy observation of a fuzzy random variable $\widetilde{X}$ $\left(\widetilde{X}\,(w) = \widetilde{x}\right)$ then the α-level set $\widetilde{x}_\alpha$ is $\widetilde{x}_\alpha = \left[\widetilde{x}^L_\alpha, \widetilde{x}^U_\alpha\right]$.

By a fuzzy observation we mean an imprecise data. We can see that $\widetilde{x}^L_\alpha$ and $\widetilde{x}^U_\alpha$ are the observations of $\widetilde{X}^L_\alpha$ and $\widetilde{X}^U_\alpha$ respectively. From theorem 2.4 $\widetilde{X}^L_\alpha\,(w) = \widetilde{x}^L_\alpha$ and $\widetilde{X}^U_\alpha\,(w) = \widetilde{x}^U_\alpha$ are continuous with respect to α for fixed w. Thus $\left[\widetilde{x}^L_\alpha, \widetilde{x}^U_\alpha\right]$ is continuously shrinking with respect to α. $\left[\widetilde{x}^L_\alpha, \widetilde{x}^U_\alpha\right]$ is the disjoint union of $\left[\widetilde{x}^L_\alpha, \widetilde{x}^L_1\right]$ and $\left(\widetilde{x}^U_1, \widetilde{x}^U_\alpha\right]$. Therefore for any real number $x \in \left[\widetilde{x}^L_\alpha, \widetilde{x}^U_\alpha\right]$ we have $x = \widetilde{x}^L_\beta$ or $x = \widetilde{x}^U_\beta$ for some $\beta \geq \alpha$. This confirm the existence of a suitable α-level set for which x coincides with the lower end of that α-level set or with the upper end of that α-level set. Hence for any $x \in \left[\widetilde{x}^L_\alpha, \widetilde{x}^U_\alpha\right]$, we can associate an $F\left(\widetilde{x}^L_\beta\right)$ or $F\left(\widetilde{x}^U_\beta\right)$ with x.

If $\widetilde{f}$ is a fuzzy valued function then $\widetilde{f}_\alpha$ is a set valued function for all $\alpha \in [0, 1]$. $\widetilde{f}$ is called (fuzzy-valued) measurable if and only if $\widetilde{f}_\alpha$ is (set-valued) measurable for all $\alpha \in [0, 1]$.

To make fuzzy random variables more governable mathematically a more strong sense of measurability for fuzzy valued function is required. $\widetilde{f}\,(x)$ is called a closed-fuzzy-valued function if $\widetilde{f} : X \to \Im_{cl}$ (the set of all closed fuzzy numbers). Let $\widetilde{f}\,(x)$ be a closed fuzzyvalued function defined on X. From H.C. Wu [11] the following two statements are equivalent.

(i) $\widetilde{f}_{\alpha}^{L}(x)$ and $\widetilde{f}_{\alpha}^{U}(x)$ are (real- valued) measurable for all $\alpha \in [0,1]$.
(ii) $\widetilde{f}(x)$ is (fuzzy-valued) measurable and one of $\widetilde{f}_{\alpha}^{L}(x)$ and $\widetilde{f}_{\alpha}^{U}(x)$ is (real valued) measurable for all $\alpha \in [0,1]$.
Then $\widetilde{f}(x)$ is called strongly measurable if one of the above two conditions is satisfied. It is easy to see that the strong measurability implies measurability.
Let (X,$\mathcal{M}, \mu$) be a measure space and (R, $\mathcal{B}$) be a Borel measurable space.
Let $f : X \to \mathcal{P}(\mathcal{R})$ be a set valued function. For $K \subseteq R$, the inverse image of f is defined by

$$f^{-1}(k) = \{x \in X, f(x) \cap k \neq \emptyset\}$$

Let (X,$\Updownarrow, \mu$) be a complete σ-finite measure space. From Hiai and Umegaki [3] the following two statements are equivalent.
a) For each Borel set $k \subseteq R$, $f^{-1}(k)$ is measurable (ie $f^{-1}(k) \in \mathcal{M}$)
b) $\{(x,y); y \in f(x)\}$ is $\mathcal{M} \times \mathcal{B}$ measurable.
If we construct an interval

$$A_{\alpha} = \left[\min\left\{\inf_{\alpha\leq\beta\leq1} F\left(\widetilde{x}_{\beta}^{L}\right), \inf_{\alpha\leq\beta\leq1} F\left(\widetilde{x}_{\beta}^{U}\right)\right\}, \max\left\{\sup_{\alpha\leq\beta\leq1} F\left(\widetilde{x}_{\beta}^{L}\right), \sup_{\alpha\leq\beta\leq1} F\left(\widetilde{x}_{\beta}^{U}\right)\right\}\right]$$

Then this interval will contain all of the distributions associated with each of $x \in \left[\widetilde{x}_{\alpha}^{L}, \widetilde{x}_{1}^{L}\right]$. We denote by $\widetilde{F}(\widetilde{x})$ the fuzzy distribution function of the fuzzy random variable $\widetilde{x}$. Then we define the membership function of $\widetilde{F}(\widetilde{x})$ for any fixed $\widetilde{x}$ by

$$\mu_{\widetilde{F}(\widetilde{x})}^{(r)} = \sup_{\alpha\leq\beta\leq1} \alpha 1_{A_{\alpha}}(r)$$

we also say that the fuzzy distribution function $\widetilde{F}(\widetilde{x})$ is induced by the distribution function $F(x)$. Since $F(x)$ is continuous from theorem 2.4 and 2.1, we can write A_{α} as

$$A_{\alpha} = \left[\min\left\{\min_{\alpha\leq\beta\leq1} F\left(\widetilde{x}_{\beta}^{L}\right), \min_{\alpha\leq\beta\leq1} F\left(\widetilde{x}_{\beta}^{U}\right)\right\}, \max\left\{\max_{\alpha\leq\beta\leq1} F\left(\widetilde{x}_{\beta}^{L}\right), \max_{\alpha\leq\beta\leq1} F\left(\widetilde{x}_{\beta}^{U}\right)\right\}\right]$$

For typographical reasons we employ the following notations.

$$F^{\min}\left(\widetilde{x}_{\beta}^{(\bullet)}\right) = \min\left\{\min_{\alpha\leq\beta\leq1} F\left(\widetilde{x}_{\beta}^{L}\right), \min_{\alpha\leq\beta\leq1} F\left(\widetilde{x}_{\beta}^{U}\right)\right\}$$

$$F^{\max}\left(\widetilde{x}_{\beta}^{(\bullet)}\right)=\max\left\{\max_{\alpha\leq\beta\leq 1}F\left(\widetilde{x}_{\beta}^{L}\right),\max_{\alpha\leq\beta\leq 1}F\left(\widetilde{x}_{\beta}^{U}\right)\right\}$$

Let $\widetilde{X}$ and $\widetilde{Y}$ be two fuzzy random variables. We say that $\widetilde{X}$ and $\widetilde{Y}$ are independent if and only if each random variable in the set

$\left\{\widetilde{X}_{\alpha}^{L},\widetilde{X}_{\alpha}^{U};0\leq\alpha\leq 1\right\}$ is independent of each random variable in the set

$\left\{\widetilde{Y}_{\alpha}^{L},\widetilde{Y}_{\alpha}^{U};0\leq\alpha\leq 1\right\}$. We say that $\widetilde{X}$ and $\widetilde{Y}$ are identically distributed if and only if $\widetilde{X}_{\alpha}^{L}$ and $\widetilde{Y}_{\alpha}^{L}$ are identically distributed and $\widetilde{X}_{\alpha}^{U}$ and $\widetilde{Y}_{\alpha}^{U}$ are identically distributed for all $\alpha\in[0,1]$.

Definition 3.2. Let $\widetilde{X}$ and $\{\widetilde{X}_n\}$ be fuzzy random variables defined on the same probability space $(\Omega,\mathcal{A},\mathcal{P})$.

(i) We say that $\{\widetilde{X}_n\}$ converges in distribution to $\widetilde{X}$ level-wise if $\left(\widetilde{X}_n\right)_{\alpha}^{L}$ and $\left(\widetilde{X}_n\right)_{\alpha}^{U}$ converge in distribution to $\widetilde{X}_{\alpha}^{L}$ and $\widetilde{X}_{\alpha}^{U}$ respectively for all $\alpha\in(0,1]$. Let $\widetilde{F}_n(\widetilde{x})$ and $\widetilde{F}(\widetilde{x})$ be the respective fuzzy distribution functions of $\widetilde{X}_n$ and $\widetilde{X}$.

(ii) We say that $\{\widetilde{X}_n\}$ converge in distribution to $\widetilde{X}$ strongly if

$$\lim_{n\to\infty}\widetilde{F}_n(\widetilde{x})= \\ s\widetilde{F}(\widetilde{x})\ \text{ i.e.}$$

$$\lim_{n\to\infty}\sup_{\alpha\leq\beta\leq 1}\left\{\max\left\{\left|\left(\widetilde{F}_n\right)_{\alpha}^{L}(\widetilde{x})-\widetilde{F}_{\alpha}^{L}(\widetilde{x})\right|,\left|\left(\widetilde{F}_n\right)_{\alpha}^{U}(\widetilde{x})-\widetilde{F}_{\alpha}^{U}(\widetilde{x})\right|\right\}\right\}=0$$

(iii) We say that $\{\widetilde{X}_n\}$ converges in distribution to $\widetilde{X}$ weakly if

$$\lim_{n\to\infty}\widetilde{F}_n(\widetilde{x})= \\ w\widetilde{F}(\widetilde{x})\ \text{ i.e.}$$

$$\lim_{n\to\infty}\left(\widetilde{F}_n\right)_{\alpha}^{L}(\widetilde{x})=\widetilde{F}_{\alpha}^{L}(\widetilde{x})\ \text{ and}$$

$$\lim_{n\to\infty}\left(\widetilde{F}_n\right)_{\alpha}^{U}(\widetilde{x})=\widetilde{F}_{\alpha}^{U}(\widetilde{x})\ \text{ for all } \alpha\in[0,1].$$

4. HELLYS THEOREMS

In this section based on the theoretical framework of section 3 and section 4 we have established the Hellys theorem and Helly Bray theorem for fuzzy valued functions and fuzzy distribution functions.

Theorem 4. (Hellys Theorem) If (i) non-decreasing sequence of fuzzy probability distribution function $\{F_n(x)\}$ converges to the fuzzy probability distribution function $F(x)$ (ii) the fuzzy valued function $g(x)$ is everywhere continuous and (iii) a, b are continuity points of $F(x)$ then

$$\lim_{n\to\infty}\int_a^b g_\alpha^L(x)\,dF_n^{\min}\left(x_\beta^{(\bullet)}\right)\wedge g_\alpha^U(x)\,dF_n^{\min}\left(x_\beta^{(\bullet)}\right)=$$

$$=\int_a^b g_\alpha^L(x)\,dF^{\min}\left(x_\beta^{(\bullet)}\right)\wedge g_\alpha^U(x)\,dF^{\max}\left(x_\beta^{(\bullet)}\right) \tag{3.1}$$

Proof. By stipulation $F_n\left(x_\beta^L\right)$ and $F_n\left(x_\beta^U\right)$ is non-decreasing.

$$\therefore\ F\left(x_\beta^L\right) \text{ and } F\left(x_\beta^U\right) \text{ is also decreasing.}$$

For all $n \geq 1$, $F_n\left(x_\beta^L+h\right)-F_n\left(x_\beta^L\right)\geq 0$; if $h>0$.

$$F_n\left(x_\beta^U+h\right)-F_n\left(x_\beta^U\right)\geq 0;\ \text{if } h>0.$$

Letting $n\to\infty$ we have

$$F\left(x_\beta^L+h\right)-F\left(x_\beta^L\right)\geq 0;\ \text{if } h>0$$

$$F\left(x_\beta^U+h\right)-F\left(x_\beta^U\right)\geq 0;\ \text{if } h>0$$

we take $a=x_0<x_1<x_2<.....<x_k=b$ where $x_0, x_1, ...$ are the continuity points of the fuzzy probability distribution function F. then for $\alpha\leq\beta\leq 1$

$$\int_a^b g_\alpha^L(x)\,dF^{\min}\left(x_\beta^{(\bullet)}\right)\wedge g_\alpha^U(x)\,dF^{\max}\left(x_\beta^{(\bullet)}\right)=$$

$$=\sum_{i=0}^{k-1}\int_{x_i}^{x_{i+1}} g_\alpha^L(x)\,dF^{\min}\left(x_\beta^{(\bullet)}\right)\wedge g_\alpha^U(x)\,dF^{\max}\left(x_\beta^{(\bullet)}\right)=$$

$$= \sum_{i=0}^{k-1} \left(\int_{x_i}^{x_{i+1}} \left(g_\alpha^L(x) - g_\alpha^L(x_i)\right) dF^{\min}\left(x_\beta^{(\bullet)}\right) + \int_{x_i}^{x_{i+1}} g_\alpha^L(x_i)\, dF^{\min}\left(x_\beta^{(\bullet)}\right) \right) \wedge$$

$$\wedge \sum_{i=0}^{k-1} \left(\int_{x_i}^{x_{i+1}} \left(g_\alpha^U(x) - g_\alpha^U(x_i)\right) dF^{\max}\left(x_\beta^{(\bullet)}\right) + \int_{x_i}^{x_{i+1}} g_\alpha^U(x_i)\, dF^{\max}\left(x_\beta^{(\bullet)}\right) \right) =$$

$$= \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} \left(g_\alpha^L(x) - g_\alpha^L(x_i)\right) dF^{\min}\left(x_\beta^{(\bullet)}\right) +$$

$$+ \sum_{i=0}^{k-1} g_\alpha^L(x_i) \int_{x_i}^{x_{i+1}} dF^{\min}\left(x_\beta^{(\bullet)}\right) \wedge$$

$$\wedge \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} \left(g_\alpha^U(x) - g_\alpha^U(x_i)\right) dF^{\max}\left(x_\beta^{(\bullet)}\right) +$$

$$+ \sum_{i=0}^{k-1} g_\alpha^U(x_i) \int_{x_i}^{x_{i+1}} dF^{\max}\left(x_\beta^{(\bullet)}\right) =$$

$$= \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} \left(g_\alpha^L(x) - g_\alpha^L(x_i)\right) dF^{\min}\left(x_\beta^{(\bullet)}\right) +$$

$$+ \sum_{i=0}^{k-1} g_\alpha^L(x_i) \left(F^{\min}(x_{i+1}) - F^{\min}(x_i)\right) \wedge$$

$$\wedge \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} \left(g_\alpha^U(x) - g_\alpha^U(x_i)\right) dF^{\max}\left(x_\beta^{(\bullet)}\right) +$$

$$+ \sum_{i=0}^{k-1} g_\alpha^U(x_i) \left(F^{\max}(x_{i+1}) - F^{\max}(x_i)\right)$$

By stipulation the fuzzy valued function g is continuous every where. i.e for each $\alpha \in [0,1]$.

$$\left|g_\alpha^L(x) - g_\alpha^L(x_i)\right| < \frac{\frac{\varepsilon}{3}}{F(b) - F(a)} \text{ for } x_i \le x \le x_{i+1}.$$

We take $|\theta_1| \le 1$. Then

$$\int_a^b g_\alpha^L(x)\, dF^{\min}\left(x_\beta^{(\bullet)}\right) \wedge g_\alpha^U(x)\, dF^{\max}\left(x_\beta^{(\bullet)}\right) \le \frac{\theta_1 \varepsilon}{3} +$$

$$+ \sum_{i=0}^{k-1} g_\alpha^L(x_i)\left(F^{\min}(x_{i+1}) - F^{\min}(x_i)\right) \wedge$$

$$\wedge \sum_{i=0}^{k-1} g_\alpha^U(x_i)\left(F^{\max}(x_{i+1}) - F^{\max}(x_i)\right) \tag{3.2}$$

Similarly

$$\int_a^b g_\alpha^L(x)\, dF_n^{\min}\left(x_\beta^{(\bullet)}\right) \wedge g_\alpha^U(x)\, dF_n^{\max}\left(x_\beta^{(\bullet)}\right) \le$$

$$\le \frac{\theta_2 \varepsilon}{3} + \sum_{i=0}^{k-1} g_\alpha^L(x_i)\left(F_n^{\min}(x_{i+1}) - F_n^{\min}(x_i)\right) \wedge$$

$$\wedge \sum_{i=0}^{k-1} g_\alpha^U(x_i)\left(F_n^{\max}(x_{i+1}) - F_n^{\max}(x_i)\right) \tag{3.3}$$

Since $F_n^{\min}\left(x_\beta^{(\bullet)}\right) \to F^{\min}\left(x_\beta^{(\bullet)}\right)$ at continuity points of F and $F_n^{\max}\left(x_\beta^{(\bullet)}\right) \to F^{\max}\left(x_\beta^{(\bullet)}\right)$at continuity points of F

$$F_n^{\min}(x_{i+1}) - F^{\min}(x_{i+1}) < \frac{\varepsilon}{6\sum g_\alpha^L(x_i)} \text{ for all } i$$

and large values of n. Similarly

$$F_n^{\max}(x_{i+1}) - F^{\max}(x_{i+1}) < \frac{\varepsilon}{6\sum g_\alpha^U(x_i)} \text{ for all } i$$

and large values of n.
Hence letting $n \to \infty$ the absolute difference of (3.2) and (3.3) is

$$\int_a^b g_\alpha^L(x)\left|dF_n^{\min}\left(x_\beta^{(\bullet)}\right)-dF^{\min}\left(x_\beta^{(\bullet)}\right)\right|\wedge$$

$$\wedge g_\alpha^U(x)\left|dF^{\max}\left(x_\beta^{(\bullet)}\right)-dF^{\max}\left(x_\beta^{(\bullet)}\right)\right|\leq$$

$$\leq\frac{\varepsilon}{3}\left|\theta_2-\theta_1\right|+\sum_{i=0}^{k-1}g_\alpha^L(x_i)\left|F_n^{\min}(x_{i+1})-F_n^{\min}(x_i)-F^{\min}(x_{i+1})-F^{\min}(x_i)\right|\wedge$$

$$\wedge\sum_{i=0}^{k-1}g_\alpha^U(x_i)\left|F_n^{\max}(x_{i+1})-F_n^{\max}(x_i)-F^{\max}(x_{i+1})-F^{\max}(x_i)\right|$$

Then

$$\int_a^b g_\alpha^L(x)\left|dF_n^{\min}\left(x_\beta^{(\bullet)}\right)-dF^{\min}\left(x_\beta^{(\bullet)}\right)\right|\wedge$$

$$\wedge g_\alpha^U(x)\left|dF^{\max}\left(x_\beta^{(\bullet)}\right)-dF^{\max}\left(x_\beta^{(\bullet)}\right)\right|<\in$$

This show that

$$\lim_{n\to\infty}\int_a^b g_\alpha^L(x)\,dF_n^{\min}\left(x_\beta^{(\bullet)}\right)\wedge g_\alpha^U(x)\,dF_n^{\min}\left(x_\beta^{(\bullet)}\right)=$$

$$=\int_a^b g_\alpha^L(x)\,dF^{\min}\left(x_\beta^{(\bullet)}\right)\wedge g_\alpha^U(x)\,dF^{\max}\left(x_\beta^{(\bullet)}\right)$$

which completes the proof.

Theorem 4.2. (Helly Bray Theorem) If (i) the fuzzy valued function $g(x)$ is continuous
(ii) the fuzzy probability distribution function $F_n(x)\to F(x)$ in each continuity point of $F(x)$ and
(iii) for any $\varepsilon>0$ we can find A such that

$$\int_{-\infty}^{A}\left|g_\alpha^L(x)\right|dF_n^{\min}\left(x_\beta^{(\bullet)}\right)\wedge\left|g_\alpha^U(x)\right|dF_n^{\max}\left(x_\beta^{(\bullet)}\right)+$$

$$+\int_{A}^{\infty}\left|g_{\alpha}^{L}(x)\right| dF_{n}^{\min}\left(x_{\beta}^{(\bullet)}\right) \wedge \left|g_{\alpha}^{U}(x)\right| dF_{n}^{\max}\left(x_{\beta}^{(\bullet)}\right) < \in \tag{3.4}$$

for all $n = 1, 2, 3, ...$, then

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} g_{\alpha}^{L}(x)\, dF_{n}^{\min}\left(x_{\beta}^{(\bullet)}\right) \wedge g_{\alpha}^{U}(x)\, dF_{n}^{\max}\left(x_{\beta}^{(\bullet)}\right) =$$

$$= \int_{-\infty}^{\infty} g_{\alpha}^{L}(x)\, dF^{\min}\left(x_{\beta}^{(\bullet)}\right) \wedge g_{\alpha}^{U}(x)\, dF^{\max}\left(x_{\beta}^{(\bullet)}\right)$$

Proof. The theorem is proved for $F_n(x) \to F(x)$ for all x.
Letting $n \to \infty$ and from condition (3.4) we have

$$\int_{-\infty}^{A}\left|g_{\alpha}^{L}(x)\right| dF^{\min}\left(x_{\beta}^{(\bullet)}\right) \wedge \left|g_{\alpha}^{U}(x)\right| dF^{\max}\left(x_{\beta}^{(\bullet)}\right) +$$

$$+\int_{A}^{\infty}\left|g_{\alpha}^{L}(x)\right| dF^{\min}\left(x_{\beta}^{(\bullet)}\right) \wedge \left|g_{\alpha}^{U}(x)\right| dF^{\max}\left(x_{\beta}^{(\bullet)}\right) < \in \tag{3.5}$$

By theorem (3.1)

$$\lim_{n\to\infty}\int_{a}^{b} g_{\alpha}^{L}(x)\, dF_{n}^{\min}\left(x_{\beta}^{(\bullet)}\right) \wedge g_{\alpha}^{U}(x)\, dF_{n}^{\max}\left(x_{\beta}^{(\bullet)}\right) =$$

$$= \int_{a}^{b} g_{\alpha}^{L}(x)\, dF^{\min}\left(x_{\beta}^{(\bullet)}\right) \wedge g_{\alpha}^{U}(x)\, dF^{\max}\left(x_{\beta}^{(\bullet)}\right) \tag{3.6}$$

By the expression (3.4) it follows that for continuity points $b > a$, the fuzzy valued function $g(x)$ is integrable over $(-\infty, \infty)$, with respect to the fuzzy probability distribution function $F(x)$. If in (3.6) w e take $a = -A$ and $b = A$ we have

$$\lim_{n\to\infty}\int_{-A}^{A} g_{\alpha}^{L}(x)\, dF_{n}^{\min}\left(x_{\beta}^{(\bullet)}\right) \wedge g_{\alpha}^{U}(x)\, dF_{n}^{\max}\left(x_{\beta}^{(\bullet)}\right) =$$

$$= \int_{-A}^{A} g_\alpha^L(x)\, dF^{\min}\left(x_\beta^{(\bullet)}\right) \wedge g_\alpha^U(x)\, dF^{\max}\left(x_\beta^{(\bullet)}\right) \tag{3.7}$$

$$\int_{-\infty}^{\infty} g_\alpha^L(x)\, dF_n^{\min}\left(x_\beta^{(\bullet)}\right) \wedge g_\alpha^U(x)\, dF_n^{\max}\left(x_\beta^{(\bullet)}\right) -$$

$$- \int_{-\infty}^{\infty} g_\alpha^L(x)\, dF^{\min}\left(x_\beta^{(\bullet)}\right) \wedge g_\alpha^U(x)\, dF^{\max}\left(x_\beta^{(\bullet)}\right) =$$

$$= \left(\int_{-\infty}^{-A} + \int_{-A}^{A} + \int_{A}^{\infty}\right) \left(g_\alpha^L(x)\, dF_n^{\min}\left(x_\beta^{(\bullet)}\right) \wedge g_\alpha^U(x)\, dF_n^{\max}\left(x_\beta^{(\bullet)}\right)\right) -$$

$$- \left(\int_{-\infty}^{-A} + \int_{-A}^{A} + \int_{A}^{\infty}\right) \left(g_\alpha^L(x)\, dF^{\min}\left(x_\beta^{(\bullet)}\right) \wedge g_\alpha^U(x)\, dF^{\max}\left(x_\beta^{(\bullet)}\right)\right)$$

Then

$$\lim_{n\to\infty} \left| \int_{-\infty}^{\infty} \left(g_\alpha^L(x)\, dF_n^{\min}\left(x_\beta^{(\bullet)}\right) \wedge \left(g_\alpha^U(x)\, dF_n^{\max}\left(x_\beta^{(\bullet)}\right)\right)\right) - \right.$$

$$\left. - \int_{-\infty}^{\infty} \left(g_\alpha^L(x)\, dF^{\min}\left(x_\beta^{(\bullet)}\right) \wedge g_\alpha^U(x)\, dF^{\max}\left(x_\beta^{(\bullet)}\right)\right) \right| \leq$$

$$\leq \lim_{n\to\infty} \left[\int_{-\infty}^{-A} \left(\left|g_\alpha^L(x)\right| dF_n^{\min}\left(x_\beta^{(\bullet)}\right) \wedge \left|g_\alpha^U(x)\right| dF_n^{\max}\left(x_\beta^{(\bullet)}\right)\right) + \right.$$

$$\left. + \int_{A}^{\infty} \left(\left|g_\alpha^L(x)\right| dF_n^{\min}\left(x_\beta^{(\bullet)}\right) \wedge \left|g_\alpha^U(x)\right| dF_n^{\max}\left(x_\beta^{(\bullet)}\right)\right) \right] +$$

$$+ \lim_{n\to\infty} \left| \int_{-A}^{A} \left(g_\alpha^L(x)\, dF_n^{\min}\left(x_\beta^{(\bullet)}\right) - g_\alpha^L(x)\, dF^{\min}\left(x_\beta^{(\bullet)}\right)\right) \wedge \right.$$

$$\left. \wedge \left(g_\alpha^U(x)\, dF_n^{\max}\left(x_\beta^{(\bullet)}\right) - g_\alpha^U(x)\, dF^{\max}\left(x_\beta^{(\bullet)}\right)\right) \right| < \varepsilon + \varepsilon + \varepsilon = 3\varepsilon$$

if n and A are both large. Thus we have proved the theorem if

$$F_n^{\min}\left(x_\beta^{(\bullet)}\right) \to F^{\min}\left(x_\beta^{(\bullet)}\right) \text{ and } F_n^{\max}\left(x_\beta^{(\bullet)}\right) \to F^{\max}\left(x_\beta^{(\bullet)}\right) \text{ at all } x_\beta^{(\bullet)}$$

$$\therefore \lim_{n\to\infty} \int\limits_{-\infty}^{\infty} g_\alpha^L(x)\, dF_n^{\min}\left(x_\beta^{(\bullet)}\right) \wedge g_\alpha^U(x)\, dF_n^{\max}\left(x_\beta^{(\bullet)}\right) =$$

$$= \int\limits_{-\infty}^{\infty} g_\alpha^L(x)\, dF^{\min}\left(x_\beta^{(\bullet)}\right) \wedge g_\alpha^U(x)\, dF^{\max}\left(x_\beta^{(\bullet)}\right)$$

Now we can choose only continuity points of $F(x)$ viz., $x_1, x_2, ...$ Between two discontinuity points one can choose a continuity point and for monotonic F_n, the points of discontinuities are at most countable. Hence the result is true if $F_n^{\min}(x) \to F^{\min}(x)$ and $F_n^{\max}(x) \to F^{\max}(x)$ at all continuity points of F, and the proof is complete.

REFERENCES

[1] Aumann, R.J., *Integrals of set valued functions,* J.Math. Anal. Appl. 12, 1-12,1965.

[2] Bazarra, M.S., Shetty, C.M., *Non-linear programming,* Wiley, New York, 1993.

[3] Hiai, F., Umegaki, H., *Integrals conditional expectations and martingales of multivalued functions,* J. Multivariate. Anal.7, 149-182, 1977.

[4] Kruse, R., *The strong law of large number for fuzzy random variables*, Proc. Roy. Soc. London A407, 171-182, 1986.

[5] Kwakernaak, H., *Fuzzy random variables I : Definitions and theorems,* Inform. Sci 15, 1-29, 1978.

[6] Negoita, C.V., Ralescu, D.A., *Applications of Fuzzy sets to system analysis,* Wiley, New York, 1975.

[7] Puri, M.L. and Ralescu, D.A., *The concept of normality for fuzzy random variables,* Ann. Prob. 13, 1373-1379, 1985.

[8] Puri, M.L. and Ralescu, D.A., *Fuzzy random variables,* J.Math. Anal. Appl. 114, 409-422, 1986.

[9] Wu, H.C., *The central limit theorems for fuzzy random variables*, Inform, Sci. 120, 239-256,1999.

[10] Wu, H.C., *Probability density functions of fuzzy random variables*, Fuzzy sets and systems 105, 139-158, 1999.

[11] Wu, H.C., *The laws of large numbers for fuzzy random variables*, Fuzzy sets and systems 116, 245-262, 2000.
[12] Wu, H.C., *The fuzzy estimators of fuzzy parameters based on fuzzy random variables*, European Journal of Operations research 146, 101-114, 2003.
[13] Zadeh, L.A., *Fuzzy sets*, Inform and Control 8, 338-353, 1965.
[14] Zadeh, L.A., *The concept of linguistic variable and its application to approximate reasoning I, II and III Inform.* Sci. 8, 199-249, 1975; 8, 301-357, 1975; 9, 43-80, 1975.

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About AM-HM inequality

Mihály Bencze[7]

ABSTRACT. In [1] M. Bencze introduced the following notation $G(m) = \left(\sum\limits_{i=1}^{n} a_i^m\right)\left(\sum\limits_{i=1}^{n} a_i^{-m}\right)$ where $a_i > 0$ $(i = 1, 2, ..., n)$ and by mathematical induction proved that $G(m) \geq G(m-1) \geq ... \geq G(1) \geq G(0) = n^2$.
In this paper we generalize this result and we give some refinements.

MAIN RESULTS

Theorem 1. If $a, b, c > 0$, then

$$\left(\sum a^2\right)\left(\sum \frac{1}{a^2}\right) \geq \left(\sum a\right)\left(\sum \frac{1}{a}\right) \geq 9$$

Proof. This is a classical inequality, but now we give an elementary proof.
If $x, y, z > 0$, then $\sum x^2 \geq \sum xy$.
Using this inequality, we obtain the following

$$\left(\sum a^2\right)\left(\sum \frac{1}{a^2}\right) \geq \left(\sum ab\right)\left(\sum \frac{1}{ab}\right) = abc\left(\sum \frac{1}{a}\right)\frac{1}{abc}\left(\sum a\right) =$$

$$= \left(\sum a\right)\left(\sum \frac{1}{a}\right) \geq 9$$

Theorem 2. If $a_i > 0$ $(i = 1, 2, ..., n)$, then

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} \frac{1}{a_i^2}\right) \geq \left(\sum_{cyclic} a_1 a_2 ... a_{n-2}\right)\left(\sum_{cyclic} \frac{1}{a_1 a_2 ... a_{n-2}}\right) \geq n^2$$

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Proof. We have $\sum\limits_{i=1}^{n} a_i^2 \geq \sum\limits_{cyclic} a_1 a_2$, which is equivalent with $\sum\limits_{cyclic} (a_1 - a_2)^2 \geq 0$, therefore:

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} \frac{1}{a_i^2}\right) \geq \left(\sum_{cyclic} a_1 a_2\right)\left(\sum_{cyclic} \frac{1}{a_1 a_2}\right) =$$

$$= \left(\sum_{cyclic} a_1 a_2 ... a_{n-2}\right)\left(\sum_{cyclic} \frac{1}{a_1 a_2 ... a_{n-2}}\right) \geq n^2$$

Theorem 3. If $a_i > 0$ $(i = 1, 2, ..., n)$, then

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} \frac{1}{a_i^2}\right) \geq \frac{4}{(n-1)^2}\left(\sum_{1\leq i<j\leq n} a_i a_j\right)\left(\sum_{1\leq i<j\leq n} \frac{1}{a_i a_j}\right) \geq n^2$$

Proof. We have have the inequality $\sum\limits_{i=1}^{n} a_i^2 \geq \frac{2}{n-1} \sum\limits_{1\leq i<j\leq n} a_i a_j$ which is equivalent with $\sum\limits_{1\leq i<j\leq n} (a_i - a_j)^2 \geq 0$, therefore:

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} \frac{1}{a_i^2}\right) \geq \frac{4}{(n-1)^2}\left(\sum_{1\leq i<j\leq n} a_i a_j\right)\left(\sum_{1\leq i<j\leq n} \frac{1}{a_i a_j}\right) \geq n^2$$

Theorem 4. If $a_i > 0$ $(i = 1, 2, ..., n)$ and $k \in \{1, 2, ..., n\}$, then

$$\left(\sum_{i=1}^{n} a_i\right)^k \left(\sum_{i=1}^{n} \frac{1}{a_i}\right)^k \geq \frac{n^{2k}}{\binom{n}{k}^2}\left(\sum_{cyclic} x_1 x_2 ... x_k\right)\left(\sum_{cyclic} \frac{1}{x_1 x_2 ... x_k}\right) \geq n^{2k}$$

Proof. Using the Mac Laurian inequality we have:
$\sum\limits_{cyclic} x_1 x_2 ... x_k \leq \frac{\binom{n}{k}}{n^k}\left(\sum\limits_{i=1}^{n} x_i\right)^k$, therefore

$$\left(\sum_{i=1}^{n} a_i\right)^k \left(\sum_{i=1}^{n} \frac{1}{a_i}\right)^k \geq \frac{n^{2k}}{\binom{n}{k}^2}\left(\sum_{cyclic} x_1 ... x_k\right)\left(\sum_{cyclic} \frac{1}{x_1 x_2 ... x_k}\right) \geq n^{2k}$$

Theorem 5. If $a_i, p_i > 0$ $(i = 1, 2, ..., n)$, $k \in N$, and

$$F(k) = \left(\sum_{i=1}^{n} p_i a_i^k\right)\left(\sum_{i=1}^{n} \frac{p_i}{a_i^k}\right), \text{ then}$$

$$F(k) \geq \frac{1}{\left(\sum\limits_{i=1}^{n} p_i\right)^2} F(k-1) F(1) \geq \frac{1}{\left(\sum\limits_{i=1}^{n} p_i\right)^4} F(k-2) F^2(1) \geq ...$$

$$\geq \frac{1}{\left(\sum\limits_{i=1}^{n} p_i\right)^{2k-2}} F^k(1) \geq \left(\sum_{i=1}^{n} p_i\right)^2$$

Proof. Because $\left(a_1^k, a_2^k, ..., a_n^k\right)$ and $\left(a_1^{k-1}, a_2^{k-1}, ..., a_n^{k-1}\right)$ are same ordoned, then from Chebyshev's inequality we get

$$\frac{\sum\limits_{i=1}^{n} p_i a_i^k}{\sum\limits_{i=1}^{n} p_i} \geq \frac{\sum\limits_{i=1}^{n} p_i a_i^{k-1}}{\sum\limits_{i=1}^{n} p_i} \cdot \frac{\sum\limits_{i=1}^{n} p_i a_i}{\sum\limits_{i=1}^{n} a_i} \text{ or}$$

$$\sum_{i=1}^{n} p_i a_i^k \geq \frac{1}{\sum\limits_{i=1}^{n} p_i}\left(\sum_{i=1}^{n} p_i a_i^{k-1}\right)\left(\sum_{i=1}^{n} p_i a_i\right) \text{ and}$$

$$\sum_{i=1}^{n} \frac{p_i}{a_i^k} \geq \frac{1}{\sum\limits_{i=1}^{n} p_i}\left(\sum_{i=1}^{n} \frac{p_i}{a_i^{k-1}}\right)\left(\sum_{i=1}^{n} \frac{p_i}{a_i}\right)$$

therefore after multiplication yields.

$$F(k) \geq \frac{1}{\left(\sum\limits_{i=1}^{n} p_i\right)^2} F(k-1) F(1)$$

and iterating this we finish the proof of the theorem.

Corollary 5.1. If $a_i, p_i > 0$ $(i = 1, 2, ..., n)$, $k \in N$, and

$$F(k) = \left(\sum_{i=1}^{n} p_i a_i^k\right)\left(\sum_{i=1}^{n} \frac{p_i}{a_i^k}\right), \text{ then}$$

$$F\left(k\right)\geq F\left(k-1\right)\geq ...\geq F\left(1\right)\geq F\left(0\right)=\left(\sum_{i=1}^{n}p_i\right)^2$$

Proof. From Theorem 5 we have

$$F\left(k\right)\geq \frac{1}{\left(\sum\limits_{i=1}^{n}p_i\right)^2}F\left(k-1\right)F\left(1\right)\geq F\left(k-1\right),\text{ because}$$

$$F\left(1\right)\geq \left(\sum_{i=1}^{n}p_i\right)^2.$$

Iterating $F\left(k\right)\geq F\left(k-1\right)$ we get the Corollary.

Theorem 6. If $a_i,p_i>0\ \left(i=1,2,...,n\right),\lambda_1\geq\lambda_2\geq..\geq\lambda_k\geq 0,$

$$F\left(\lambda_r\right)=\left(\sum_{i=1}^{n}p_ia_i^{\lambda_r}\right)\left(\sum_{i=1}^{n}\frac{p_i}{a_i^{\lambda_r}}\right),\text{ then}$$

$$F\left(\lambda_1\right)\geq \frac{1}{\left(\sum\limits_{i=1}^{n}p_i\right)^2}F\left(\lambda_2\right)F\left(\lambda_1-\lambda_2\right)\geq$$

$$\geq \frac{1}{\left(\sum\limits_{i=1}^{n}p_i\right)^4}F\left(\lambda_3\right)F\left(\lambda_1-\lambda_2\right)F\left(\lambda_2-\lambda_3\right)\geq ...$$

$$\geq \frac{1}{\left(\sum\limits_{i=1}^{n}p_i\right)^{2k-2}}F\left(\lambda_k\right)F\left(\lambda_1-\lambda_2\right)F\left(\lambda_2-\lambda_3\right)...F\left(\lambda_{k-1}-\lambda_k\right)\geq\left(\sum_{i=1}^{n}p_i\right)^2$$

Proof. Because $\left(a_1^{\lambda_1},a_2^{\lambda_2},...,a_n^{\lambda_1}\right)$ and $\left(a_1^{\lambda_2},a_2^{\lambda_2},...,a_n^{\lambda_2}\right)$ are same ordoned, from Chebishev‘s inequality we get

$$\frac{\sum\limits_{i=1}^{n}p_ia_i^{\lambda_1}}{\sum\limits_{i=1}^{n}p_i}\geq\frac{\sum\limits_{i=1}^{n}p_ia_i^{\lambda_2}}{\sum\limits_{i=1}^{n}p_i}\cdot\frac{\sum\limits_{i=1}^{n}p_ia_i^{\lambda_1-\lambda_2}}{\sum\limits_{i=1}^{n}p_i}\text{ and}$$

$$F\left(\lambda_1\right) \geq \frac{1}{\left(\sum\limits_{i=1}^{n} p_i\right)^2} F\left(\lambda_2\right) F\left(\lambda_1-\lambda_2\right)$$

iterating this we obtain the result.

Corollary 6.1. If $a_i, p_i > 0$, $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_k \geq 0$, and

$$F\left(\lambda_r\right) = \left(\sum_{i=1}^{n} p_i a_i^{\lambda_r}\right)\left(\sum_{i=1}^{n} \frac{p_i}{a_i^{\lambda_r}}\right), \text{ then}$$

$$F\left(\lambda_1\right) \geq F\left(\lambda_2\right) \geq ... \geq F\left(\lambda_k\right) \geq \left(\sum_{i=1}^{n} p_i\right)^2$$

Proof. From Theorem 6 we have

$$F\left(\lambda_1\right) \geq \frac{1}{\left(\sum\limits_{i=1}^{n} p_i\right)^2} F\left(\lambda_1\right) F\left(\lambda_1-\lambda_2\right) \geq F\left(\lambda_1\right), \text{ because}$$

$$F\left(\lambda_1-\lambda_2\right) \geq \left(\sum_{i=1}^{n} p_i\right)^2.$$

Iterating this we finish the proof.

Corollary 6.2. If $a_i, p_i > 0\ (i = 1, 2, ..., n), t_j > 0\ (j = 1, 2, ..., m)$, and

$$F\left(t_j\right) = \left(\sum_{i=1}^{n} p_i a_i^{t_j}\right)\left(\sum_{i=1}^{n} \frac{p_i}{a_i^{t_j}}\right), \text{ then}$$

$$\prod_{j=1}^{m} F\left(t_j\right) \leq \left(\sum_{i=1}^{n} p_i\right)^{2m-2} F\left(\sum_{j=1}^{m} t_j\right)$$

Proof. We have

$$F\left(t_1\right) F\left(t_2\right) \leq \left(\sum_{i=1}^{n} p_i\right)^2 F\left(t_1+t_2\right)$$

and after iteration we get the result.

Corollary 6.3. If $a_i, p_i > 0$ $(i = 1, 2, ..., n)$, $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_k \geq 0$ and

$$F(\lambda_r) = \left(\sum_{i=1}^{n} p_i a_i^{\lambda_r}\right)\left(\sum_{i=1}^{n} \frac{p_i}{a_i^{\lambda_r}}\right), \text{ then}$$

$$F(\lambda_1) \geq \frac{1}{\left(\sum\limits_{i=1}^{n} p_i\right)^2} F(\lambda_2) F(\lambda_2 - \lambda_3) \geq F(\lambda_2) \geq \frac{1}{\left(\sum\limits_{i=1}^{n} p_i\right)^2} F(\lambda_3) F(\lambda_3 - \lambda_4) \geq$$

$$\geq F(\lambda_3) \geq ... \geq \frac{1}{\left(\sum\limits_{i=1}^{n} p_i\right)^2} F(\lambda_{k-1}) F(\lambda_{k-1} - \lambda_k) \geq F(\lambda_k) \geq \left(\sum_{i=1}^{n} p_i\right)^2.$$

Proof. We have

$$F(\lambda_1) \geq \frac{1}{\left(\sum\limits_{i=1}^{n} p_i\right)^2} F(\lambda_2) F(\lambda_1 - \lambda_2) \geq F(\lambda_2), \text{ because}$$

$$F(\lambda_1 - \lambda_2) \geq \left(\sum_{i=1}^{n} p_i\right)^2.$$

The result holds by iteration.

Corollary 6.4. If $x, y, z, u, v, t > 0$, then

$$\left(\sum xu^2\right)\left(\sum \frac{x}{u^2}\right) \geq \frac{\left(\sum xu\right)^2 \left(\sum \frac{x}{u}\right)^2}{\left(\sum x\right)^2} \geq \left(\sum xu\right)\left(\sum \frac{x}{u}\right) \geq \left(\sum x\right)^2.$$

Proof. In Corollary 6.3 we take $n = 3, \lambda_1 = 2, \lambda_2 = 1$.

Corollary 6.5. If $x, y, z > 0$, then

$$\left(\sum x^3\right)\left(\sum \frac{1}{x}\right) \geq \frac{9\left(\sum x^2\right)^2}{\left(\sum x\right)^2} \geq 3\sum x^2 \geq \left(\sum x\right)^2.$$

Proof. In Corollary 6.4 we take $u = x, v = y, z = t$.

Corollary 6.6. In all triangle ABC holds

1). $\frac{\left(s^2-3r^2-6Rr\right)\left(s^2+r^2+4Rr\right)}{2Rr} \geq 9\left(\frac{s^2-r^2-4Rr}{s}\right)^2 \geq 6\left(s^2-r^2-4Rr\right) \geq 4s^2$

2). $\frac{\left(s^2-12Rr\right)(4R+r)}{r} \geq 9\left(\frac{s^2-2r^2-8Rr}{s}\right)^2 \geq 3\left(s^2-2r^2-8Rr\right) \geq s^2$

3). $\frac{(4R+r)^3-12s^2R}{r} \geq 9\left(\frac{(4R+r)^2-2s^2}{4R+r}\right)^2 \geq 3\left((4R+r)^2-2s^2\right) \geq (4R+r)^2$

4). $\frac{\left((2R-r)\left((4R+r)^2-3s^2\right)+6Rr^2\right)\left(s^2+r^2-8Rr\right)}{32R^3r^2} \geq 9\left(\frac{8R^2+r^2-s^2}{4(2R-r)R}\right)^2 \geq \frac{3\left(8R^2+r^2-s^2\right)}{8R^2} \geq$
$\left(\frac{2R-r}{2R}\right)^2$

5). $\frac{\left((4R+r)^3-3s^2(2R+r)\right)\left(s^2+(4R+r)^2\right)}{32R^3s^2} \geq 9\left(\frac{(4R+r)^2-s^2}{4(4R+r)R}\right)^2 \geq$
$\geq \frac{3\left((4R+r)^2-s^2\right)}{8R^2} \geq \left(\frac{4R+r}{2R}\right)^2$

Proof. In Corollary 6.5 we take
$(x,y,z) \in \left\{(a,b,c);(s-a,s-b,s-c);(r_a,r_b,r_c);\left(\sin^2\frac{A}{2},\sin^2\frac{B}{2},\sin^2\frac{C}{2}\right);\right.$
$\left.\left(\cos^2\frac{A}{2},\cos^2\frac{B}{2},\cos^2\frac{C}{2}\right)\right\}$

Corollary 6.7. If $x,y,z>0$, then

$$\left(\sum x^3\right)\left(\sum\frac{1}{x^3}\right)\left(\sum x\right)\left(\sum\frac{1}{x}\right) \geq \frac{81\left(\sum x^2\right)\left(\sum\frac{1}{x^2}\right)^2}{\left(\sum x\right)^2\left(\sum\frac{1}{x}\right)^2} \geq$$

$$\geq 3\left(\sum x^2\right)\left(\sum\frac{1}{x^2}\right) \geq \left(\sum x\right)^2\left(\sum\frac{1}{x}\right)^2 \geq 81.$$

Proof. We multiplying the inequalities from Corollary 6.5 for (x,y,z) and for $\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right)$.

Corollary 6.8. If $f,g:R\to(0,+\infty)$ are integrable on $[a,b]$, $\lambda_1\geq\lambda_2\geq...\geq\lambda_k\geq 0$ and

$$G\left(\lambda_r\right)=\left(\int_a^b g(x)\,f^{\lambda_r}(x)\,dx\right)\left(\int_a^b\frac{g(x)\,dx}{f^{\lambda_r}(x)}\right),\text{ then}$$

$$G\left(\lambda_1\right)\geq\frac{G\left(\lambda_2\right)G\left(\lambda_1-\lambda_2\right)}{\left(\int_a^b g(x)\,dx\right)^2}\geq\frac{G\left(\lambda_3\right)G\left(\lambda_1-\lambda_2\right)G\left(\lambda_2-\lambda_3\right)}{\left(\int_a^b g(x)\,dx\right)^4}\geq...$$

$$\geq\frac{G\left(\lambda_k\right)G\left(\lambda_1-\lambda_2\right)...G\left(\lambda_{k-1}-\lambda_k\right)}{\left(\int_a^b g(x)\,dx\right)^{2k-2}}\geq\left(\int_a^b g(x)\,dx\right)$$

Proof. In Theorem 6 we take $\alpha_i = a + \frac{(b-a)i}{n}$ $(i = 1, 2, ..., n)$, $a_i = f(\alpha_i)$, $p_i = g(\alpha_i)$, $\alpha_i - \alpha_{i-1} = \frac{b-a}{n}$ and after then we take $n \to \infty$.

Corollary 6.9. If $f, g : R \to (0, +\infty)$ are integrable on $[a, b]$, $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_k \geq 0$ and

$$G(\lambda_r) = \left(\int_a^b g(x) f^{\lambda_r}(x)\, dx\right)\left(\int_a^b \frac{g(x)\, dx}{f^{\lambda_r}(x)}\right), \text{ then}$$

$$G(\lambda_1) \geq G(\lambda_2) \geq ... \geq G(\lambda_k) \geq \left(\int_a^b g(x)\, dx\right)^2.$$

Proof. In Corollary 6.1 we apply the proof of Corollary 6.8.

Corollary 6.10. If $f, g : R \to (0, +\infty)$ are integrable on $[a, b]$, $t_j > 0$ $(j = 1, 2, ..., m)$ and

$$G(t_j) = \left(\int_a^b g(x) f^{t_j}(x)\, dx\right)\left(\int_a^b \frac{g(x)\, dx}{f^{t_j}(x)}\right), \text{ then}$$

$$\prod_{j=1}^{m} G(t_j) \leq \left(\int_a^b g(x)\, dx\right)^{2m-2} G\left(\sum_{j=1}^{m} t_j\right)$$

Proof. See the proofs of Corollary 6.2 and Corollary 6.8.

Corollary 6.11. If $f, g : R \to (0, +\infty)$ are integrable on $[a, b]$, $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_k \geq 0$ and

$$G(\lambda_r) = \left(\int_a^b g(x) f^{\lambda_r}(x)\, dx\right)\left(\int_a^b \frac{g(x)\, dx}{f^{\lambda_r}(x)}\right), \text{ then}$$

$$G(\lambda_1) \geq \frac{G(\lambda_2)\, G(\lambda_2 - \lambda_3)}{\left(\int_a^b g(x)\, dx\right)^2} \geq G(\lambda_2) \geq \frac{G(\lambda_3)\, G(\lambda_3 - \lambda_4)}{\left(\int_a^b g(x)\, dx\right)^2} \geq G(\lambda_3) \geq ...$$

$$\geq \frac{G\left(\lambda_{k-1}\right) G\left(\lambda_{k-1}-\lambda_k\right)}{\left(\int\limits_a^b g\left(x\right) dx\right)^2} \geq G\left(\lambda_k\right) \geq \left(\int\limits_a^b g\left(x\right) dx\right)^2$$

Proof. See the proofs of Corollary 6.3 and Corollary 6.8.

Open Question 1. If $a_i > 0$ $(i = 1, 2, ..., n)$ and

$$F\left(k\right) = \left(\sum_{cyclic} \frac{a_1^k}{a_1^{k-1} + a_2 a_3 ... a_k}\right) \left(\sum_{cyclic} \frac{a_2 a_3 ... a_k}{a_1\left(a_1^{k-1} + a_2 a_3 ... a_k\right)}\right), \text{ then}$$

$$F\left(k\right) \geq F\left(k-1\right) \geq ... \geq F\left(0\right).$$

Remark. *Using the inequality* $\frac{x^3}{x^2+yz} \geq \frac{4x-y-z}{4}$ *we obtain that*

$$F\left(3\right) \geq \frac{1}{4}\left(\sum_{i=1}^n a_i\right)\left(\sum_{i=1}^n \frac{1}{a_i}\right) \geq \frac{n^2}{4}.$$

Open Question 2. If $a_i > 0$ $(i = 1, 2, ..., n)$ and

$$G\left(k\right) = \left(\sum_{cyclic} \frac{a_1^{k-2}}{a_1^{k-1} + a_2 a_3 ... a_k}\right) \left(\sum_{cyclic} \frac{a_1 a_2 ... a_k}{a_1\left(a_1^{k-1} + a_2 a_3 ... a_k\right)}\right), \text{ then}$$

$$G\left(k\right) \geq G\left(k-1\right) \geq ... \geq G\left(0\right).$$

Remark. *Using the inequality* $\frac{x^2}{x^2+yz} \leq \frac{1}{4}\left(\frac{1}{y} + \frac{1}{z}\right)$ *we obtain*

$$G\left(3\right) \leq \frac{1}{4}\left(\sum_{i=1}^n a_i\right)\left(\sum_{i=1}^n \frac{1}{a_i}\right).$$

Open Question 3. If $a_i > 0$ $(i = 1, 2, ..., n)$ and

$$H\left(k\right) = \left(\sum_{i=1}^n a_i^{k-2}\right)\left(\sum_{i=1}^n \frac{1}{a_i^{k-1}}\right), \text{ then}$$

1). $F\left(k\right) \geq \frac{1}{4} H\left(k\right) \geq G\left(k\right)$

2). $F(k) \geq \frac{1}{4}H(k) \geq G(K) \geq F(k-1) \geq \frac{1}{4}H(k-1) \geq G(k-1) \geq ...$

Open Question 4. If $a_i > 0$ $(i = 1, 2, ..., n)$ and

$$L(k) = \left(\sum_{cyclic} \frac{a_1^k}{a_1^{k-1} + a_2^{k-1}}\right)\left(\sum_{cyclic} \frac{a_2^{k-1}}{a_1\left(a_1^{k-1} + a_2^{k-1}\right)}\right), \text{ then}$$

$$L(k) \geq L(k-1) \geq .. \geq L(0).$$

Remark. *Using the inequality* $\frac{x^2}{x+y} \geq \frac{3x-y}{4}$ *we obtain*

$$L(2) \geq \frac{1}{4}\left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} \frac{1}{a_i}\right)$$

Open Question 5. If $a_i > 0$ $(i = 1, 2, ..., n)$ and

$$M(k) = \left(\sum_{cyclic} \frac{a_1^{k-1}}{a_1^k + a_2^k}\right)\left(\sum_{cyclic} \frac{a_1 a_2^k}{a_1^k + a_2^k}\right), \text{ then}$$

$$M(k) \geq M(k-1) \geq ... \geq M(0).$$

Remark. *Using the inequality* $\frac{x}{x^2+y^2} \geq \frac{1}{2y}$ *we obtain*

$$M(2) \leq \frac{1}{4}\left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} \frac{1}{a_i}\right).$$

Open Question 6. If $a_i > 0$ $(i = 1, 2, ..., n)$ and

$$H(k) = \left(\sum_{i=1}^{n} a_i^{k-2}\right)\left(\sum_{i=1}^{n} \frac{1}{a_i^{k-2}}\right), \text{ then}$$

1). $L(k) \geq \frac{1}{4}H(k) \geq M(k)$

2). $L(k) \geq \frac{1}{4}H(k) \geq M(k) \geq L(k-1) \geq \frac{1}{4}H(k-1) \geq M(k-1) \geq ...$

Open Question 7. If $a_i > 0$ $(i = 1, 2, ..., n)$ and

$$N(k) = \left(\sum_{cyclic} \frac{a_1^{k+1}}{a_1 + a_2}\right)\left(\sum_{cyclic} \frac{a_2}{a_1^k (a_1 + a_2)}\right), \text{ then}$$

$$N(k) \geq N(k-1) \geq ... \geq N(0).$$

Remark. *We have* $N(1) \geq \frac{1}{4}H(1)$ *and* $N(2) \geq \frac{1}{4}H(2)$ *etc.*

REFERENCES

[1] Bencze, M., *Egyenlőtlenségekről,* (In Hungarian), Matematikai Lapok (Kolozsvár), Nr. 2, 1976, pp. 49-54.
[2] Octogon Mathematical Magazine (1993-2009)
[3] Bencze, M., *A new proof of CBS inequality,* Octogon Mathematical Magazine, Vol. 10, Nr. 2, October 2002, pp. 841-842.

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A Generalization of the logarithmic and the Gauss means

Mihály Bencze [8]

ABSTRACT. In this paper we present a generalization of the classical logarithm and Gauss means, and we give some interesting applications.

MAIN RESULTS
1. THE LOGARITHMIC MEAN

The logarithmic mean of two positive numbers a and b is the number $L(a,b)$ defined as

$$L(a,b)=\begin{cases}\frac{a-b}{\ln a-\ln b} & \text{if} \quad a\neq b\\ a & \text{if} \quad a=b\end{cases}$$

see [1]. G. Hardy, J.E. Littlewood and G. Pólya have discovered many applications of logarithmic mean. Their book Inequalities has had quite a few succesors, and yet new properties of these means continue to be discovered.

Definition 1.1. If $a_k>0$ $(k=1,2,...,n)$ and $a_i\neq a_j$ $(i,j\in\{1,2,...,n\},i\neq j)$ $n\geq 2$, then

$$L(a_1,a_2,...,a_n)=\frac{1}{\left(-(n-1)\sum\limits_{k=1}^{n}u_k\ln a_k\right)^{\frac{1}{n-1}}} \tag{1.1}$$

denote the logarithmic mean of positive numbers $a_1,a_2,...,a_n$, where

$$u_k=\frac{1}{(a_1-a_k)...(a_{k-1}-a_k)(a_{k+1}-a_k)...(a_n-a_k)}$$

for all $k\in\{1,2,...,n\}$.
If $n=2$, then $u_1=\frac{-1}{a_2-a_1}$, $u_2=\frac{1}{a_2-a_1}$ and

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$L\left(a_1, a_2\right) = \frac{1}{u_1 \ln a_1 + u_2 \ln a_2} = \frac{a_2 - a_1}{\ln a_2 - \ln a_1}$, therefore we reobtain the classical logarithmic mean.

Remark 1.2. *By a simple calculation we get:*

$$\sum_{k=1}^{n} u_k = 0 \tag{1.2}$$

Theorem 1.3. We have the following relation:

$$\frac{1}{L\left(a_1, a_2, ..., a_n\right)} = \left((n-1) \int_0^\infty \frac{dx}{\prod\limits_{k=1}^{n} \left(x + a_k\right)} \right)^{\frac{1}{n-1}} \tag{1.3}$$

Proof. We have the following descomposition:

$$\frac{1}{\prod\limits_{k=1}^{n} \left(x + a_k\right)} = -\sum_{k=1}^{n} \frac{u_k}{x + a_k},$$

therefore

$$\int_0^\infty \frac{dx}{\prod\limits_{k=1}^{n} \left(x + a_k\right)} = -\sum_{k=1}^{n} u_k \int_0^\infty \frac{dx}{x + a_k} = -\sum_{k=1}^{n} u_k \ln\left(x + a_k\right)\Big|_0^\infty =$$

$$= -\ln \prod_{k=1}^{n} \left(x + a_k\right)^{u_k}\Big|_0^\infty = \sum_{k=1}^{n} u_k \ln a_k$$

Because $\sum\limits_{k=1}^{n} u_k = 0$, then we obtain

$$\frac{1}{L\left(a_1, a_2, ..., a_n\right)} = \left((n-1) \int_0^\infty \frac{dx}{\prod\limits_{k=1}^{n} \left(x + a_k\right)} \right)^{\frac{1}{n-1}} = \left(-(n-1) \sum_{k=1}^{n} u_k \ln a_k \right)^{\frac{1}{n-1}}$$

which finish the proof.

In following we denote $A\left(a_1, a_2, ..., a_n\right) = \frac{1}{n} \sum\limits_{k=1}^{n} a_k$ the arithmetic and $G\left(a_1, a_2, ..., a_n\right) = \sqrt[n]{\prod\limits_{k=1}^{n} a_k}$ the geometric mean.

Remark 1.3. *The mean L can also be expressed in terms of a divided difference, of order n-1, of the logarithmic function*

Theorem 1.4. We have the following inequalities:

$$G\left(a_1, a_2, ..., a_n\right) \leq L\left(a_1, a_2, ..., a_n\right) \leq A\left(a_1, a_2, ..., a_n\right) \tag{1.4}$$

Proof. Using the $AM - GM$ inequality we obtain

$$\prod_{k=1}^{n}\left(x + a_k\right) \leq (x + A)^n$$

From Huygen's inequality we get

$$\prod_{k=1}^{n}\left(x + a_k\right) \geq (x + G)^n,$$

therefore

$$\frac{1}{(n-1)\,A^{n-1}} = \int\limits_0^\infty \frac{dx}{(x+A)^n} \leq \int\limits_0^\infty \frac{dx}{\prod\limits_{k=1}^{n}\left(x + a_k\right)} = \frac{1}{(n-1)\,L^{n-1}} \leq$$

$$\leq \int\limits_0^\infty \frac{dx}{(x+G)^n} = \frac{1}{(n-1)\,G^{n-1}}$$

which finish the proof.

Corollary 1.4.1. If $x_k \in R$ $(k = 1, 2, ..., n)$ are different, then we have the following inequalities:

$$e^{A(x_1, x_2, ..., x_n)} \leq \frac{1}{(n-1)\left(\sum\limits_{k=1}^{n} \frac{-x_k}{(e^{x_1}-e^{x_2})...(e^{x_{k-1}}-e^{x_k})(e^{x_{k+1}}-e^{x_k})...(e^{x_n}-e^{x_1})}\right)^{\frac{1}{n-1}}} \leq$$

$$\leq A\left(e^{x_1}, e^{x_2}, ..., e^{x_n}\right) \tag{1.5}$$

Proof. In Theorem 1.4 we take

$$a_k = e^{x_k} \;\; (k = 1, 2, ..., n)$$

Corollary 1.4.2. If $t \geq 0$, then

$$tanht \leq t \leq sinht \tag{1.6}$$

Proof. In (1.5) we take $n = 2$, and after elementary calculus we get

$$tanh\frac{x_1 - x_2}{2} \leq \frac{x_1 - x_2}{2} \leq sinh\frac{x_1 - x_2}{2},$$

and in these we denote $t = \frac{x_1 - x_2}{2} \geq 0$.
These inequalities are fundamental inequalities in analysis, therefore Corollary 1.4.1 offer a lot of generalizations of these.

Corollary 1.4.3. If $a_k > 0$ $(k = 1, 2, ..., n)$ are different, then we have the following inequalities:

$$G\left(a_1, a_2, ..., a_n\right) \leq \frac{1}{\left(-\left(n-1\right)\sum\limits_{k=1}^{n} \frac{\ln\frac{a_1}{a_k} ... \ln\frac{a_{k-1}}{a_k} \ln\frac{a_{k+1}}{a_k} ... \ln\frac{a_n}{a_k} \ln a_k}{L(a_1,a_k)...L(a_{k-1},a_k)L(a_{k+1},a_k)...L(a_n,a_k)}\right)^{\frac{1}{n-1}}} \leq$$

$$\leq A\left(a_1, a_2, ..., a_n\right) \tag{1.7}$$

Proof. In Theorem 1.4 we consider the substitutions

$$a_i - a_j = \frac{L\left(a_i, a_j\right)}{\ln\frac{a_i}{a_j}} \quad \left(i, j \in \left\{1, 2, ..., n\right\},\ i \neq j\right) \tag{1.8}$$

Using the inequalities $G\left(a_i, a_j\right) \leq L\left(a_i, a_j\right) \leq A\left(a_i, a_j\right)$ we obtain from corollary 1.4.3 a cathegory of new inequalities.

Remark 1.5. *If in Corollary 1.4.3 we replace a_k by a_k^t $(k = 1, 2, ..., n)$, respectively, then we obtain a lot of new inequalities.*
By example, if we consider $n = 2$, then we obtain

$$G_t\left(a_1, a_2\right) = G\left(a_1^t, a_2^t\right) \leq \frac{a_2^t - a_1^t}{t\left(\ln a_2 - \ln a_1\right)} \leq A\left(a_1^t, a_2^t\right) = A_t\left(a_1, a_2\right) \tag{1.9}$$

or

$$\overline{G}_t\left(a_1, a_2\right) = tG_t\left(a_1, a_2\right)\frac{a_2 - a_1}{a_2^t - a_1^t} \leq L\left(a_1, a_2\right) \leq tA_t\left(a_1, a_2\right)\frac{a_2 - a_1}{a_2^t - a_1^t} =$$

$$= \overline{A}_t\left(a_1, a_2\right) \tag{1.10}$$

with properties: $\overline{G}_{-t}\left(a_1, a_2\right) = \overline{G}_t\left(a_1, a_2\right), \overline{A}_{-t}\left(a_1, a_2\right) = \overline{A}_t\left(a_1, a_2\right),$
$\overline{G}_0\left(a_1, a_2\right) = \overline{A}_0\left(a_1, a_2\right) = L\left(a_1, a_2\right), \overline{G}_1\left(a_1, a_2\right) = G\left(a_1, a_2\right),$
$\overline{A}_1\left(a_1, a_2\right) = A\left(a_1, a_2\right).$
For fixed a_1 *and* a_2, $\overline{G}_t\left(a_1, a_2\right)$ *is a decreasing function of* $|t|$, *and* $\overline{A}_t\left(a_1, a_2\right)$ *is an increasing function of* $|t|$, *therefore* $\overline{G}_t\left(a_1, a_2\right) \leq L\left(a_1, a_2\right) \leq \overline{A}_t\left(a_1, a_2\right)$ *is a very fundamental inequality (see [3]).*
In same way the idea of Corollary 1.4.3 and of remark 1.5 can be continued with hard calculus, and we introduced in same way the new means $\overline{G}_t\left(a_1, a_2, ..., a_n\right)$ *and* $\overline{A}_t\left(a_1, a_2, ..., a_n\right)$ *for which we obtain the fundamental inequalities* $\overline{G}_t\left(a_1, a_2, ..., a_n\right) \leq L\left(a_1, a_2, ..., a_n\right) \leq \overline{A}_t\left(a_1, a_2, ..., a_n\right)$ *which offer for all* $t \in R$ *a lot of new refinements for the inequalities proved in Theorem 1.4.*

Remark 1.6. *If in Remark 1.5 we choose* $t = 2^{-m}$ $(m \in N)$, *then we obtain*

$$\overline{G}_{2^{-m}}\left(a_1, a_2\right) \leq L\left(a_1, a_2\right) \leq \overline{A}_{2^{-m}}\left(a_1, a_2\right) \tag{1.11}$$

After elementary calculus we get

$$\overline{G}_{2^{-(m+1)}}\left(a_1, a_2\right) \prod_{k=1}^{m} A\left(a_1^{2^{-k}}, a_2^{2^{-k}}\right) \leq L\left(a_1, a_2\right) \leq$$

$$\leq \overline{A}_{2^{-(m+1)}}\left(a_1, a_2\right) \prod_{k=1}^{m} A\left(a_1^{2^{-k}}, a_2^{2^{-k}}\right)$$

If we let $m \to \infty$ in two formulas above, then

$$L\left(a_1, a_2\right) = \prod_{k=1}^{\infty} A\left(a_1^{2^{-k}}, a_2^{2^{-k}}\right) \tag{1.12}$$

(See [3])
Using these inductively for

$$\overline{G}_{\overline{2}^m}\left(a_1, a_2, ..., a_n\right) \leq L\left(a_1, a_2, ..., a_n\right) \leq \overline{A}_{\overline{2}^m}\left(a_1, a_2, ..., a_n\right)$$

we obtained a product formula for $L\left(a_1, a_2, ..., a_n\right)$.

2. THE GAUSS MEAN

Given positive numbers a and b, inductively define two sequences as
$a_0 = a, b_0 = b$, $a_{n+1} = A\left(a_n, b_n\right), b_{n+1} = G\left(a_n, b_n\right).$
Then $\left(a_n\right)_{n\geq 0}$ is decreasing, and $\left(b_n\right)_{n\geq 0}$ is increasing. All a_n and b_n are between a and b. So both sequence converge. By induction we obtain $a_{n+1-b_{n+1}} \leq \frac{1}{2}\left(a_n - b_n\right)$, and hence the sequences converge to a common limit, denoted by $AG\left(a,b\right)$ which is called the Gauss arithmetic-geometric mean.
Gauss showed that

$$\frac{1}{AG\left(a,b\right)} = \frac{2}{\pi}\int\limits_0^\infty \frac{dx}{\sqrt{\left(x^2+a^2\right)\left(x^2+b^2\right)}} \tag{2.1}$$

and $G\left(a,b\right) \leq AG\left(a,b\right) \leq A\left(a,b\right).$

Definition 2.1. If $a_k > 0\ \left(k = 1,2,...,n\right)$, then the Gauss mean of $a_1, a_2, ..., a_n$ is defined in following way

$$\frac{1}{AG\left(a_1,a_2,...,a_n\right)} = \frac{2}{\pi}\int\limits_0^\infty \left(\frac{1}{\prod\limits_{k=1}^n \left(x^2+a_k^2\right)}\right)^{\frac{1}{n}} dx. \tag{2.2}$$

If $Q\left(a_1,a_2,...,a_n\right) = \sqrt{\frac{1}{n}\sum\limits_{k=1}^n a_k^2}$, then we obtain the following:

Theorem 2.2. We have the following inequalities:

$$G\left(a_1,a_2,...,a_n\right) \leq AG\left(a_1,a_2,...,a_n\right) \leq Q\left(a_1,a_2,...,a_n\right) \tag{2.3}$$

Proof. Using the AM-GM inequality we have

$$\prod_{k=1}^n \left(x^2+a_k^2\right)^{\frac{1}{n}} \leq x^2 + Q^2$$

From Huygen's inequality we obtain

$$\prod_{k=1}^n \left(x^2+a_k^2\right)^{\frac{1}{n}} \geq x^2 + G^2$$

therefore

$$\frac{1}{Q\left(a_1,a_2,...,a_n\right)}=\frac{2}{\pi}\int\limits_0^{\infty}\frac{dx}{x^2+Q^2}\leq\frac{2}{\pi}\int\limits_0^{\infty}\frac{dx}{\left(x^2+a_k^2\right)^{\frac{1}{n}}}=AG\left(a_1,a_2,...,a_n\right)\leq$$

$$\leq\frac{2}{\pi}\int\limits_0^{\infty}\frac{dx}{x^2+G^2}=\frac{1}{G\left(a_1,a_2,...,a_n\right)}$$

Definition 2.3. If $a_k>0$ $(k=1,2,...,n)$, then

$$\frac{1}{B_\alpha\left(a_1,a_2,...,a_n\right)}=c_\alpha\left((n-1)\int\limits_0^{\infty}\frac{dx}{\prod\limits_{k=1}^{n}\left(x^\alpha+a_k^\alpha\right)^{\frac{1}{\alpha}}}\right)^{\frac{1}{n-1}} \tag{2.4}$$

when c_α is a constant, depend only of $\alpha\in R$. This mean generalize the logarithmic and the Gauss means too.

Theorem 2.4. If $a_k>0$ $(k=1,2,...,n)$ and $\alpha\in(-\infty,0]\cup[1,+\infty)$, then $L\left(a_1,a_2,...,a_n\right)\leq c_\alpha 2^{\frac{n(\alpha-1)}{(n-1)\alpha}}B_\alpha\left(a_1,a_2,...,a_n\right)$ and if $\alpha\in(0,1)$ then holds the reverse inequality.

Proof. We have the inequalities

$$\left(x^\alpha+a_k^\alpha\right)^{\frac{1}{\alpha}}\geq\frac{x+a_k}{2^{1-\frac{1}{\alpha}}}$$

$(k=1,2,...,n)$, therefore

$$\frac{1}{B_\alpha\left(a_1,a_2,...,a_n\right)}=c_\alpha\left((n-1)\int\limits_0^{\infty}\frac{dx}{\prod\limits_{k=1}^{n}\left(x^\alpha+a_k^\alpha\right)^{\frac{1}{\alpha}}}\right)^{\frac{1}{n-1}}\leq$$

$$\leq c_\alpha 2^{\frac{n(\alpha-1)}{(n-1)\alpha}}\left((n-1)\int\limits_0^{\infty}\frac{dx}{\prod\limits_{k=1}^{n}\left(x+a_k\right)}\right)^{\frac{1}{n-1}}$$

If $\alpha=1$ and $c_1=1$, then $B_1\left(a_1,a_2,...,a_n\right)=L\left(a_1,a_2,...,a_n\right)$ and if $\alpha=2$, $c_2=\frac{2}{\pi}$, $n=2$, then $B_2\left(a_1,a_2\right)=AG\left(a_1,a_2\right)$.

REFERENCES

[1] Hardy, G., Littlewood, J.E. and Pólya, G., *Inequalities,* Cambridge University Press, Second edition, 1952.
[2] Bullen,P.S., Mitrinovic, D.S. and Vasic, P.M., *Means and their inequalities,* D. Reidel, 1998.
[3] Bhatia, R., *The logarithmic mean,* Resonance, Vol. 13, june 2008, pp. 583-594.

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On some algebraic properties of generalized groups

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ABSTRACT. Some results that are true in classical groups are investigated in generalized groups and are shown to be either generally true in generalized groups or true in some special types of generalized groups. Also, it is shown that a Bol groupoid and a Bol quasigroup can be constructed using a non-abelian generalized group.

1. INTRODUCTION

Generalized group is an algebraic structure which has a deep physical background in the unified guage theory and has direct relation with isotopies. Mathematicians and Physicists have been trying to construct a suitable unified theory for twistor theory, isotopies theory and so on. It was known that generalized groups are tools for constructions in unified geometric theory and electroweak theory. Electorweak theories are essentially structured on Minkowskian axioms and gravitational theories are constructed on Riemannian axioms. According to Araujo et. al. [4], generalized group is equivalent to the notion of completely simple semigroup. Some of the structures and properties of generalized groups have been studied by Vagner [22], Molaei [16], [15], Mehrabi, Molaei and Oloomi [19], Molaei and Hoseini [20] and Agboola [1]. Smooth generalized groups were introduced in Agboola [3] and later on, Agboola [2] also presented smooth generalized subgroups while Molaei [17] and Molaei and Tahmoresi [18] considered the notion of topological generalized groups. Solarin and Sharma [21] were able to construct a Bol loop using a group with a non-abelian subgroup and recently, Chein and Goodaire [6] gave a new construction of Bol loops for odd case. Kuku [14], White [24] and Jacobson [11] contain most of the results on classical groups while for more on loops and their properties, readers should check [20, 5, 7, 8, 9, 12, 23].

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The aim of this study is to investigate if some results that are true in classical group theory are also true in generalized groups and to find a way of constructing a Bol structure (i.e Bol loop or Bol quasigroup or Bol groupoid) using a non-abelian generalized group.
It is shown that in a generalized group G, $(a^{-1})^{-1} = a$ for all $a \in G$. In a normal generalized group G, it is shown that the anti-automorphic inverse property $(ab)^{-1} = b^{-1}a^{-1}$ for all $a, b \in G$ holds under a necessary condition. A necessary and sufficient condition for a generalized group(which obeys the cancellation law and in which $e(a) = e(ab^{-1})$ if and only if $ab^{-1} = a$) to be idempotent is established. The basic theorem used in classical groups to define the subgroup of a group is shown to be true for generalized groups. The kernel of any homomorphism(at a fixed point) mapping a generalized group to another generalized group is shown to be a normal subgroup. Furthermore, the homomorphism is found to be an injection if and only if its kernel is the set of the identity element at the fixed point. Given a generalized group G with a generalized subgroup H, it is shown that the factor set G/H is a generalized group. The direct product of two generalized group is shown to be a generalized group. Furthermore, necessary conditions for a generalized group G to be isomorphic to the direct product of any two abelian generalized subgroups is shown. It is shown that a Bol groupoid can be constructed using a non-abelian generalized group with an abelian generalized subgroup. Furthermore, if is established that if the non-abelian generalized group obeys the cancellation law, then a Bol quasigroup with a left identity element can be constructed.

2. PRELIMINARIES

Definition 2.1. A generalized group G is a non-empty set admitting a binary operation called multiplication subject to the set of rules given below.
(i) $(xy)z = x(yz)$ for all $x, y, z \in G$.
(ii) For each $x \in G$ there exists a unique $e(x) \in G$ such that $xe(x) = e(x)x = x$ (existence and uniqueness of identity element).
(iii) For each $x \in G$, there exists $x^{-1} \in G$ such that $xx^{-1} = x^{-1}x = e(x)$ (existence of inverse element).

Definition 2.2. Let L be a non-empty set. Define a binary operation $(\cdot)$ on L. If $x \cdot y \in L$ for all $x, y \in L$, $(L, \cdot)$ is called a groupoid.
If the equations $a \cdot x = b$ and $y \cdot a = b$ have unique solutions relative to x and y respectively, then $(L, \cdot)$ is called a quasigroup. Furthermore, if there exists

a element $e \in L$ called the identity element such that for all $x \in L$, $x \cdot e = e \cdot x = x$, $(L, \cdot)$ is called a loop.

Definition 2.3. A loop is called a Bol loop if and only if it obeys the identity

$$((xy)z)y = x((yz)y).$$

Remark 2.1. *One of the most studied type of loop is the Bol loop.*

PROPERTIES OF GENERALIZED GROUPS

A generalized group G exhibits the following properties:
(i) for each $x \in G$, there exists a unique $x^{-1} \in G$.
(ii) $e(e(x)) = e(x)$ and $e(x^{-1}) = e(x)$ where $x \in G$. Then, $e(x)$ is a unique identity element of $x \in G$.

Definition 2.4. If $e(xy) = e(x)e(y)$ for all $x, y \in G$, then G is called normal generalized group.

Theorem 2.1. For each element x in a generalized group G, there exists a unique $x^{-1} \in G$.
The next theorem shows that an abelian generalized group is a group.

Theorem 2.2. Let G be a generalized group and $xy = yx$ for all $x, y \in G$. Then G is a group.

Theorem 2.3. A non-empty subset H of a generalized group G is a generalized subgroup of G if and only if for all $a, b \in H$, $ab^{-1} \in H$.
If G and H are two generalized groups and $f : G \to H$ is a mapping then Mehrabi, Molaei and Oloomi [19] called f a homomorphism if $f(ab) = f(a)f(b)$ for all $a, b \in G$.
They also stated the following results on homomorphisms of generalized groups. These results are established in this work.

Theorem 2.4. Let $f : G \to H$ be a homomorphism where G and H are two distinct generalized groups. Then:
(i) $f(e(a)) = e(f(a))$ is an identity element in H for all $a \in G$.
(ii) $f(a^{-1}) = (f(a))^{-1}$.
(iii) If K is a generalized subgroup of G, then $f(K)$ is a generalized subgroup of H.

(iv) If G is a normal generalized group, then the set

$$\{(e(g), f(g)) \ : \ g \in G\}$$

with the product

$$(e(a), f(a))(e(b), f(b)) := (e(ab), f(ab))$$

is a generalized group denoted by $\cup f(G)$.

3. MAIN RESULTS

Results on Generalized Groups and Homomorphisms

Theorem 3.1. Let G be a generalized group. For all $a \in G$, $(a^{-1})^{-1} = a$.
Proof. $(a^{-1})^{-1}a^{-1} = e(a^{-1}) = e(a)$. Post multiplying by a, we obtain

$$[(a^{-1})^{-1}a^{-1}]a = e(a)a. \tag{1}$$

From the L. H. S.,

$$(a^{-1})^{-1}(a^{-1}a) = (a^{-1})^{-1}e(a) = (a^{-1})^{-1}e(a^{-1}) = (a^{-1})^{-1}e((a^{-1})^{-1}) = (a^{-1})^{-1}. \tag{2}$$

Hence from (1) and (2), $(a^{-1})^{-1} = a$.

Theorem 3.2. Let G be a generalized group in which the left cancellation law holds and $e(a) = e(ab^{-1})$ if and only if $ab^{-1} = a$. G is a idempotent generalized group if and only if $e(a)b^{-1} = b^{-1}e(a) \ \forall \ a, b \in G$.
Proof. $e(a)b^{-1} = b^{-1}e(a) \Leftrightarrow (ae(a))b^{-1} = ab^{-1}e(a) \Leftrightarrow ab^{-1} = ab^{-1}e(a) \Leftrightarrow e(a) = e(ab^{-1}) \Leftrightarrow ab^{-1} = a \Leftrightarrow ab^{-1}b = ab \Leftrightarrow ae(b) = ab \Leftrightarrow a^{-1}ae(b) = a^{-1}ab \Leftrightarrow e(a)e(b) = e(a)b \Leftrightarrow e(b) = b \Leftrightarrow b = bb$.

Theorem 3.3. Let G be a normal generalized group in which $e(a)b^{-1} = b^{-1}e(a) \ \forall \ a, b \in G$. Then, $(ab)^{-1} = b^{-1}a^{-1} \ \forall \ a, b \in G$.
Proof. Since $(ab)^{-1}(ab) = e(ab)$, then by multiplying both sides of the equation on the right by $b^{-1}a^{-1}$ we obtain

$$[(ab)^{-1}ab]b^{-1}a^{-1} = e(ab)b^{-1}a^{-1}. \tag{3}$$

So,

$$[(ab)^{-1}ab]b^{-1}a^{-1} = (ab)^{-1}a(bb^{-1})a^{-1} = (ab)^{-1}a(e(b)a^{-1}) =$$

$$= (ab)^{-1}(aa^{-1})e(b) = (ab)^{-1}(e(a)e(b)) = (ab)^{-1}e(ab) =$$

$$= (ab)^{-1}e((ab)^{-1}) = (ab)^{-1}. \tag{4}$$

Using (3) and (4), we get
$[(ab)^{-1}ab]b^{-1}a^{-1} = (ab)^{-1} \Rightarrow e(ab)(b^{-1}a^{-1}) = (ab)^{-1} \Rightarrow (ab)^{-1} = b^{-1}a^{-1}$.

Theorem 3.4. Let H be a non-empty subset of a generalized group G. The following are equivalent.
(i) H is a generalized subgroup of G.
(ii) For $a, b \in H$, $ab^{-1} \in H$.
(iii) For $a, b \in H$, $ab \in H$ and for any $a \in H$, $a^{-1} \in H$.
Proof.
(i)$\Rightarrow$ (ii) If H is a generalized subgroup of G and $b \in G$, then $b^{-1} \in H$. So by closure property, $ab^{-1} \in H \ \forall \ a \in H$.
(ii)$\Rightarrow$ (iii) If $H \neq \phi$, and $a, b \in H$, then we have $bb^{-1} = e(b) \in H$, $e(b)b^{-1} = b^{-1} \in H$ and $ab = a(b^{-1})^{-1} \in H$ i.e $ab \in H$.
(iii)$\Rightarrow$ (i) $H \subseteq G$ so H is associative since G is associative. Obviously, for any $a \in H$, $a^{-1} \in H$. Let $a \in H$, then $a^{-1} \in H$. So, $aa^{-1} = a^{-1}a = e(a) \in H$. Thus, H is a generalized subgroup of G.

Theorem 3.5. Let $a \in G$ and $f : G \to H$ be an homomorphism. If $\ker f$ at a is denoted by

$$\ker f_a = \{x \in G \ : \ f(x) = f(e(a))\}.$$

Then,
(i) $\ker f_a \lhd G$.
(ii) f is a monomorphism if and only if $\ker f_a = \{e(a) \ : \ \forall \ a \in G\}$.
Proof.
(i) It is necessary to show that $\ker f_a \leq G$. Let $x, y \in \ker f_a \leq G$, then $f(xy^{-1}) = f(x)f(y^{-1}) = f(e(a))(f(e(a)))^{-1} = f(e(a))f(e(a)^{-1}) = f(e(a))f(e(a)) = f(e(a))$. So, $xy^{-1} \in \ker f_a$. Thus, $\ker f_a \leq G$. To show that $\ker f_a \lhd G$, since $y \in \ker f_a$, then by the definition of $\ker f_a$, $f(xyx^{-1}) = f(x)f(y)f(x^{-1}) = f(e(a))f(e(a))f(e(a))^{-1} = f(e(a))f(e(a))f(e(a)) = f(e(a)) \Rightarrow xyx^{-1} \ker f_a$. So, $\ker f_a \lhd G$.
(ii) $f : G \to H$. Let $\ker f_a = \{e(a) \ : \ \forall \ a \in G\}$ and $f(x) = f(y)$, this implies that

$$f(x)f(y)^{-1} = f(y)f(y)^{-1} \Rightarrow f(xy^{-1}) = e(f(y)) = f(e(y)) \Rightarrow$$

$$\Rightarrow xy^{-1} \in \ker f_y \Rightarrow xy^{-1} = e(y) \tag{5}$$

and

$$f(x)f(y)^{-1} = f(x)f(x)^{-1} \Rightarrow f(xy^{-1}) = e(f(x)) = f(e(x)) \Rightarrow$$

$$\Rightarrow xy^{-1} \in \ker f_x \Rightarrow xy^{-1} = e(x). \tag{6}$$

Using (5) and (6), $xy^{-1} = e(y) = e(x) \Leftrightarrow x = y$. So, f is a monomorphism. Conversely, if f is mono, then $f(y) = f(x) \Rightarrow y = x$. Let $k \in \ker f_a \ \forall \ a \in G$. Then, $f(k) = f(e(a)) \Rightarrow k = e(a)$. So, $\ker f_a = \{e(a) \ : \ \forall \ a \in G\}$.

Theorem 3.6. Let G be a generalized group and H a generalized subgroup of G. Then G/H is a generalized group called the quotient or factor generalized group of G by H.
Proof. It is necessary to check the axioms of generalized group on G/H.
Associativity. Let $a, b, c \in G$ and $aH, bH, cH \in G/H$. Then
$aH(bH \cdot cH) = (aH \cdot bH)cH$, so associativity law holds.
Identity. If $e(a)$ is the identity element for each $a \in G$, then $e(a)H$ is the identity element of aH in G/H since
$e(a)H \cdot aH = e(a) \cdot aH = aH \cdot e(a) = aH$. Therefore identity element exists and is unique for each elements aH in G/H.
Inverse. $(aH)(a^{-1}H) = (aa^{-1})H = e(a)H = (a^{-1}a)H = (a^{-1}H)(aH)$ shows that $a^{-1}H$ is the inverse of aH in G/H.
So the axioms of generalized group are satisfied in G/H.

Theorem 3.7. Let G and H be two generalized groups. The direct product of G and H denoted by

$$G \times H = \{(g, h) \ : \ g \in G \text{ and } h \in H\}$$

is a generalized group under the binary operation $\circ$ such that

$$(g_1, h_1) \circ (g_2, h_2) = (g_1g_2, h_1h_2).$$

Proof. This is achieved by investigating the axioms of generalized group for the pair $(G \times H, \circ)$.

Theorem 3.8. Let G be a generalized group with two abelian generalized subgroups N and H of G such $G = NH$. If $N \subseteq COM(H)$ or

$H \subseteq COM(N)$ where $COM(N)$ and $COM(H)$ represent the commutators of N and H respectively, then $G \cong N \times H$.
Proof. Let $a \in G$. Then $a = nh$ for some $n \in N$ and $h \in H$. Also, let $a = n_1h_1$ for some $n_1 \in N$ and $h_1 \in H$. Then $nh = n_1h_1$ so that $e(nh) = e(n_1h_1)$, therefore $n = n_1$ and $h = h_1$. So that $a = nh$ is unique. Define $f : G \to H$ by $f(a) = (n, h)$ where $a = nh$. This function is well defined in the previous paragraph which also shows that f is a one-one correspondence. It remains to check that f is a group homomorphism. Suppose that $a = nh$ and $b = n_1h_1$, then $ab = nhn_1h_1$ and $hn_1 = n_1h$. Therefore,
$f(ab) = f(nhn_1h_1) = f(nn_1hh_1) = (nn_1, hh_1) = (n, h)(n_1, h_1) = f(a)f(b)$.
So, f is a group homomorphism. Hence a group isomorphism since it is a bijection.

Construction of Bol Algebraic Structures

Theorem 3.9. Let H be a subgroup of a non-abelian generalized group G and let $A = H \times G$. For $(h_1, g_1), (h_2, g_2) \in A$, define

$$(h_1, g_1) \circ (h_2, g_2) = (h_1h_2, h_2g_1h_2^{-1}g_2)$$

then $(A, \circ)$ is a Bol groupoid.
Proof. Let $x, y, z \in A$. By checking, it is true that $x \circ (y \circ z) \neq (x \circ y) \circ z$. So, $(A, \circ)$ is non-associative. H is a quasigroup and a loop(groups are quasigroups and loops) but G is neither a quasigroup nor a loop(generalized groups are neither quasigroups nor a loops) so A is neither a quasigroup nor a loop but is a groupoid because H and G are groupoids.
Let us now verify the Bol identity:

$$((x \circ y) \circ z) \circ y = x \circ ((y \circ z) \circ y)$$

$$\text{L. H. S.} = ((x \circ y) \circ z) \circ y = (h_1h_2h_3h_2, h_2h_3h_2g_1h_2^{-1}g_2h_3^{-1}g_3h_2^{-1}g_2).$$

$$\text{R. H. S.} = x \circ ((y \circ z) \circ y) =$$

$$= (h_1h_2h_3h_2, h_2h_3h_2g_1h_2^{-1}(h_3^{-1}h_2^{-1}h_2h_3)g_2h_3^{-1}g_3h_2^{-1}g_2) =$$
$$= (h_1h_2h_3h_2, h_2h_3h_2g_1h_2^{-1}g_2h_3^{-1}g_3h_2^{-1}g_2).$$

So, L. H. S.=R. H. S.. Hence, $(A, \circ)$ is a Bol groupoid.

Corollary 3.1. Let H be a abelian generalized subgroup of a non-abelian generalized group G and let $A = H \times G$. For $(h_1, g_1), (h_2, g_2) \in A$, define

$$(h_1, g_1) \circ (h_2, g_2) = (h_1h_2, h_2g_1h_2^{-1}g_2)$$

then $(A, \circ)$ is a Bol groupoid.

Proof. By Theorem 2.2, an abelian generalized group is a group, so H is a group. The rest of the claim follows from Theorem 3.9.

Corollary 3.2. Let H be a subgroup of a non-abelian generalized group G such that G has the cancellation law and let $A = H \times G$. For $(h_1, g_1), (h_2, g_2) \in A$, define

$$(h_1, g_1) \circ (h_2, g_2) = (h_1h_2, h_2g_1h_2^{-1}g_2)$$

then $(A, \circ)$ is a Bol quasigroup with a left identity element.

Proof. The proof of this goes in line with Theorem 3.9. A groupoid which has the cancellation law is a quasigroup, so G is quasigroup hence A is a quasigroup. Thus, $(A, \circ)$ is a Bol quasigroup with a left identity element since by kunen [13], every quasigroup satisfying the right Bol identity has a left identity.

Corollary 3.3. Let H be a abelian generalized subgroup of a non-abelian generalized group G such that G has the cancellation law and let $A = H \times G$. For $(h_1, g_1), (h_2, g_2) \in A$, define

$$(h_1, g_1) \circ (h_2, g_2) = (h_1h_2, h_2g_1h_2^{-1}g_2)$$

then $(A, \circ)$ is a Bol quasigroup with a left identity element.

Proof. By Theorem 2.2, an abelian generalized group is a group, so H is a group. The rest of the claim follows from Theorem 3.2.

REFERENCES

[1] Agboola, A. A. A., (2004), *Certain properties of generalized groups*, Proc. Jang. Math. Soc. 7, 2, 137–148.

[2] Agboola, A. A. A., (2004), *Smooth generalized subgroups and homomorphisms*, Advanc. Stud. Contemp. Math. 9, 2, 183–193.

[3] Agboola, A. A. A., (2004), *Smooth generalized groups* , Nig. Math. Soc. 7, 2, 137–148.

[4] Araujo, J., and Konieczny, j., (2002), *Molaei's Generalized Groups are Completely Simple Semigroups* , Bul. Inst. Politeh. Jassy, Sect. I. Mat. Mec. Teor. Fiz., 48(52) No. 1–2 , 1–5.
[5] Bruck, R. H., (1966), *A survey of binary systems*, Springer-Verlag, Berlin-Göttingen-Heidelberg, 185pp.
[6] Chein, O., and Goodaire, E. G., (2005), *A new construction of Bol loops: the "odd" case*, Quasigroups and Related Systems, 13, 1, 87–98.
[7] Chein, O., Pflugfelder, H. O., and Smith, J. D. H., (1990), *Quasigroups and Loops : Theory and Applications*, Heldermann Verlag, 568pp.
[8] Dene, J., and Keedwell, A. D., (1974), *Latin squares and their applications*, Academic Press, 549pp.
[9] Goodaire, E. G., Jespers, E., and Milies, C. P., (1996), *Alternative Loop Rings*, NHMS(184), Elsevier, 387pp.
[10] Ilori, S. A., and Akinleye, O., (1993), *Elementary abstract and linear algebra*, Ibadan University Press, 549pp.
[11] Jacobson, N., (1980), *Basic Algebra I*, W. H. Freeman and Company, San Francisco, 472pp.
[12] Jaiyéọlá, T. G., (2009), *A Study of New Concepts in Smarandache Quasigroups and Loops*, Books on Demand, ProQuest Information and Learning, 300 N. Zeeb Road, USA, 127pp.
[13] Kunen, K., (1996), *Moufang Quasigroups*, J. Alg. 183, 231–234.
[14] Kuku, A. O., (1992), *Abstract algebra*, Ibadan University press, 419pp.
[15] Molaei, M. R., (1999), *Generalized actions*, Proceedings of the First International Conference on Geometry, Integrability and Quantization, Coral Press Scientific Publishing Proceedings of the First International Conference on Geometry, 175–180.
[16] Molaei, M. R., (1999), *Generalized groups*, Bull. Inst. Polit. Di. Iase Fasc. 3, 4, 21–24.
[17] Molaei, M. R., (2000), *Topological generalized groups*, Int. Jour. Appl. Math. 2, 9, 1055–1060.
[18] Molaei, M. R., and Tahmoresi, A., (2004), *Connected topological generalized groups*, General Mathematics Vol. 12, No. 1, 13-?22.
[19] Mehrabi, M. R., and Oloomi, A., (2000), *Generalized subgroups and homomorphisms*, Arabs Jour. Math. Sc. 6, 1–7.
[20] Pflugfelder, H. O., (1990), *Quasigroups and Loops : Introduction*, Sigma series in Pure Math. 7, Heldermann Verlag, Berlin, 147pp.
[21] Solarin, A. R. T. and Sharma, B. L. (1981), *On the construction of Bol loops*, Scientific Annals of Al.I. Cuza. Univ. 27, 13–17.
[22] Vagner, V., (Wagner) (1952), *Generalized Groups*, Doklady Akademiý

Nauk SSSR,84, 1119–1122(Russian).
[23] Vasantha Kandasamy, W. B., (2002), *Smarandache loops*, Department of Mathematics, Indian Institute of Technology, Madras, India, 128pp.
[24] White, A., (1988), *An introduction to abstract algebra*, 7, Leicester Place, London.

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New identities and inequalities in triangle

Mihály Bencze[10]

ABSTRACT. In this paper we present some new identities and inequalities in triangle.

MAIN RESULTS

Theorem 1. In all triangle ABC holds the following identities
1). $\frac{1}{r_a}+\frac{1}{r_b}=\frac{2}{h_c}$ and his permutations
2). $\sum\frac{1}{h_ar_a}=\frac{4R+r}{s^2r}$ or $\sum\frac{a}{r_a}=\frac{2(4R+r)}{s}$
3). $\sum\frac{r_a}{h_cr_c}=\frac{(4R+r)^2-s^2}{2s^2r}$
4). $\sum\frac{r_a}{h_a}=\frac{2R-r}{r}$ or $\sum ar_a=2s\left(2R-r\right)$
5). $\sum\frac{r_ar_b}{h_c}=4R+r$
Proof.
1). $\frac{1}{r_a}+\frac{1}{r_b}=\frac{s-a}{sr}+\frac{s-b}{sr}=\frac{c}{sr}=\frac{2c}{ch_c}=\frac{2}{h_c}$
2). $2\sum\frac{1}{h_ar_a}=\sum\left(\frac{1}{r_ar_b}+\frac{1}{r_ar_c}\right)=2\sum\frac{1}{r_ar_b}=\frac{2(4R+r)}{s^2r}$ so $\sum\frac{1}{h_ar_a}=\frac{4R+r}{s^2r}$
3). $2\sum\frac{r_b}{h_ar_a}=\sum\left(\frac{1}{r_a}+\frac{r_b}{r_ar_c}\right)=\frac{(4R+r)^2-s^2}{s^2r}$ so $\sum\frac{r_a}{h_cr_c}=\frac{(4R+r)^2-s^2}{2s^2r}$
4) $2\sum\frac{r_a}{h_a}=\sum\left(\frac{r_a}{r_b}+\frac{r_a}{r_c}\right)=\sum\frac{r_b+r_c}{r_a}=\frac{2(2R-r)}{r}$ or $\sum\frac{r_a}{h_a}=\frac{2R-r}{r}$. Because $R\geq 2r$, then $\sum\frac{r_a}{h_a}\geq 3$, which is a result of H. Demir (see [1])
5). $2\sum\frac{r_br_c}{h_a}=\sum\left(r_a+r_b\right)=2\sum r_a-2\left(4R+r\right)$ or $\sum\frac{r_br_c}{h_a}=4R+r$
Corollary 1.1. In all triangle ABC holds

$$\frac{s^2+r^2+4Rr}{2R}\leq\sum\sqrt{r_ar_b}\leq 4R+r$$

Proof. Using the AM-GM-HM inequality we have

$$\frac{2}{\frac{1}{r_a}+\frac{1}{r_b}}\leq\sqrt{r_ar_b}\leq\frac{r_a+r_b}{2}\text{ or }h_c\leq\sqrt{r_ar_b}\leq\frac{r_a+r_b}{2}\text{ so}$$

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$$\frac{s^2+r^2+4Rr}{2R}=\sum h_c\le\sum\sqrt{r_ar_b}\le\sum\frac{r_a+r_b}{2}=\sum r_a=4R+r$$

Corollary 1.2. In all triangle ABC holds
1). $h_c\le\sqrt{r_ar_b}$
2). $h_ah_bh_c\le r_ar_br_c$
3). $\sum\sqrt{r_a}\ge\frac{2s\sqrt{r}}{R}$
4). $\sum\frac{1}{\sqrt{r_ar_b}}\le\frac{1}{r}$
5). $\sum\frac{1}{\sqrt{r_a}}\le\frac{s^2+r^2+4Rr}{4sr\sqrt{r}}$
6). $\prod\left(\sqrt{r_a}+\sqrt{r_b}\right)\ge\frac{s\sqrt{r}\left(s^2+r^2+2Rr\right)}{R^2}$
7). $\sum\frac{1}{\left(\sqrt{r_a}+\sqrt{r_b}\right)\sqrt{r_c}}\le\frac{\left(s^2+r^2+4Rr\right)^2+8s^2Rr}{4s^2r(s^2+r^2+2Rr)}$

Proof.
1). $\frac{2}{h_c}=\frac{1}{r_a}+\frac{1}{r_b}\ge\frac{2}{\sqrt{r_ar_b}}$ so $h_c\le\sqrt{r_ar_b}$
2). $\prod h_c\le\prod\sqrt{r_ar_b}=\prod r_a$
3). $h_ah_b\le\sqrt{r_br_cr_cr_a}=s\sqrt{r}\sqrt{r_c}$ so $\sum\sqrt{r_c}\ge\frac{1}{s\sqrt{r}}\sum h_ah_b=\frac{2s\sqrt{r}}{R}$
4). $\sum\frac{1}{\sqrt{r_ar_b}}\le\sum\frac{1}{h_a}=\frac{1}{r}$
5). $\sum\frac{1}{\sqrt{r_c}}\le s\sqrt{r}\sum\frac{1}{h_ah_b}=\frac{s^2+r^2+4Rr}{4sr\sqrt{r}}$
6). We have $h_a+h_b\le\sqrt{r_br_c}+\sqrt{r_cr_a}=\sqrt{r_c}\left(\sqrt{r_a}+\sqrt{r_b}\right)$ so
$\prod\left(\sqrt{r_a}+\sqrt{r_b}\right)\ge\prod\frac{h_a+h_b}{\sqrt{r_c}}=\frac{s\sqrt{r}\left(s^2+r^2+2Rr\right)}{R^2}$
7). $\sum\frac{1}{\left(\sqrt{r_a}+\sqrt{r_b}\right)\sqrt{r_c}}\le\sum\frac{1}{h_a+h_b}=\frac{\left(s^2+r^2+4Rr\right)^2+8s^2Rr}{4s^2r(s^2+r^2+2Rr)}$

Corollary 1.3. In all triangle ABC holds
1). $\left(\frac{1}{r_a}\right)^\alpha+\left(\frac{1}{r_b}\right)^\alpha\ge2\left(\frac{1}{h_c}\right)^\alpha$ and his permutations
2). $\sum\left(\frac{1}{h_ar_a}\right)^\alpha\ge3\left(\frac{4R+r}{3s^2r}\right)^\alpha$
3). $\sum\left(\frac{r_a}{h_cr_c}\right)^\alpha\ge3\left(\frac{(4R+r)^2-s^2}{6s^2r}\right)^\alpha$
4). $\sum\left(\frac{r_a}{h_a}\right)^\alpha\ge3\left(\frac{2R-r}{3r}\right)^\alpha$ which is a refinement of H. Guggenheimer‘s inequality (see [1])
5). $\sum\left(\frac{r_ar_b}{h_c}\right)^\alpha\ge3\left(\frac{4R+r}{3}\right)^\alpha$ for all $\alpha\in(-\infty,0]\cup[1,+\infty)$ and for $\alpha\in(0,1)$ holds the reverse inequalities.

Proof. The function $f(x)=x^\alpha$ for $\alpha\in(-\infty,0]\cup[1,+\infty)$ is convex, and using the Jensen‘s inequality for all identities in Theorem 1 we get the desired inequalities.

Corollary 1.4. In all triangle ABC holds
1). $\sum \left(\sqrt{r_a}\right)^{\alpha} \geq 3\left(\frac{2s\sqrt{r}}{3R}\right)^{\alpha}$
2). $\sum \left(\frac{1}{\sqrt{r_a r_b}}\right)^{\beta} \leq 3\left(\frac{1}{3r}\right)^{\beta}$
3). $\sum \left(\frac{1}{\sqrt{r_a}}\right)^{\beta} \leq 3\left(\frac{s^2+r^2+4Rr}{12sr\sqrt{r}}\right)^{\beta}$
4). $\sum \left(\frac{1}{\left(\sqrt{r_a}+\sqrt{r_b}\right)\sqrt{r_c}}\right)^{\beta} \leq 3\left(\frac{\left(s^2+r^2+4Rr\right)^2+8s^2Rr}{12s^2r(s^2+r^2+2Rr)}\right)^{\beta}$
for all $\alpha \in (-\infty, 0] \cup [1, +\infty)$ and $\beta \in (0, 1)$.
Proof. See the proof of Corollary 1.3.

Corollary 1.5. In all triangle ABC holds

$$\sum h_a h_b \leq s\sqrt{r}\sum \sqrt{r_c} \leq \sum r_a r_b$$

which is a refinement of A. Makowski‘s (see [1]) result.
Proof.

$$\sum h_a h_b \leq \sum \sqrt{r_b r_c r_c r_a} = s\sqrt{r}\sum \sqrt{r_c} \leq \sum \frac{r_b r_c + r_c r_a}{2} = \sum r_a r_b$$

The result can be written in the following way:

$$\frac{s^2+r^2+4Rr}{2Rs\sqrt{r}} \leq \sum \sqrt{r_a} \leq s^2$$

Corollary 1.6. In all triangle ABC holds

$$\sum h_a \leq \sum \sqrt{r_b r_c} \leq \sum r_a$$

a new refinement for L. Carliz‘s problem (see [1]).
Proof. We have

$$\sum h_a \leq \sum \sqrt{r_b r_c} \leq \sum \frac{r_b + r_c}{2} = \sum r_a$$

Corollary 1.7. In all triangle ABC holds
1). $\frac{4R+r}{r(4R^2+4Rr+3r^2)} \leq \sum \frac{1}{h_a r_a} \leq \frac{4R+r}{r^2(16R-5r)}$
2). $\max\left\{\frac{6R^2+2Rr-r^2}{s^2r}; \frac{(4R+r)^2-s^2}{2r(4R^2+4Rr+3r^2)}\right\} \leq \sum \frac{r_a}{h_c r_c} \leq$
$\min\left\{\frac{8R^2-4Rr+3r^2}{s^2r}; \frac{(4R+r)^2-s^2}{2r^2(16R-5r)}\right\}$

Proof. In Theorem 1 for the identities 2) and 3) we use the inequality

$$r\left(16R-5r\right)\leq s^2\leq 4R^2+4Rr+3r^2$$

which holds from $IH^2=3r^2+4Rr+4R^2-s^2\geq 0$ and
$9GI^2=s^2+5r^2-16Rr\geq 0$.

Corollary 1.8. In all triangle ABC holds

$$\sum h_a^2\leq\sum r_br_c\leq\sum r_a^2 \text{ or}$$

$$\frac{\left(s^2+r^2+4Rr\right)^2-16s^2Rr}{4R^2}\leq s^2\leq\left(4R+r\right)^2-s^2$$

Proof. We have

$$\sum h_a^2\leq\sum r_br_c\leq\sum\frac{r_b^2+r_c^2}{2}=\sum r_a^2$$

Corollary 1.9. In all triangle ABC holds
1). $\sum h_a^2h_b^2\leq s^2r\sum r_c\leq\sum r_a^2r_b^2$ or
$\frac{2r^2\left(s^2-r^2-4Rr\right)}{R^2}\leq r\left(4R+r\right)\leq s^2-8Rr-2r^2$
2). $\sum h_a^{2\alpha}\leq\sum\left(r_br_c\right)^\alpha\leq\sum r_a^{2\alpha}$ for all $\alpha\geq 0$
3). $\sum\left(h_ah_b\right)^{2\alpha}\leq\left(s^2r\right)^\alpha\sum r_a^\alpha\leq\sum\left(r_br_c\right)^{2\alpha}$ for all $\alpha\geq 0$
4). $\sum\frac{h_a^2+h_b^2}{r_a+r_b}\leq 4R+r$
5). $\sum\frac{r_c}{h_a^2+h_b^2}\geq\frac{(4R+r)^2+s^2}{4s^2R}$
6). $\sum\frac{h_c^2+h_b^2}{r_c}\leq 8\left(4R+r\right)$
7). $\max\left\{\sum\frac{h_a^2}{r_b};\sum\frac{h_a^2}{r_c}\right\}\leq 4R+r$
8). $\min\left\{\sum\frac{r_b}{h_a^2};\sum\frac{r_c}{h_a^2}\right\}\geq\frac{1}{r}$
9). $\max\left\{h_a^2r_a;h_b^2r_b;h_c^2r_c\right\}\leq s^2r$
10). $\sum h_a^2r_a\leq 3s^2r$
11). $\sum\frac{h_a^2}{r_a}\leq\frac{s^2-8Rr-2r^2}{r}$
12). $\sum h_a^4\leq s^2\left(s^2-8Rr-2r^2\right)$
13). $\sum\frac{r_a}{h_a^2}\geq\frac{(4R+r)^2-2s^2}{s^2r}$
14). $\sum\frac{1}{h_a^4}\geq\frac{(4R+r)^2-2s^2}{s^4r^2}$
15). $\sum\frac{\left(h_a^2+h_b^2\right)\left(h_b^2+h_c^2\right)}{r_cr_a}\leq\left(4R+r\right)^2+s^2$
16). $\max\left\{\sum\frac{h_a^6}{r_b^3};\sum\frac{h_a^6}{r_c^3}\right\}\leq\left(4R+r\right)^3-12s^2R$

17). $\prod\left(h_a^2+h_b^2\right) \le \frac{4R}{r}$

18). $\sum \frac{r_cr_a}{\left(h_a^2+h_b^2\right)\left(h_b^2+h_c^2\right)} \ge \frac{4R+r}{2s^2R}$

19). $\sum \frac{h_a^2+h_b^2}{r_c^2} \le \frac{2(2R-r)}{r}$

20). $\sum \frac{r_c^2}{h_a^2+h_b^2} \ge \frac{(4R+r)\left((4R+r)^2+s^2\right)}{4s^2R} - 3$

21). $\sum \frac{h_a^2}{r_b+r_c} \le \frac{s^2+4Rr+r^2}{4R}$

22). $\sum \frac{r_b+r_c}{h_a^2} \ge \frac{2}{r}$

23). $\sum \frac{\left(h_a^2+h_b^2\right)^2}{r_c} \le 4s^2\left(R+r\right)$

24). $\sum \frac{\left(h_a^2+h_b^2\right)\left(h_b^2+h_c^2\right)}{r_a^2r_c^2} \le \frac{(4R+r)^3-s^2(8R-r)}{s^2r}$

Proof.

1). $\sum h_a^2h_b^2 \le \sum r_br_cr_cr_a = s^2r\sum r_c \le \sum \frac{r_b^2r_c^2+r_c^2r_a^2}{2} = \sum r_a^2r_b^2$

4). We have $h_a^2+h_b^2 \le \left(r_a+r_b\right)r_c$ so $\sum \frac{h_a^2+h_b^2}{r_a+r_b} \le \sum r_c = 4R+r$

5). $\sum \frac{r_c}{h_a^2+h_b^2} \ge \sum \frac{1}{r_a+r_b} = \frac{(4R+r)^2+s^2}{4s^2R}$

6). $\sum \frac{h_a^2+h_b^2}{r_c} \le \sum\left(r_a+r_b\right) = 8\left(4R+r\right)$

7). We have $\frac{h_a^2}{r_b} \le r_c$ and $\frac{h_a^2}{r_c} \le r_b$ so $\sum \frac{h_a^2}{r_b} \le \sum r_c = 4R+r$ and $\sum \frac{h_a^2}{r_c} \le \sum r_b = 4R+r.$

8). We have $\frac{1}{r_c} \le \frac{r_b}{h_a^2}$ and $\frac{1}{r_b} \le \frac{r_c}{h_a^2}$ so $\sum \frac{r_b}{h_a^2} \ge \sum \frac{1}{r_c} = \frac{1}{r}$ and $\sum \frac{r_c}{h_a^2} \ge \sum \frac{1}{r_b} = \frac{1}{r}.$

9). We have $h_a^2 \le r_br_c$ so $h_a^2r_a \le r_ar_br_c = s^2r$

10). $\sum h_a^2r_a \le \sum s^2r = 3s^2r$

11). $\sum \frac{h_a^2}{r_a} \le \sum \frac{r_br_c}{r_a} = \frac{s^2-8Rr-2r^2}{r}$

12). $\sum h_a^4 \le \sum r_b^2r_c^2 = s^2\left(s^2-8Rr-2r^2\right)$

13). $\sum \frac{r_a}{h_a^2} \ge \sum \frac{r_a}{r_br_c} = \frac{(4R+r)^2-2s^2}{s^2r}$

14). $\sum \frac{1}{h_a^4} \ge \sum \frac{1}{r_b^2r_c^2} = \frac{(4R+r)^2-2s^2}{s^4r^2}$

15). $\sum \frac{\left(h_a^2+h_b^2\right)\left(h_b^2+h_c^2\right)}{r_cr_a} \le \sum\left(r_a+r_b\right)\left(r_b+r_c\right) = \left(4R+r\right)^2+s^2$

16). $\sum \frac{h_a^6}{r_c^3} \le \sum r_b^3 = \left(4R+r\right)^3 - 12s^2R$

17). $\prod\left(h_a^2+h_b^2\right) \le \frac{\prod(r_a+r_b)}{\prod r_c} = \frac{4R}{r}$

18). $\sum \frac{r_cr_a}{\left(h_a^2+h_b^2\right)\left(h_b^2+h_c^2\right)} \ge \sum \frac{1}{(r_a+r_b)(r_b+r_c)} = \frac{4R+r}{2s^2R}$

19). $\sum \frac{h_a^2+h_b^2}{r_c^2} \le \sum \frac{r_a+r_b}{r_c} = \frac{2(2R-r)}{r}$

20). $\sum \frac{r_c^2}{h_a^2+h_b^2} \ge \sum \frac{r_c}{r_a+r_b} = \frac{(4R+r)\left((4R+r)^2+s^2\right)}{4s^2R} - 3$

21). $\sum \frac{h_a^2}{r_b+r_c} \leq \sum \frac{r_b r_c}{r_b+r_c} = \frac{s^2+4Rr+r^2}{4R}$
22). $\sum \frac{r_b+r_c}{h_a^2} \geq \sum \frac{r_b+r_c}{r_b r_c} = \frac{2}{r}$
23). $\sum \frac{\left(h_a^2+h_b^2\right)^2}{r_a r_b r_c^2} \leq \sum \frac{(r_a+r_b)^2}{r_a r_b} = \frac{4(R+r)}{r}$ or
$\sum \frac{\left(h_a^2+h_b^2\right)^2}{r_c} \leq r_a r_b r_c \frac{4(R+r)}{r} = 4s^2 \left(R+r\right)$
24). $\sum \frac{\left(h_a^2+h_b^2\right)\left(h_b^2+h_c^2\right)}{r_a^2 r_c^2} \leq \sum \frac{(r_a+r_b)(r_b+r_c)}{r_a r_c} = \frac{(4R+r)^3-s^2(8R-r)}{s^2 r}$

Corollary 1.10. In all triangle ABC holds
1). $(4R+r)(2R-r) \geq s^2 r$
2). $\min\left\{\sum b^2 h_a r_a; \sum c^2 h_a r_a\right\} \geq \frac{4s^4 r}{4R+r}$
3). $\min\left\{\sum h_a r_a r_b^3; \sum h_a r_a r_c^3; \sum h_a r_a^3\right\} \geq (4R+r) s^2 r$
4). $\min\left\{\sum h_a r_a m_a^4; \sum h_a r_a m_b^4; \sum h_a r_a m_c^4\right\} \geq \frac{9s^2 r\left(s^2-r^2-4Rr\right)^2}{4(4R+r)}$
5). $\min\left\{\sum h_a r_a \sin^4 \frac{A}{2}; \sum h_a r_a \sin^4 \frac{B}{2}; \sum h_a r_a \sin^4 \frac{C}{2}\right\} \geq \frac{s^2 r(2R-r)^2}{4R^2(4R+r)}$
6). $\min\left\{\sum h_a r_a \cos^4 \frac{A}{2}; \sum h_a r_a \cos^4 \frac{B}{2}; \sum h_a r_a \cos^4 \frac{C}{2}\right\} \geq \frac{s^2 r(4R+r)}{4R^2}$
7). $\min\left\{\sum \frac{h_a}{r_a}; \sum \frac{h_a r_a}{r_b^2}; \sum \frac{h_a r_a}{r_c^2}\right\} \geq \frac{s^2}{r(4R+r)}$ and $\sum \frac{h_a}{r_a} \geq \frac{9r}{2R-r}$
8). $\min\left\{\sum h_a^3 r_a; \sum h_a r_a h_b^2; \sum h_a r_a h_c^2\right\} \geq \frac{s^2 r\left(s^2+r^2+4Rr\right)^2}{4R^2(4R+r)}$
9). $\sum r_a h_b h_c \geq \frac{2s^4 r}{R(4R+r)}$ and $\min\left\{\sum r_a h_b^2 h_a^3; \sum r_a h_c^2 h_a^3\right\} \geq \frac{4s^6 r^3}{R^2(4R+r)}$

Proof.
1). $\frac{4R+r}{s^2 r} = \sum \frac{1}{h_a r_a} = \sum \frac{a^2}{a^2 h_a r_a} \geq \frac{(\sum a)^2}{\sum a^2 h_a r_a} = \frac{4s^2}{2sr \sum a r_a} = \frac{1}{2R-r}$, therefore
$(4R+r)(2R-r) \geq s^2 r$
2). $\frac{4R+r}{s^2 r} = \sum \frac{1}{h_a r_a} = \sum \frac{b^2}{b^2 h_a r_a} \geq \frac{(\sum b)^2}{\sum b^2 h_a r_a} = \frac{4s^2}{\sum b^2 h_a r_a}$, therefore
$\sum b^2 h_a r_a \geq \frac{4s^4 r}{4R+r}$
3). $\frac{4R+r}{s^2 r} = \sum \frac{1}{h_a r_a} = \sum \frac{r_a^2}{h_a r_a^3} \geq \frac{(\sum r_a)^2}{\sum h_a r_a^3} = \frac{(4R+r)^2}{\sum h_a r_a^3}$, therefore
$\sum h_a r_a^3 \geq (4R+r) s^2 r$ and $\frac{4R+r}{s^2 r} = \sum \frac{r_b^2}{h_a r_a r_b^2} \geq \frac{(\sum r_b)^2}{\sum h_a r_a h_b^2} = \frac{(4R+r)^2}{\sum h_a r_a r_b^2}$, therefore
$\sum h_a r_a r_b^2 \geq (4R+r) s^2 r$
4). $\frac{4R+r}{s^2 r} = \sum \frac{m_a^4}{h_a r_a m_a^4} \geq \frac{\left(\sum m_a^2\right)^2}{\sum h_a r_a m_a^4} = \frac{9\left(s^2-r^2-4Rr\right)^2}{4\sum h_a r_a m_a^4}$, therefore
$\sum h_a r_a m_a^4 \geq \frac{9s^2 r\left(s^2-r^2-4Rr\right)^2}{4(4R+r)}$
5). $\frac{4R+r}{s^2 r} = \sum \frac{\sin^4 \frac{A}{2}}{h_a r_a \sin^4 \frac{A}{2}} \geq \frac{\left(\sum \sin^2 \frac{A}{2}\right)^2}{\sum h_a r_a \sin^4 \frac{A}{2}}$, therefore $\sum h_a r_a \sin^4 \frac{A}{2} \geq \frac{s^2 r(2R-r)^2}{4R^2(4R+r)}$
6). $\frac{4R+r}{s^2 r} = \sum \frac{\cos^4 \frac{A}{2}}{h_a r_a \cos^4 \frac{A}{2}} \geq \frac{\left(\sum \cos^2 \frac{A}{2}\right)^2}{\sum h_a r_a \cos^4 \frac{A}{2}}$, therefore $\sum h_a r_a \cos^4 \frac{A}{2} \geq \frac{s^2 r(4R+r)}{4R^2}$

7). $\frac{4R+r}{s^2r}=\sum\frac{\frac{1}{r_a^2}}{\frac{h_a}{r_a}}\geq\frac{\left(\sum\frac{1}{r_a}\right)^2}{\frac{h_a}{r_a}}\geq\frac{1}{r_2\sum\frac{h_a}{h_b}}$, therefore $\sum\frac{h_a}{r_a}\geq\frac{s^2}{r(4R+r)}$ and
$\frac{4R+r}{s^2r}=\sum\frac{\frac{1}{r_b^2}}{\frac{h_ar_a}{r_b^2}}\geq\frac{\left(\sum\frac{1}{r_b}\right)^2}{\sum\frac{h_ar_a}{r_b^2}}=\frac{1}{r^2\sum\frac{h_ar_a}{r_b^2}}$, therefore $\sum\frac{h_ar_a}{r_b^2}\geq\frac{s^2}{r(4R+r)}$
An another way we have $\left(\sum\frac{r_a}{h_a}\right)\left(\sum\frac{h_a}{r_a}\right)\geq 9$, therefore
$\sum\frac{h_a}{r_a}\geq\frac{9}{\sum\frac{r_a}{h_a}}=\frac{9r}{2R-r}$

8). $\frac{4R+r}{s^2r}=\sum\frac{h_a^2}{h_a^3r_a}\geq\frac{(\sum h_a)^2}{\sum h_a^3r_a}$, therefore $\sum h_a^3r_a\geq\frac{s^2r\left(s^2+r^2+4Rr\right)^2}{4R^2(4R+r)}$ and
$\frac{4R+r}{s^2r}=\sum\frac{h_b^2}{h_ar_ah_b^2}\geq\frac{(\sum h_b)^2}{\sum h_ar_ah_b^2}$, therefore $\sum h_ar_ah_b^2\geq\frac{s^2r\left(s^2+r^2+4Rr\right)^2}{4R^2(4R+r)}$
9). $\frac{4R+r}{s^2r}=\sum\frac{h_b^2h_c^2}{h_ah_b^2h_c^2r_a}\geq\frac{(\sum h_ah_b)^2}{h_ah_bh_c\sum r_ah_bh_c}$, therefore $\sum r_ah_bh_c\geq\frac{2s^4r}{R(4R+r)}$ and
$\frac{4R+r}{s^2r}=\sum\frac{h_a^2h_b^2}{r_ah_a^3h_b^2}\geq\frac{(\sum h_ah_b)^2}{\sum r_ah_a^3h_b^2}$, therefore $\sum r_ah_a^3h_b^2\geq\frac{4s^6r^3}{R^2(4R+r)}$

Corollary 1.11. In all triangle ABC holds
1). $\sum r_ar_ch_c\geq\frac{2s^2r(4R+r)^2}{(4R+r)^2-s^2}$
2). $\sum h_ar_c\geq\frac{2s^4}{(4R+r)^2-s^2}$
3). $\sum r_a^3r_ch_c\geq\frac{2s^2r\left((4R+r)^2-2s^2\right)^2}{(4R+r)^2-s^2}$
4). $\sum r_a^2r_b^3h_c\geq\frac{2s^4\left(s^2-8Rr-2r^2\right)^2}{(4R+r)^2-s^2}$
5). $\sum r_a^5r_ch_c\geq\frac{2s^2r\left((4R+r)^3-12s^2R\right)^2}{(4R+r)^2-s^2}$
6). $\sum\frac{r_a^2r_b^2h_c}{r_c}\geq\frac{2s^2r\left(s^2-8Rr-2r^2\right)^2}{(4R+r)^2-s^2}$
7). $\sum\frac{h_cr_c}{r_a}\left(r_a+r_b\right)^2\left(r_b+r_c\right)^2\geq\frac{2s^2r\left((4R+r)^2+s^2\right)^2}{(4R+r)^2-s^2}$
8). $\sum\frac{h_cr_cm_a^4}{r_a}\geq\frac{9s^2r\left(s^2-r^2-4Rr\right)^2}{2\left((4R+r)^2-s^2\right)}$
9). $\sum\frac{h_cr_c}{r_a}\sin^4\frac{A}{2}\geq\frac{s^2r(2R-r)^2}{2R^2\left((4R+r)^2-s^2\right)}$
10). $\sum\frac{h_cr_c}{r_a}\cos^4\frac{A}{2}\geq\frac{s^2r(4R+r)^2}{2R^2\left((4R+r)^2-s^2\right)}$
Proof.
1). $\frac{(4R+r)^2-s^2}{2s^2r}=\sum\frac{r_a}{h_cr_c}=\sum\frac{r_a^2}{h_cr_cr_a}\geq\frac{(\sum r_a)^2}{\sum h_cr_cr_a}$, therefore
$\sum h_cr_cr_a\geq\frac{2s^2r(4R+r)^2}{(4R+r)^2-s^2}$
2). $\frac{(4R+r)^2-s^2}{2s^2r}=\sum\frac{r_a^2r_b^2}{h_cr_cr_ar_b^2}\geq\frac{(\sum r_ar_b)^2}{r_ar_br_c\sum h_cr_b}$ therefore $\sum h_cr_b\geq\frac{2s^4}{(4R+r)^2-s^2}$

3). $\frac{(4R+r)^2-s^2}{2s^2r}=\sum\frac{r_a^4}{h_cr_cr_a^3}\geq\frac{\left(\sum r_a^2\right)^2}{\sum h_cr_cr_a^3}$ therefore $\sum h_cr_cr_a^3\geq\frac{2s^2r\left((4R+r)^2-2s^2\right)^2}{(4R+r)^2-s^2}$

4). $\frac{(4R+r)^2-s^2}{2s^2r}=\sum\frac{r_a^4r_b^4}{h_cr_cr_a^3r_b^4}\geq\frac{\left(\sum r_a^2r_b^2\right)^2}{r_ar_br_c\sum h_cr_a^2r_b^2}$ therefore $\sum h_cr_a^2r_b^3\geq\frac{2s^4\left(s^2-8Rr-2r^2\right)^2}{(4R+r)^2-s^2}$

5). $\frac{(4R+r)^2-s^2}{2s^2r}=\sum\frac{r_a^6}{h_cr_cr_a^5}\geq\frac{\left(\sum r_a^3\right)^2}{\sum h_cr_cr_a^5}$ therefore $\sum h_cr_cr_a^5\geq\frac{2s^2r\left((4R+r)^3-12s^2R\right)^2}{(4R+r)^2-s^2}$

6). $\frac{(4R+r)^2-s^2}{2s^2r}=\sum\frac{\frac{r_a^2r_b^2}{r_c^2}}{\frac{h_cr_a^2r_b^2}{r_c}}\geq\frac{\sum\left(\frac{r_ar_b}{r_c}\right)^2}{\sum\frac{h_cr_a^2r_b^2}{r_c}}$ therefore $\sum\frac{h_cr_a^2r_b^2}{r_c}\geq\frac{2s^2r\left(s^2-8Rr-2r^2\right)^2}{(4R+r)^2-s^2}$

7). $\frac{(4R+r)^2-s^2}{2s^2r}=\sum\frac{(r_a+r_b)^2(r_b+r_c)^2}{\frac{h_cr_c}{r_a}(r_a+r_b)^2(r_b+r_c)^2}\geq\frac{(\sum(r_a+r_b)(r_b+r_c))^2}{\sum\frac{h_cr_c}{r_a}(r_a+r_b)^2(r_b+r_c)^2}$, therefore $\sum\frac{h_cr_c}{r_a}\left(r_a+r_b\right)^2\left(r_b+r_c\right)^2\geq\frac{2s^2r\left((4R+r)^2+s^2\right)^2}{(4R+r)^2-s^2}$

8). $\frac{(4R+r)^2-s^2}{2s^2r}=\sum\frac{m_a^4}{\frac{h_cr_cm_a^4}{r_a}}\geq\frac{\left(\sum m_a^2\right)^2}{\sum\frac{h_cr_cm_a^4}{r_a}}$, therefore $\sum\frac{h_cr_cm_a^4}{r_a}\geq\frac{9s^2r\left(s^2-r^2-4Rr\right)^2}{2\left((4R+r)^2-s^2\right)}$

9). $\frac{(4R+r)^2-s^2}{2s^2r}=\sum\frac{\sin^4\frac{A}{2}}{\frac{h_cr_c}{r_a}\sin^4\frac{A}{2}}\geq\frac{\left(\sum\sin^2\frac{A}{2}\right)^2}{\sum\frac{h_cr_c}{r_a}\sin^4\frac{A}{2}}$ therefore $\sum\frac{h_cr_c}{r_a}\sin^4\frac{A}{2}\geq\frac{s^2r(2R-r)^2}{2R^2\left((4R+r)^2-s^2\right)}$

10). $\frac{(4R+r)^2-s^2}{2s^2r}=\sum\frac{\cos^4\frac{A}{2}}{\frac{h_cr_c}{r_a}\cos^4\frac{A}{2}}\geq\frac{\left(\sum\cos^2\frac{A}{2}\right)^2}{\sum\frac{h_cr_c}{r_a}\cos^4\frac{A}{2}}$ therefore $\sum\frac{h_cr_c}{r_a}\cos^4\frac{A}{2}\geq\frac{s^2r(4R+r)^2}{2R^2\left((4R+r)^2-s^2\right)}$

Corollary 1.12. In all triangle ABC holds

1). $\sum h_ar_a\geq\frac{r(4R+r)^2}{2R-r}$

2). $\sum h_ar_ar_b^2\geq\frac{rs^4}{2R-r}$

3). $\sum h_ar_a^3\geq\frac{r\left((4R+r)^2-2s^2\right)}{2R-r}$

4). $\sum h_ar_a^5\geq\frac{r\left((4R+r)^3-12s^2R\right)}{2R-r}$

5). $\left(\sum\sqrt{ar_a}\right)^2\leq\min\left\{6s\left(2R-r\right);2s\left(4R+r\right)\right\}$

6). $\sum\frac{h_a}{r_a}\sin^4\frac{A}{2}\geq\frac{r(2R-r)}{4R^2}$

7). $\sum\frac{h_a}{r_a}\cos^4\frac{A}{2}\geq\frac{r(4R+r)^2}{4R^2(2R-r)}$

8). $\left(\sum\sqrt{r_ar_b}\right)^2\leq\frac{(4R+r)\left(s^2+r^2+4Rr\right)}{2R}$

9). $\sum\frac{h_ar_a}{r_b}\geq 4R+r$

10). $\sum h_cr_ar_b\geq\frac{s^4}{4R+r}$

11). $\sum \frac{h_c r_a^3}{r_b} \geq \frac{\left((4R+r)^2-2s^2\right)^2}{4R+r}$

12). $\sum \frac{h_c r_a^5}{r_b} \geq \frac{\left((4R+r)^3-12s^2R\right)^2}{4R+r}$

13). $\sum \frac{h_r}{r_a r_b} \sin^4 \frac{A}{2} \geq \frac{(2R-r)^2}{4R^2(4R+r)}$

14). $\sum \frac{h_c}{r_a r_b} \cos^4 \frac{A}{2} \geq \frac{4R+r}{4R^2}$

15). $\sum \frac{h_c}{r_a r_b} \sin^4 \frac{A}{2} \sin^4 \frac{B}{2} \geq \frac{\left(s^2+r^2-8Rr\right)^2}{256R^4(4R+r)}$

16). $\sum \frac{h_c}{r_a r_b} \cos^4 \frac{A}{2} \cos^4 \frac{B}{2} \geq \frac{\left(s^2+(4R+r)^2\right)^2}{256R^4(4R+r)}$

Proof.

1). $\frac{2R-r}{r} = \sum \frac{r_a}{h_a} = \sum \frac{r_a^2}{h_a r_a} \geq \frac{\left(\sum r_a\right)^2}{\sum h_a r_a}$, therefore $\sum h_a r_a \geq \frac{r(4R+r)^2}{2R-r}$

2). $\frac{2R-r}{r} = \sum \frac{r_a^2 r_b^2}{h_a r_a r_b^2} \geq \frac{\left(\sum r_a r_b\right)^2}{\sum h_a r_a r_b^2}$, therefore $\sum h_a r_a r_b^2 \geq \frac{rs^4}{2R-r}$

3). $\frac{2R-r}{r} = \sum \frac{r_a^4}{h_a r_a^3} \geq \frac{\left(r_a^2\right)^2}{\sum h_a r_a^3}$, therefore $\sum h_a r_a^3 \geq \frac{r\left((4R+r)^2-2s^2\right)}{2R-r}$

4). $\frac{2R-r}{r} = \sum \frac{r_a^6}{h_a r_a^5} \geq \frac{\left(\sum r_a^3\right)^2}{\sum h_a r_a^5}$ therefore $\sum h_a r_a^5 \geq \frac{r\left((4R+r)^3-12s^2R\right)}{2R-r}$

5). $\left(\sum \sqrt{a r_a}\right)^2 \leq \left(\sum a\right)\left(\sum r_a\right) = 2s\,(4R+r)$ and $\frac{2R-r}{r} = \sum \frac{a r_a}{a h_a} \geq \frac{\left(\sum \sqrt{a r_a}\right)^2}{\sum a h_a}$, therefore $\left(\sum \sqrt{a r_a}\right)^2 \leq 6s\,(2R-r)$

6). $\frac{2R-r}{r} = \sum \frac{\sin^4 \frac{A}{2}}{\frac{h_a}{r_a} \sin^4 \frac{A}{2}} \geq \frac{\left(\sum \sin^2 \frac{A}{2}\right)^2}{\sum \frac{h_a}{r_a} \sin^4 \frac{A}{2}}$, therefore $\sum \frac{h_a}{r_a} \sin^4 \frac{A}{2} \geq \frac{r(2R-r)}{4R^2}$

7). $\frac{2R-r}{r} = \sum \frac{\cos^4 \frac{A}{2}}{\frac{h_a}{r_a} \cos^4 \frac{A}{2}} \geq \frac{\left(\sum \cos^2 \frac{A}{2}\right)^2}{\sum \frac{h_a}{r_a} \cos^4 \frac{A}{2}}$, therefore $\sum \frac{h_a}{r_a} \cos^4 \frac{A}{2} \geq \frac{r(4R+r)}{4R^2(2R-r)}$

8). $4R+r = \sum \frac{r_a r_b}{h_c} \geq \frac{\left(\sum \sqrt{r_a r_b}\right)^2}{\sum h_c}$, therefore $\left(\sum \sqrt{r_a r_b}\right)^2 \leq \frac{(4R+r)\left(s^2+r^2+4Rr\right)}{2R}$

9). $4R+r = \sum \frac{r_a^2}{\frac{h_c r_a}{r_b}} \geq \frac{\left(\sum r_a\right)^2}{\sum \frac{h_c r_a}{r_b}}$, therefore $\sum \frac{h_c r_a}{r_b} \geq 4R+r$

10). $4R+r = \sum \frac{r_a^2 r_b^2}{h_c r_a r_b} \geq \frac{\left(\sum r_a r_b\right)^2}{\sum h_c r_a r_b}$, therefore $\sum h_c r_a r_b \geq \frac{s^4}{4R+r}$

11). $4R+r = \sum \frac{r_a^4}{\frac{h_c r_a^3}{r_b}} \geq \frac{\left(\sum r_a^2\right)^2}{\sum \frac{h_c r_a^3}{r_b}}$, therefore $\sum \frac{h_c r_a^3}{r_b} \geq \frac{\left((4R+r)^2-2s^2\right)^2}{4R+r}$

12). $4R+r = \sum \frac{r_a^6}{\frac{h_c r_a^5}{r_b}} \geq \frac{\left(\sum r_a^3\right)^2}{\sum \frac{h_c r_a^5}{r_b}}$, therefore $\sum \frac{h_c r_a^5}{r_b} \geq \frac{\left((4R+r)^3-12s^2R\right)^2}{4R+r}$

13). $4R+r = \sum \frac{\sin^4 \frac{A}{2}}{\frac{h_c}{r_a r_b} \sin^4 \frac{A}{2}} \geq \frac{\left(\sum \sin^2 \frac{A}{2}\right)^2}{\sum \frac{h_c}{r_a r_b} \sin^4 \frac{A}{2}}$, therefore

$\sum \frac{h_c}{r_a r_b} \sin^4 \frac{A}{2} \geq \frac{(2R-r)^2}{4R^2(4R+r)}$

14). $4R+r = \sum \frac{\cos^4 \frac{A}{2}}{\frac{h_c}{r_a r_b} \cos^4 \frac{A}{2}} \geq \frac{\left(\sum \cos^2 \frac{A}{2}\right)^2}{\sum \frac{h_c}{r_a r_b} \cos^4 \frac{A}{2}}$, therefore $\sum \frac{h_c}{r_a r_b} \cos^4 \frac{A}{2} \geq \frac{4R+r}{4R^2}$

15).$4R+r = \sum \frac{\sin^4 \frac{A}{2} \sin^4 \frac{B}{2}}{\frac{h_c}{r_a r_b} \sin^4 \frac{A}{2} \sin^4 \frac{B}{2}} \geq \frac{\left(\sum \sin^2 \frac{A}{2} \sin^2 \frac{B}{2}\right)^2}{\sum \frac{h_c}{r_a r_b} \sin^4 \frac{A}{2} \sin^4 \frac{B}{2}}$, therefore

$\sum \frac{h_c}{r_a r_b} \sin^4 \frac{A}{2} \sin^4 \frac{B}{2} \geq \frac{\left(s^2+r^2-8Rr\right)^2}{256R^4(4R+r)}$

16). $4R+r = \sum \frac{\cos^4 \frac{A}{2} \cos^4 \frac{B}{2}}{\frac{h_c}{r_a r_b} \cos^4 \frac{A}{2} \cos^4 \frac{B}{2}} \geq \frac{\left(\sum \cos^2 \frac{A}{2} \cos^2 \frac{B}{2}\right)^2}{\sum \frac{h_c}{r_a r_b} \cos^4 \frac{A}{2} \cos^4 \frac{B}{2}}$, therefore

$\sum \frac{h_c}{r_a r_b} \cos^4 \frac{A}{2} \cos^4 \frac{B}{2} \geq \frac{\left(s^2+(4R+r)^2\right)^2}{256R^4(4R+r)}$

Theorem 2. In all triangle ABC holds

1). $r_a + r_b = \frac{cs}{r_c}$ and his permutations

2). $\frac{as}{r_a} + r_a = \frac{bs}{r_b} + r_b = \frac{cs}{r_c} + r_c = 4R + r$

3). $\sum \frac{xh_a - yr}{h_a} = 3x - y$ for all $x, y \in R$

4). $\sum \frac{xr_a - yr}{r_a} = 3x - y$ for all $x, y \in R$

5). $\sum \frac{h_a+r}{h_a-r} = \frac{3\left(3s^2-r^2\right)}{s^2+r^2+2Rr}$

6). $\sum \frac{h_a-r_a}{h_a+r_a} = \frac{7r^2+10Rr-s^2}{s^2+r^2+2Rr}$

7). $\sum h_a r_a = \frac{r\left(s^2+(4R+r)^2\right)}{2R}$ or $\sum \frac{r_a}{a} = \frac{s^2+(4R+r)^2}{4sR}$

8). $\sum \frac{1}{h_a r_a} = \frac{4R+r}{s^2 r}$

Proof.

1). $r_a + r_b = \frac{sr}{s-a} + \frac{sr}{s-b} = \frac{c(s-c)}{r} = \frac{cs}{r_c}$ etc.

2). $r_a + r_b + r_c = 4R + r = \frac{cs}{r_c} + r_c$

3). $\sum \frac{xh_a - yr}{h_a} = \sum \left(x - yr\frac{1}{h_a}\right) = 3x - y$

4). $\sum \frac{xr_a - yr}{h_a} = \sum \left(x - yr\frac{1}{r_a}\right) = 3x - y$

5). $\sum \frac{h_a+r}{h_a-r} = \sum \frac{\frac{2sr}{a}+r}{\frac{2sr}{a}-r} = \sum \frac{2s+a}{2s-a} = \sum \left(1 + \frac{3a}{b+c}\right) = \frac{3\left(3s^2-r^2\right)}{s^2+r^2+2Rr}$

6). $\sum \frac{h_a-r_a}{h_a+r_a} = \sum \frac{\frac{2sr}{a}-\frac{sr}{s-a}}{\frac{2sr}{a}+\frac{sr}{s-a}} = \sum \left(1 - \frac{2a}{b+c}\right) = \frac{7r^2+10Rr-s^2}{s^2+r^2+2Rr}$

7). $\sum h_a r_a = \sum \frac{2sr}{a} \cdot \frac{sr}{s-a} = 2s^2r^2 \sum \frac{1}{a(s-a)} =$

$= 2sr^2 \sum \left(\frac{1}{a} + \frac{1}{s-a}\right) = \frac{r(s^2+(4R+r)^2)}{2R}$

8). $\sum \frac{1}{h_a r_a} = \frac{1}{2s^2r^2} \sum a(s-a) = \frac{4R+r}{s^2 r}$

Remark. *Because* $s^2 \geq 7r^2 + 10Rr$ *therefore* $\sum \frac{h_a-r_a}{h_a+r_a} \leq 0$ *this is a result of Cosnita, C., and Turtoiu, F., (see [1]).*

Remark. *Using 7) and 8) we have*

$$\left(\sum h_a r_a\right)\left(\sum \frac{1}{h_a r_a}\right) \geq 9$$

or

$$\left(s^2+(4R+r)^2\right)(4R+r)\geq 18s^2R$$

Corollary 2.1. In all triangle ABC holds

1). $r_a^\alpha + r_b^\alpha \geq 2\left(\frac{cs}{2r_c}\right)^\alpha$ and his permutation

2). $\min\left\{r_a^\alpha + \left(\frac{as}{r_a}\right)^\alpha ; r_b^\alpha + \left(\frac{bs}{r_b}\right)^\alpha ; r_c^\alpha + \left(\frac{cs}{r_c}\right)^\alpha\right\} \geq 3\left(\frac{4R+r}{3}\right)^\alpha$ for all $\alpha \in (-\infty, 0] \cup [1, +\infty)$ and for $\alpha \in (0,1)$ holds the reverse inequalities

3). $\sum\left(\frac{h_a+r}{h_a-r}\right)^\alpha \geq 3\left(\frac{3s^2-r^2}{s^2+r^2+2Rr}\right)^\alpha$

4). $\sum\left(h_a r_a\right)^\alpha \geq 3\left(\frac{r\left(s^2+(4R+r)^2\right)}{6R}\right)^\alpha$

5). $\sum\left(\frac{1}{h_a r_a}\right)^\alpha \geq 3\left(\frac{4R+r}{3s^2 r}\right)^\alpha$

Corollary 2.2. Let ABC be a triangle, and $\{xh_a, xh_b, xh_c\} \geq yr$, where $x, y \geq 0$, then

1). $\left(\sum\sqrt{xh_a - yr}\right)^2 \leq \frac{(3x-y)\left(s^2+r^2+4Rr\right)}{2R}$

2). $\left(\sum\sqrt{(xh_a - yr)\,h_a}\right)^2 \leq \frac{(3x-y)\left(\left(s^2+r^2+4Rr\right)^2-16s^2Rr\right)}{4R^2}$

3). $\left(\sum\sqrt{(xh_a - yr)\,h_b}\right)^2 \leq \frac{2(3x-y)s^2r}{R}$

4). $\left(\sum\sqrt{(xh_a - yr)\,r_a}\right)^2 \leq \frac{(3x-y)r\left(s^2+(4R+r)^2\right)}{2R}$

Proof.

1). $3x-y=\sum\frac{xh_a-yr}{h_a} \geq \frac{\left(\sum\sqrt{xh_a-yr}\right)^2}{\sum h_a}$ therefore $\left(\sum\sqrt{xh_a - yr}\right)^2 \leq \frac{(3x-y)\left(s^2+r^2+4Rr\right)}{2R}$

2). $3x-y=\sum\frac{(xh_a-yr)h_a}{h_a^2} \geq \frac{\left(\sum\sqrt{(xh_a-yr)h_a}\right)^2}{\sum h_a^2}$, therefore $\left(\sum\sqrt{(xh_a - yr)\,h_a}\right)^2 \leq \frac{(3x-y)\left(\left(s^2+r^2+4Rr\right)^2-16s^2Rr\right)}{4R^2}$

3). $3x-y=\sum\frac{(xh_a-yr)h_b}{h_a h_b} \geq \frac{\left(\sum\sqrt{(xh_a-yr)h_b}^2\right)}{\sum h_a h_b}$, therefore $\left(\sum\sqrt{(xh_a - yr)\,h_b}\right)^2 \leq \frac{2(3x-y)s^2r}{R}$

4). $3x-y=\sum\frac{(xh_a-yr)r_a}{h_a r_a} \geq \frac{\left(\sum\sqrt{(xh_a-yr)r_a}\right)^2}{\sum h_a r_a}$, therefore $\left(\sum\sqrt{(xh_a - yr)\,r_a}\right)^2 \leq \frac{(3x-y)\left(s^2+(4R+r)^2\right)}{2R}$

Corollary 2.3. Let ABC be a triangle, and $\{xr_a, xr_b, xr_c\} \geq yr$, where $x, y \geq 0$, then

1). $\left(\sum \sqrt{xr_a - yr}\right)^2 \leq (3x-y)(4R+r)$

2). $\left(\sum \sqrt{(xr_a - yr) r_a}\right)^2 \leq (3x-y)\left((4R+r)^2 - 2s^2\right)$

3). $\left(\sum \sqrt{(xr_a - yr) r_b}\right)^2 \leq (3x-y) s^2$

4). $\left(\sum \sqrt{(xr_a - yr) h_b}\right)^2 \leq \frac{(3x-y)\left(s^2+(4R+r)^2\right)}{2R}$

Proof.

1). $3x - y = \sum \frac{xr_a - yr}{r_a} \geq \frac{\left(\sum \sqrt{xr_a - yr}\right)^2}{\sum r_a}$ therefore $\left(\sum \sqrt{xr_a - yr}\right)^2 \leq (3x-y)(4R+r)$

2). $3x - y = \sum \frac{(xr_a - yr) r_a}{r_a^2} \geq \frac{\left(\sum \sqrt{(xr_a - yr) r_a}\right)^2}{\sum r_a^2}$, therefore $\left(\sum \sqrt{(xr_a - yr) r_a}\right)^2 \leq (3x-y)\left((4R+r)^2 - 2s^2\right)$

3). $3x - y = \sum \frac{(xr_a - yr) r_b}{r_a r_b} \geq \frac{\left(\sum \sqrt{(xr_a - yr) r_b}\right)^2}{\sum r_a r_b}$, therefore $\left(\sum \sqrt{(xr_a - yr) r_b}\right)^2 \leq (3x-y) s^2$

4). $3x - y = \sum \frac{(xr_a - yr) h_b}{r_a h_a} \geq \frac{\left(\sum \sqrt{(xr_a - yr) h_b}\right)^2}{\sum r_a h_a}$, therefore $\left(\sum \sqrt{(xr_a - yr) h_b}\right)^2 \leq \frac{(3x-y)\left(s^2+(4R+r)^2\right)}{2R}$

Corollary 2.4. In all triangle ABC holds

1). $\frac{48r(3R-r)}{s^2+r^2+2Rr} \leq \sum \frac{h_a+r}{h_a-r} \leq \frac{12\left(3R^2+3Rr+2r^2\right)}{s^2+r^2+2Rr}$

2). $\frac{6r(R-2r)}{s^2+r^2+2Rr} \leq \sum \frac{r_a-h_a}{r_a+h_a} \leq \frac{2\left(2R^2-3Rr-2r^2\right)}{s^2+r^2+2Rr}$

3). $\frac{8R^2+12Rr-2r^2}{R} \leq \sum h_a r_a \leq \frac{2\left(5R^2+3Rr+r^2\right)}{R}$

4). $\frac{4R+r}{r^2(16R-5r)} \leq \sum \frac{1}{h_a r_a} \leq \frac{4R+r}{r(4R^2+4Rr+3r^2)}$

Proof. In Theorem 2 for 5), 6), 7) and 8) we use the inequalities

$$r(16R-5r) \leq s^2 \leq 4R^2 + 4Rr + 3r^2$$

Corollary 2.5. In all triangle ABC holds

1). $\left(\sum \sqrt{(h_a + r) h_b}\right)^2 \leq \frac{3r\left(3s^2-r^2\right)\left(3s^2-r^2-4Rr\right)}{2R(s^2+r^2+2Rr)}$

2). $\left(\sum \sqrt{\frac{(h_a+r)r_a}{h_a+r}}\right)^2 \leq \frac{3(4R+r)\left(3s^2-r^2\right)}{s^2+r^2+2Rr}$

3). $\left(\sum\sqrt{\frac{h_a+r}{h_a-r}}\sin^2\frac{A}{2}\right)^2 \le \frac{3\left(3s^2-r^2\right)\left(8R^2+r^2-s^2\right)}{8R^2(s^2+r^2+2Rr)}$

4). $\left(\sum\sqrt{\frac{h_a+r}{h_a-r}}\cos^2\frac{A}{2}\right)^2 \le \frac{3\left(3s^2-r^2\right)\left((4R+r)^2-s^2\right)}{8R^2(s^2+r^2+2Rr)}$

5). $\left(\sum\sqrt{\frac{h_a+r}{h_a-r}}\sin\frac{A}{2}\right)^2 \le \frac{3\left(3s^2-r^2\right)(2R-r)}{2R(s^2+r^2+2Rr)}$

6). $\left(\sum\sqrt{\frac{h_a+r}{h_a-r}}\cos\frac{A}{2}\right)^2 \le \frac{3\left(3s^2-r^2\right)(4R+r)}{2R(s^2+r^2+2Rr)}$

7). $\left(\sum\sqrt{\frac{(h_a+r)(r_a-h_a)}{(h_a-r)(r_a+h_a)}}\right)^2 \le \frac{3\left(3s^2-r^2\right)\left(s^2-7r^2-10Rr\right)}{(s^2+r^2+2Rr)^2}$

8). $\left(\sum r_a\sqrt{h_a}\right)^2 \le \frac{r(4R+r)\left(s^2+(4R+r)^2\right)}{2R}$

Proof.

1). $\frac{3\left(3s^2-r^2\right)}{s^2+r^2+2Rr} = \sum\frac{(h_a+r)h_b}{(h_a-r)h_b} \ge \frac{\left(\sum\sqrt{(h_a+r)h_b}\right)^2}{\sum(h_a-r)h_b}$ therefore
$\left(\sum\sqrt{(h_a+r)\,h_b}\right)^2 \le \frac{3r\left(3s^2-r^2\right)\left(3s^2-r^2-4Rr\right)}{2R(s^2+r^2+2Rr)}$

2). $\frac{3\left(3s^2-r^2\right)}{s^2+r^2+2Rr} = \sum\frac{\frac{(h_a+r)r_a}{h_a-r}}{r_a} \ge \frac{\left(\sum\sqrt{\frac{(h_a+r)r_a}{h_a-r}}\right)^2}{\sum r_a}$ therefore
$\left(\sum\sqrt{\frac{(h_a+r)r_a}{h_a-r}}\right)^2 \le \frac{3(4R+r)\left(3s^2-r^2\right)}{s^2+r^2+2Rr}$

3). $\frac{3\left(3s^2-r^2\right)}{s^2+r^2+2Rr} = \sum\frac{\frac{h_a+r}{h_a-r}\sin^4\frac{A}{2}}{\sin^4\frac{A}{2}} \ge \frac{\left(\sum\sqrt{\frac{h_a+r}{h_a-r}}\sin^2\frac{A}{2}\right)^2}{\sum\sin^4\frac{A}{2}}$ therefore
$\left(\sum\sqrt{\frac{h_a+r}{h_a-r}}\sin^2\frac{A}{2}\right)^2 \le \frac{3\left(3s^2-r^2\right)\left(8R^2+r^2-s^2\right)}{8R^2(s^2+r^2+2Rr)}$

4). $\left(\sum\sqrt{\frac{h_a+r}{h_a-r}}\cos^2\frac{A}{2}\right)^2 \le \left(\sum\frac{h_a+r}{h_a-r}\right)\left(\sum\cos^4\frac{A}{2}\right) = \frac{3\left(3s^2-r^2\right)\left((4R+r)^2-s^2\right)}{8R^2(s^2+r^2+2Rr)}$

5). $\left(\sum\sqrt{\frac{h_a+r}{h_a-r}}\sin\frac{A}{2}\right)^2 \le \left(\sum\frac{h_a+r}{h_a-r}\right)\left(\sum\sin^2\frac{A}{2}\right) = \frac{3\left(3s^2-r^2\right)(2R-r)}{2R(s^2+r^2+2Rr)}$

6). $\left(\sum\sqrt{\frac{h_a+r}{h_a-r}}\cos\frac{A}{2}\right)^2 \le \left(\sum\frac{h_a+r}{h_a-r}\right)\left(\sum\cos^2\frac{A}{2}\right) = \frac{3\left(3s^2-r^2\right)(4R+r)}{2R(s^2+r^2+2Rr)}$

7). $\left(\sum\sqrt{\frac{(h_a+r)(r_a-h_a)}{(h_a-r)(r_a+h_a)}}\right)^2 \le \left(\sum\frac{h_a+r}{h_a-r}\right)\left(\sum\frac{r_a-h_a}{r_a+h_a}\right) = \frac{3\left(3s^2-r^2\right)\left(s^2-7r^2-10Rr\right)}{(s^2+r^2+2Rr)^2}$

8). $\frac{r\left(s^2+(4R+r)^2\right)}{2R} = \sum\frac{h_a r_a^2}{r_a} \ge \frac{\left(\sum r_a\sqrt{h_a}\right)^2}{\sum r_a}$, therefore

$\left(\sum r_a\sqrt{h_a}\right)^2 \leq \frac{r(4R+r)\left(s^2+(4R+r)^2\right)}{2R}$

REFERENCES

[1] Bottema,O., *Geometric inequalities,* Gröningen, 1969.
[2] Octogon Mathematical Magazine, 1993-2009.

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A cathegory of inequalities

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ABSTRACT. In this paper we present some inequalities which generate a cathegory of elementary inequalities.

MAIN RESULTS

Theorem 1. If $a, b, c, d, x, y, z > 0$ then

$$(a+c)\,x^2 + by^2 + cz^2 \geq \left(\sqrt{a}x + \sqrt{b}y\right)\left(\sqrt{c}x + \sqrt{d}y\right)$$

Proof. We have

$$(a+c)\,x^2 + by^2 + cz^2 = \left(\left(\sqrt{a}x\right)^2 + \left(\sqrt{b}y\right)^2\right) + \left(\sqrt{c}x\right)^2 + \left(\sqrt{d}z\right)^2 \geq$$

$$\geq \frac{\left(\sqrt{a}x + \sqrt{b}y\right)^2}{2} + \frac{\left(\sqrt{c}x + \sqrt{d}z\right)^2}{2} \geq \left(\sqrt{a}x + \sqrt{b}y\right)\left(\sqrt{c}x + \sqrt{d}z\right)$$

Corollary 1.1. If $a, b, c, d, x_k > 0$ $(k = 1, 2, ..., n)$, then

1). $\sum\limits_{cyclic} \frac{(a+c)x_1^2 + bx_2^2 + cx_3^2}{\left(\sqrt{a}x_1 + \sqrt{b}x_2\right)\left(\sqrt{c}x_1 + \sqrt{d}x_3\right)} \geq n$

2). $\sum\limits_{cyclic} \frac{(a+c)x_1^2 + bx_2^2 + cx_3^2}{\sqrt{a}x_1 + \sqrt{b}x_2} \geq \left(\sqrt{c} + \sqrt{d}\right) \sum\limits_{k=1}^{n} x_k$

3). $\sum\limits_{cyclic} \frac{(a+c)x_1^2 + bx_2^2 + cx_3^2}{\sqrt{c}x_1 + \sqrt{d}x_3} \geq \left(\sqrt{a} + \sqrt{b}\right) \sum\limits_{k=1}^{n} x_k$

4). $\sum\limits_{cyclic} \frac{(a+c)x_1^2 + bx_2^2 + cx_3^2}{\sqrt{a}x_1 + \sqrt{b}x_2} \geq n \sqrt[n]{\prod\limits_{cyclic} \left(\sqrt{c}x_1 + \sqrt{d}x_3\right)} \geq 2n\sqrt[4]{cd}\sqrt[n]{\prod\limits_{k=1}^{n} x_k}$

5). $\sum\limits_{cyclic} \frac{(a+c)x_1^2 + bx_2^2 + cx_3^2}{\sqrt{c}x_1 + \sqrt{d}x_3} \geq n \sqrt[n]{\prod\limits_{cyclic} \left(\sqrt{a}x_1 + \sqrt{b}x_2\right)} \geq 2n\sqrt[4]{ab}\sqrt[n]{\prod\limits_{k=1}^{n} x_k}$

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Proof. We have

1). $\sum\limits_{cyclic} \frac{(a+c)x_1^2+bx_2^2+cx_3^2}{\left(\sqrt{a}x_1+\sqrt{b}x_2\right)\left(\sqrt{c}x_1+\sqrt{d}x_3\right)} \geq \sum 1 = n$

2). $\sum\limits_{cyclic} \frac{(a+c)x_1^2+bx_2^2+cx_3^2}{\sqrt{a}x_1+\sqrt{b}x_2} \geq \sum \left(\sqrt{c}x_1+\sqrt{d}x_3\right) = \left(\sqrt{c}+\sqrt{d}\right)\sum\limits_{k=1}^{n} x_k$

3). $\sum\limits_{cyclic} \frac{(a+c)x_1^2+bx_2^2+cx_3^2}{\sqrt{c}x_1+\sqrt{d}x_3} \geq \sum \left(\sqrt{a}x_1+\sqrt{b}x_2\right) = \left(\sqrt{a}+\sqrt{b}\right)\sum\limits_{k=1}^{n} x_k$

4). $\sum\limits_{cyclic} \frac{(a+c)x_1^2+bx_2^2+cx_3^2}{\sqrt{a}x_1+\sqrt{b}x_2} \geq \sum \left(\sqrt{c}x_1+\sqrt{d}x_3\right) \geq$

$\geq n\sqrt[n]{\prod\limits_{cyclic}\left(\sqrt{c}x_1+\sqrt{d}x_3\right)} \geq 2n\sqrt[4]{cd}\sqrt[n]{\prod\limits_{k=1}^{n} x_k}$

5). $\sum\limits_{cyclic} \frac{(a+c)x_1^2+bx_2^2+cx_3^2}{\sqrt{c}x_1+\sqrt{d}x_3} \geq \sum \left(\sqrt{a}x_1+\sqrt{b}x_2\right) \geq n\sqrt[n]{\prod\left(\sqrt{a}x_1+\sqrt{b}x_2\right)} \geq$

$\geq 2n\sqrt[4]{ab}\sqrt[n]{\prod\limits_{k=1}^{n} x_k}$

Theorem 2. If $a_i, b_i > 0 \ (i = 1, 2, ..., k-1)$, $x_j > 0 \ (j = 1, 2, ..., k)$, then

$$(a_1 + a_2 + ... + a_{k-1})\, x_1^{k-1} + b_1 x_2^{k-1} + b_2 x_3^{k-1} + ... + b_{k-1} x_k^{k-1} \geq$$

$$\geq \frac{k-1}{2^{k-2}} \left(\sqrt[k-1]{a_1}x_1 + \sqrt[k-1]{b_1}x_2\right)\left(\sqrt[k-1]{a_2}x_1 + \sqrt[k-1]{b_2}x_3\right) \cdot ...$$

$$\cdot \left(\sqrt[k-1]{a_{k-1}}x_1 + \sqrt[k-1]{b_{k-1}}x_k\right)$$

Proof. We have

$$(a_1 + a_2 + ... + a_{k-1})\, x_1^{k-1} + b_1 x_2^{k-1} + b_2 x_3^{k-1} + ... + b_{k-1} x_k^{k-1} =$$

$$= \sum \left(a_1 x_1^{k-1} + b_1 x_2^{k-1}\right) \geq \sum \frac{1}{2^{k-2}} \left(\sqrt[k-1]{a_1}x_1 + \sqrt[k-1]{b_1}x_2\right)^{k-1} \geq$$

$$\geq \frac{k-1}{2^{k-2}} \prod \left(\sqrt[k-1]{a_1}x_1 + \sqrt[k-1]{b_1}x_2\right)$$

Corollary 2.1. If $x_i > 0 \ (i = 1, 2, ..., n)$, $a_j, b_j > 0 \ (j = 1, 2, ..., k-1)$, $k \in \{2, 3, ..., n\}$, then

1). $\sum\limits_{cyclic} \frac{(a_1+...+a_{k-1})x_1^{k-1}+b_1x_2^{k-1}+...+b_{k-1}x_k^{k-1}}{\left(\sqrt[k-1]{a_1}x_1+\sqrt[k-1]{b_1}x_2\right)...\left(\sqrt[k-1]{a_1}x_1+\sqrt[k-1]{b_{k-1}}x_k\right)} \geq \frac{n(k-1)}{2^{k-2}}$

2). $\sum\limits_{cyclic} \frac{(a_1+...+a_{k-1})x_1^{k-1}+b_1x_2^{k-1}+...+b_{k-1}x_k^{k-1}}{\left(\sqrt[k-1]{a_1}x_1+\sqrt[k-1]{b_1}x_2\right)...\left(\sqrt[k-1]{a_1}x_1+\sqrt[k-1]{b_{k-2}}x_{k-1}\right)} \geq$

$\geq \left(\sqrt[k-1]{a_1} + \sqrt[k-1]{b_{k-1}} \right) \sum\limits_{i=1}^{n} x_i$

Theorem 3. If $x_i > 0$ $(i = 1, 2, ..., k)$ and $\alpha, \beta \geq 0$, then

$$\frac{\sum\limits_{i=1}^{k} x_i^{\alpha+\beta}}{\sum\limits_{i=1}^{k} x_i^{\alpha}} \geq \frac{1}{k} \sum_{i=1}^{k} x_i^{\beta}$$

Proof. This inequality holds from Chebishev‘s inequality.

Corollary 3.1. If $x_i > 0$ $(i = 1, 2, ..., n)$, $\alpha, \beta \geq 0$ and $k \in \{2, 3, ..., n\}$, then

1). $\sum\limits_{cyclic} \frac{x_1^{\alpha+\beta} + ... + x_k^{\alpha+\beta}}{x_1^{\alpha} + ... + x_k^{\alpha}} \geq \sum\limits_{i=1}^{k} x_i^{\beta}$

2). $\sum\limits_{cyclic} \frac{x_1^{\alpha+\beta} + ... + x_k^{\alpha+\beta}}{\left(x_1^{\alpha} + ... + x_k^{\alpha}\right)\left(x_1^{\beta} + ... + x_k^{\beta}\right)} \geq \frac{n}{k}$

3). $\sum\limits_{cyclic} \frac{x_1^{\alpha+\beta} + ... + x_k^{\alpha+\beta}}{x_1^{\alpha} + ... + x_k^{\alpha}} \geq \sum \left(x_1 ... x_k\right)^{\frac{\beta}{k}}$

Theorem 4. If $x_i > 0$ $(i = 1, 2, ..., k)$, $\alpha \in (-\infty, 0] \cup [1, +\infty)$, then

$$x_1^{\alpha} + ... + x_k^{\alpha} \geq k^{1-\alpha} \left(x_1 ... x_k\right)^{\alpha}$$

and if $\alpha \in (0, 1)$, then holds the reverse inequality.
Proof. This is a consequence of Jensen‘s inequality.

Corollary 4.1. If $x_i > 0$ $(i = 1, 2, ..., n)$ and $k \in \{2, 3, ..., n\}$, then

1). $\sum\limits_{cyclic} \frac{x_1^{\alpha} + ... + x_k^{\alpha}}{\left(x_1 + ... + x_k\right)^{\alpha}} \geq n k^{1-\alpha}$

2). $\sum\limits_{cyclic} \frac{x_1^{\alpha} + ... + x_k^{\alpha}}{\left(x_1 + ... + x_k\right)^{\alpha-1}} \geq k^{2-\alpha} \sum\limits_{i=1}^{n} x_i$

for all $\alpha \in (-\infty, 0] \cup [1, +\infty)$ and holds the reverse inequalities for all $\alpha \in (0, 1)$.

Theorem 5. If $a, b > 0$ and $x, y > 0$ then

$$\frac{a\left(x^2 + y^2\right) - bxy}{a\left(x^2 + y^2\right) + bxy} \geq \frac{2a - b}{2a + b}$$

Proof. After elementary calculus we get

$$(x-y)^2 \geq 0$$

Corollary 5.1. If $x_i > 0$ $(i = 1, 2, ..., n)$ and $a, b > 0$, then
1). $\sum\limits_{cyclic} \frac{a\left(x_1^2+x_2^2\right)-bx_1x_2}{a\left(x_1^2+x_2^2\right)+bx_1x_2} \geq \frac{n(2a-b)}{2a+b}$
2). $\sum\limits_{cyclic} \frac{a\left(x_1^2+x_2^2\right)+bx_1x_2}{a\left(x_1^2+x_2^2\right)-bx_1x_2} \leq \frac{n(2a+b)}{2a-b}$

Theorem 6. If $x \in \left(0, \frac{\pi}{2}\right)$ and $0 < b < a$, then
1). $a - b\sin x \geq \sqrt{a^2-b^2}\cos x$
2). $a - b\cos x \geq \sqrt{a^2-b^2}\sin x$
Proof. After elementary computation we get
1). $(b - a\sin x)^2 \geq 0$
2). $(b - a\cos x)^2 \geq 0$

Corollary 6.1. If $x \in \left(0, \frac{\pi}{2}\right)$, $0 < b < a$, $x_k \in \left(0, \frac{\pi}{2}\right)$ $(k = 1, 2, ..., n)$, then
1). $(a - b\sin x)(a - b\cos x) \geq \left(a^2-b^2\right)\sin x\cos x$
2). $\frac{a-b\sin x}{\cos x} + \frac{a-b\cos x}{\sin x} \geq 2\sqrt{a^2-b^2}$
3). $\sum\limits_{k=1}^{n} \frac{a-b\sin x_k}{\cos x_k} \geq n\sqrt{a^2-b^2}$
4). $\sum\limits_{k=1}^{n} \frac{a-b\cos x_k}{\sin x_k} \geq n\sqrt{a^2-b^2}$
5). $\sum\limits_{cyclic} \frac{a-b\sin x_1}{\cos x_2} \geq n\sqrt{a^2-b^2}$
6). $\sum\limits_{cyclic} \frac{a-b\cos x_1}{\sin x_2} \geq n\sqrt{a^2-b^2}$
7). $\left(a+\sqrt{a^2-b^2}\right) tg^2\frac{x}{2} + a - \sqrt{a^2-b^2} \geq 2btg\frac{x}{2}$
for all $x \in R$
Theorem 7. If $x, y > 0$, then

$$\frac{x^4+x^2y^2+y^4}{xy\left(x^2+y^2\right)} \geq \frac{3}{2}$$

Proof. After elementary computation we get

$$(x-y)^2\left(2x^2+xy+2y^2\right) \geq 0$$

Corollary 7.1. If $x_k > 0$ $(k = 1, 2, ..., n)$, then
1). $\sum\limits_{cyclic} \frac{x_1^4+x_1^2x_2^2+x_2^4}{x_1x_2\left(x_1^2+x_2^2\right)} \geq \frac{3n}{2}$

2). $\sum\limits_{cyclic} \frac{x_1^4+x_1^2x_2^2+x_2^4}{x_1\left(x_1^2+x_2^2\right)} \geq \frac{3}{2}\sum\limits_{k=1}^{n} x_k$

3). $\sum\limits_{cyclic} \frac{x_1^4+x_1^2x_2^2+x_2^4}{x_2\left(x_1^2+x_2^2\right)} \geq \frac{3}{2}\sum\limits_{k=1}^{n} x_k$

4). $\sum\limits_{cyclic} \frac{x_1^4+x_1^2x_2^2+x_2^4}{x_1x_2} \geq 3\sum\limits_{k=1}^{n} x_k^2$

Theorem 8. If $a > 0$, $x > -a$, then

$$\frac{x+2a}{\sqrt{x+a}} \geq 2\sqrt{a}$$

Proof. After elementary computation we get $x^2 \geq 0$.

Corollary 8.1. If $x_k > -a, a > 0$ $(k = 1, 2, ..., n)$, then

1). $\sum\limits_{k=1}^{n} \frac{x_k+2a}{\sqrt{x_k+a}} \geq 2n\sqrt{a}$

2). $\sum\limits_{k=1}^{n} \frac{x_1+2a}{\sqrt{x_2+a}} \geq 2n\sqrt{a}$

3). $\sum (x_1+2a)\sqrt{x_2+a} \geq 2\sqrt{a}\left(n+\sum\limits_{k=1}^{n} x_k\right)$

4). $\sum (x_k+2a)\sqrt{x_k+a} \geq 2\sqrt{a}\left(n+\sum\limits_{k=1}^{n} x_k\right)$

Proof. We have

2). $\sum\limits_{cyclic} \frac{x_1+2a}{\sqrt{x_2+a}} \geq n\sqrt[n]{\prod\limits_{k=1}^{n} \frac{x_k+2a}{\sqrt{x_k+a}}} \geq 2n\sqrt{a}$

Theorem 9. If $a, b, x > 0$, then

$$\frac{\sqrt{x+a}+\sqrt{x+b}}{\sqrt{x}+\sqrt{x+a+b}} \geq 1$$

Proof. After elementary computation we get $ab \geq 0$.

Corollary 9.1. If $a, b, x_k > 0$ $(k = 1, 2, ..., n)$, then

1). $\sum\limits_{k=1}^{n} \frac{\sqrt{x_k+a}+\sqrt{x_k+b}}{\sqrt{x_k}+\sqrt{x_k+a+b}} \geq n$

2). $\sum\limits_{cyclic} \frac{\sqrt{x_1+a}+\sqrt{x_1+b}}{\sqrt{x_2}+\sqrt{x_2+a+b}} \geq n$

3). $\sum\limits_{k=1}^{n} \left(\sqrt{x_k+a}+\sqrt{x_k+b}\right)\left(\sqrt{x_k+a+b}-\sqrt{x_k}\right) \geq n\left(a+b\right)$

4). $\sum\limits_{cyclic}\left(\sqrt{x_1+a}+\sqrt{x_1+b}\right)\left(\sqrt{x_2+a+b}-\sqrt{x_2}\right)\geq n\left(a+b\right)$

5). $\frac{\sqrt{2a}+\sqrt{a+b}}{\sqrt{a}+\sqrt{2a+b}}+\frac{\sqrt{a+b}+\sqrt{2b}}{\sqrt{b}+\sqrt{a+b}}\geq 2$

Theorem 10. If $x>0$, then

$$\sqrt{x^2+9}\sin\frac{\pi}{x}>3$$

Proof. If $x>2$, then $\frac{\pi}{x}<\frac{\pi}{2}$, $tg\frac{\pi}{x}>\frac{\pi}{x}>\frac{3}{x}$,

$$\cos^2 x=\frac{1}{1+tg^2x}<\frac{1}{1+\left(\frac{\pi}{x}\right)^2}<\frac{1}{1+\left(\frac{3}{x}\right)^2}=\frac{x^2}{x^2+9}$$

so

$$\sin\frac{\pi}{x}>\frac{3}{\sqrt{x^2+9}}$$

Corollary 10.1. If $x_k>2$ $(k=1,2,...,n)$, then

1). $\sum\limits_{k=1}^{n}\sqrt{x_k^2+9}\sin\frac{\pi}{x_k}>3n$

2). $\sum\limits_{cyclic}\sqrt{x_1^2+9}\sin\frac{\pi}{x_2}>3n$

3). $\sum\limits_{k=1}^{n}x_k^2+9n\geq 9\sum\limits_{k=1}^{n}\frac{1}{\sin^2\frac{\pi}{x_k}}$

4). $\cos\frac{\pi}{x}>\frac{3(x-2)}{\sqrt{13x^2-36x+36}}$ for all $x>2$

5). $\sum\limits_{k=1}^{n}\sqrt{13x_k^2-36x_k+36}\cos\frac{\pi}{x_k}>3\sum\limits_{k=1}^{n}x_k-6n$

6). $\sum\limits_{cyclic}\sqrt{13x_1-36x_1+36}\cos\frac{\pi}{x_2}>3\sum\limits_{k=1}^{n}x_k-6n$

7). $\sin\left(\frac{\pi}{x}+\frac{\pi}{y}\right)>\frac{9(x-2)}{\sqrt{(13x^2-36x+36)(y^2+9)}}+\frac{9(y-2)}{\sqrt{(13y^2-36y+36)(x^2+9)}}$ for all $x,y>2$

Proof. 4). If in $\sin\frac{\pi}{y}>\frac{3}{\sqrt{y^2+9}}$ we take $y=\frac{2x}{x-2}>2$ we get

$$\sin\left(\frac{\pi}{2}-\frac{\pi}{x}\right)=\cos\frac{\pi}{x}>\frac{3\left(x-2\right)}{\sqrt{13x^2-36x+36}}$$

7). $\sin\left(\frac{\pi}{x}+\frac{\pi}{y}\right)=\sin\frac{\pi}{x}\cos\frac{\pi}{y}+\cos\frac{\pi}{x}\sin\frac{\pi}{y}>$
$\frac{3}{\sqrt{x^2+9}}\cdot\frac{3(x-2)}{\sqrt{13y^2-36y+36}}+\frac{3}{\sqrt{y^2+9}}\cdot\frac{3(x-2)}{\sqrt{13x^2-36x+36}}$

Theorem 11. If $x\in\left(0,\frac{\pi}{2}\right)$ and $a,b,c,d>0$ then

$$\left(a+\frac{b}{\sin x}\right)\left(c+\frac{d}{\cos x}\right)\geq\left(\sqrt{ac}+\sqrt{2bd}\right)^2$$

Proof. We have

$$\left(a+\frac{b}{\sin x}\right)\left(c+\frac{d}{\cos x}\right)\geq\left(\sqrt{ac}+\sqrt{\frac{bd}{\sin x\cos x}}\right)^2\geq\left(\sqrt{ac}+\sqrt{2bd}\right)^2$$

Corollary 11.1. If $x_k\in\left(0,\frac{\pi}{2}\right)$ $(k=1,2,...,n)$, then
1). $\sum\limits_{k=1}^{n}\left(a+\frac{b}{\sin x_k}\right)\left(c+\frac{d}{\cos x_k}\right)\geq n\left(\sqrt{ac}+\sqrt{2bd}\right)^2$
2). $\sum\limits_{cyclic}\left(a+\frac{b}{\sin x_1}\right)\left(c+\frac{b}{\cos x_2}\right)\geq n\left(\sqrt{ac}+\sqrt{2bd}\right)^2$
3). $\left(a+\frac{b}{\sin x}\right)\left(c+\frac{b}{\cos x}\right)+\left(a+\frac{b}{\cos x}\right)\left(c+\frac{b}{\sin x}\right)\geq 2\left(\sqrt{ac}+\sqrt{2bd}\right)^2$

Corollary 11.2. If $x\in R$ and $a,b,c,d,e>0$ then
1). $\left(a+\frac{b}{e+\sin^2 x}\right)\left(c+\frac{d}{e+\cos^2 x}\right)\geq\left(\sqrt{ac}+\sqrt{\frac{3bd}{2e+1}}\right)^2$
2). $\left(a+\frac{b}{e+\sin^2 x}\right)\left(c+\frac{d}{e+\cos^2 x}\right)+\left(a+\frac{b}{e+\cos^2 x}\right)\left(c+\frac{d}{e+\sin^2 x}\right)\geq$
$\geq 2\left(\sqrt{ac}+\sqrt{\frac{3bd}{2e+1}}\right)^2$
3). $\sum\limits_{k=1}^{n}\left(a+\frac{b}{e+\sin^2 x_k}\right)\left(c+\frac{d}{e+\cos^2 x_k}\right)\geq n\left(\sqrt{ac}+\sqrt{\frac{3bd}{2e+1}}\right)^2$
4). $\sum\limits_{cyclic}\left(a+\frac{b}{e+\sin^2 x_1}\right)\left(c+\frac{d}{e+\cos^2 x_1}\right)\geq n\left(\sqrt{ac}+\sqrt{\frac{3bd}{2e+1}}\right)^2$

Proof. 1). $\left(a+\frac{b}{e+\sin^2 x}\right)\left(c+\frac{d}{e+\cos^2 x}\right)\geq\left(\sqrt{ac}+\sqrt{\frac{bd}{\left(e+\sin^2 x\right)\left(e+\cos^2 x\right)}}\right)^2\geq$
$\left(\sqrt{ac}+\sqrt{\frac{3bd}{2e+1}}\right)^2$

Theorem 12. If $a>0$ and $\frac{4a}{5}\leq x\leq a$, then

$$\frac{3a-\sqrt{a^2-x^2}}{x+4a}\geq\frac{1}{2}$$

Corollary 12.1. If $a>0$ and $\frac{4a}{5}\leq x_k\leq a$ $(k=1,2,...,n)$, then
1). $\sum\limits_{k=1}^{n}\frac{3a-\sqrt{a^2-x_k^2}}{x_k+4a}\geq\frac{n}{2}$
2). $\sum\limits_{cyclic}\frac{3a-\sqrt{a^2-x_1^2}}{x_2+4a}\geq\frac{n}{2}$
3). $\frac{3-\sin t}{4+\sin t}\geq\frac{1}{2}$ if $\arcsin\frac{4}{5}\leq t\leq\frac{\pi}{2}$
4). $\frac{3-\cot s}{4+\cot s}\geq\frac{1}{2}$ if $0\leq t\leq\arccos\frac{4}{5}$

5). $\sum\limits_{k=1}^{n} \frac{3-\sin t_k}{4+\sin t_k} \geq \frac{n}{2}$ if $\arcsin \frac{4}{5} \leq t_k \leq \frac{\pi}{2}$ $(k = 1, 2, ..., n)$

6). $\sum\limits_{cyclic} \frac{3-\sin t_1}{4+\sin t_2} \geq \frac{n}{2}$ if $\arcsin \frac{4}{5} \leq t_k \leq \frac{\pi}{2}$ $(k = 1, 2, ..., n)$

7). $\sum\limits_{k=1}^{n} \frac{3-\cos t_k}{4+\cos t_k} \geq \frac{n}{2}$ if $0 \leq t_k \leq \arccos \frac{4}{5}$ $(k = 1, 2, ..., n)$

8). $\sum\limits_{cyclic} \frac{3-\cos t_1}{4+\cos t_2} \geq \frac{n}{2}$ if $0 \leq t_k \leq \arccos \frac{4}{5}$ $(k = 1, 2, ..., n)$

9). $3na - \sum\limits_{k=1}^{n} \sqrt{a^2 - x_k^2} \geq 2a + \frac{1}{2} \sum\limits_{k=1}^{n} x_k$

10). $\sum\limits_{k=1}^{n} \frac{(x_k+4a)^2}{3a-\sqrt{a^2-x_k^2}} \leq 2 \sum\limits_{k=1}^{n} x_k + 8na$

Proof. 3). In Theorem 12 we take $x = a \sin t$

4). In Theorem 12 we take $x = a \cos t$

Theorem 13. If $x > 0$ then

$$\frac{(x+2)\ln(x+1)}{x} \geq 2$$

Proof. If $f(x) = \ln(x+1) - \frac{2x}{x+2}$, then

$$f'(x) = \frac{x^2}{(x+1)(x+2)^2} \geq 0,$$

therefore $f(x) \geq f(0) = 0$.

Corollary 13.1. If $x_k > 0$ $(k = 1, 2, ..., n)$

1). $\sum\limits_{k=1}^{n} \frac{(x_k+2)\ln(x_k+1)}{x_k} \geq 2n$

2). $\sum\limits_{cyclic} \frac{(x_1+2)\ln(x_2+1)}{x_3} \geq 2n$

3). $\sum\limits_{cyclic} (x_1+2)\ln(x_2+1) \geq 2 \sum\limits_{k=1}^{n} x_k$

4). $2n + \sum\limits_{k=1}^{n} x_k \geq 2 \sum\limits_{k=1}^{n} \frac{x_k}{\ln(x_k+1)}$

5). $n + 2 \sum\limits_{k=1}^{n} \frac{1}{x_k} \geq 2 \sum\limits_{k=1}^{n} \frac{1}{\ln(x_k+1)}$

6). If $A(a,b) = \frac{a+b}{2}, L(a,b) = \frac{b-a}{\ln b - \ln a}$, then $A(a,b) \geq L(a,b)$.

7). If $I(a,b) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}$, then $I(a,b) \geq \exp\left(2 - \frac{4}{b-a}\ln\frac{b+1}{a+1}\right)$

8). $\prod\limits_{k=1}^{n}(x_k+1)\geq \exp\left(2\sum\limits_{k=1}^{n}\frac{x_k}{x_k+2}\right)$

Proof. 6). If in $\ln(t+1)\geq\frac{2t}{t+2}$ we take $t=x-1$, then we get $\ln x\geq\frac{2(x-1)}{x+1}$ for all $x>0$.

If $x=\frac{b}{a}$, then we obtain $\frac{b+a}{2}\geq\frac{b-a}{\ln b-\ln a}$ or $A(a,b)\geq L(a,b)$.

7). $(b-a)I(a,b)=\int\limits_a^b\ln x dx\geq\int\limits_a^b\frac{2(x-1)}{x+1}dx=2(b-a)-4\ln\frac{b+1}{a+1}$

Theorem 14. If $x\in\left(0,\frac{\pi}{2}\right)$, then

$$\frac{1}{x(3-x^2)\sin 2x}\geq\frac{1}{2}$$

Proof. $\frac{2}{\sin 2x}\geq 2\geq x\left(3-x^2\right)$ but $2\geq x\left(3-x^2\right)\Leftrightarrow(x-1)^2(x+2)\geq 0$.

Corollary 14.1. If $x_k\in\left(0,\frac{\pi}{2}\right)$ $(k=1,2,...,n)$, then

1). $\sum\limits_{k=1}^{n}\frac{1}{x_k\left(3-x_k^2\right)\sin 2x_k}\geq\frac{n}{2}$

2). $\sum\limits_{cyclic}\frac{1}{x_1\left(3-x_2^2\right)\sin 2x_3}\geq\frac{n}{2}$

3). $\sum\limits_{k=1}^{n}\frac{1}{\left(3-x_k^2\right)\sin 2x_k}\geq\frac{1}{2}\sum\limits_{k=1}^{n}x_k$

4). $\sum\limits_{k=1}^{n}\frac{1}{x_k\sin 2x_k}\geq\frac{3}{2}n-\frac{1}{2}\sum\limits_{k=1}^{n}x_k^2$

5). In all acute triangle ABC holds $\sum\frac{1}{(3-A^2)\sin 2A}\geq\frac{\pi}{2}$

6). $\frac{1}{(\pi-2x)(12-\pi^2+8\pi x-4x^2)\sin 2x}\geq\frac{1}{2}$

Proof. 5). $\sum\frac{1}{(3-A^2)\sin 2A}\geq\sum\frac{A}{2}=\frac{\pi}{2}$

6). In Theorem 14 we take $x\to\frac{\pi}{2}-x$

Theorem 15. If $x\in\left[0,\frac{\pi}{2}\right)$, $a,b>0$ and

$$a+\sqrt{a^2+4ab}\geq 2b,\text{ then}\qquad (a+b\cos x)\,tgx\geq(a+b)\,x$$

Proof. If $f(x)=(a+b\cos x)\,tgx-(a+b)\,x$, then

$$f'(x)=\frac{b(1-\cos x)\left(\frac{a+\sqrt{a^2+4ab}}{2a}-\cos x\right)\left(\frac{\sqrt{a^2+4ab}-a}{2b}+\cos x\right)}{\cos^2 x}\geq 0$$

so $f(x)\geq f(0)=0$.

Corollary 15.1. If $x_k \in \left(0, \frac{\pi}{2}\right)$ $(k = 1, 2, ..., n)$, $a, b > 0$, $a + \sqrt{a^2 + 4ab} \geq 2b$, then

1). $\sum\limits_{k=1}^{n} \frac{(a+b\cos x_k)tgx_k}{x_k} \geq (a+b)\,n$

2). $\sum\limits_{cyclic} \frac{(a+b\cos x_1)tgx_2}{x_3} \geq (a+b)\,n$

3). $\sum\limits_{k=1}^{n} (a + b\cos x_k)\,tgx_k \geq (a+b)\sum\limits_{k=1}^{n} x_k$

4). $(a+b)\sum\limits_{k=1}^{n} \frac{x_k}{a+b\cos x_k} \leq \sum\limits_{k=1}^{n} tgx_k$

5). $(a+b)\sum\limits_{k=1}^{n} x_k tgx_k \leq na + b\sum\limits_{k=1}^{n} \cos x_k$

7). If $x, y > 0$, $x + \sqrt{x^2 + 4xy} \geq 2y$, then in all acute triangle ABC holds $\sum (x + y\cos A)\,tgA \geq (x+y)\,\pi$

8). If $x, y > 0$, $x + \sqrt{x^2 + 4xy} \geq 2y$, then in all acute triangle ABC holds $(x+y)\sum \frac{A}{x+y\cos A} \leq \frac{2sr}{s^2-(2R+r)^2}$

9). If $x, y > 0$ and $x + \sqrt{x^2 + 4xy} \geq 2y$, then in all acute triangle ABC holds $(x+y)\sum AtgA \leq 3x + y\left(1 + \frac{r}{R}\right)$

Theorem 16. If $x \in \left[0, \frac{\pi}{2}\right)$, $a, b > 0$ and $a \geq 4b$, then

$$(a - b\cos x)\sin x \leq (a-b)\,x$$

Proof. If $f(x) = (a-b)\,x - (a - b\cos x)\sin x$, then

$$f'(x) = 2b\,(1 - \cos x)\left(\frac{a-2b}{2b} - \cos x\right) \geq 0,$$

therefore $f(x) \geq f(0) = 0$.

Corollary 16.1. If $x_k \in \left(0, \frac{\pi}{2}\right)$ $(k = 1, 2, ..., n)$ and $a \geq 4b > 0$, then

1). $\sum\limits_{k=1}^{n} \frac{x_k}{(a-b\cos x_k)\sin x_k} \geq \frac{n}{a-b}$

2). $\sum\limits_{cyclic} \frac{x_1}{(a-b\cos x_2)\sin x_3} \geq \frac{n}{a-b}$

3). $\sum\limits_{k=1}^{n} (a - b\cos x_k)\sin x_k \leq (a-b)\sum\limits_{k=1}^{n} x_k$

4). $(a-b)\sum\limits_{k=1}^{n} \frac{x_k}{\sin x_k} \geq na - b\sum\limits_{k=1}^{n} \cos x_k$

5). In all acute triangle ABC holds $\sum (x - y\cos A)\sin A \leq (x-y)\,\pi$, where $x \geq 4y > 0$

6). In all acute triangle ABC holds $(x-y)\sum\frac{A}{\sin A}\geq 3x-\left(\frac{R+r}{R}\right)y$ for all $x\geq 4y>0$.
7). In all acute triangle ABC holds $(x-y)\sum\frac{A}{x-y\cos A}\geq\frac{s}{R}$ for all $x\geq 4y>0$.
Theorem 17. If $x\geq 0$, then

$$x(x+2)\geq 2(x+1)\ln(x+1)$$

Proof. If $f(x)=\frac{x(x+2)}{2(x+1)}-\ln(x+1)$, then $f'(x)=\frac{x^2}{2(x+1)^2}\geq 0$, therefore $f(x)\geq f(0)=0$.
Corollary 17.1. If $x_k>0$ $(k=1,2,...,n)$, then
1). $\sum\limits_{k=1}^{n}\frac{x_k(x_k+2)}{(1+x_k)\ln(x_k+1)}\geq 2$
2). $\sum\limits_{cyclic}\frac{x_1(2+x_2)}{(1+x_3)\ln(1+x_4)}\geq 2n$
3). $\sum\limits_{k=1}^{n}\frac{x_k(x_k+2)}{\ln(x_k+1)}\geq 2n+2\sum\limits_{k=1}^{n}x_k$
4). $\exp\left(\frac{1}{2}\sum\limits_{k=1}^{n}\frac{x_k(x_k+2)}{x_k+1}\right)\geq\prod\limits_{k=1}^{n}(x_k+1)$
5). If $H(a,b)=\frac{2}{\frac{1}{a}+\frac{1}{b}}$, then $H(a,b)\leq L(a,b)$ $(b>a>0)$
6). $I(a,b)\leq\exp\left(\frac{b+a}{4}-\frac{1}{2(b-a)}\ln\frac{b}{a}\right)$, where $b>a>0$
Proof. 5). In $\ln(x+1)\leq\frac{x(x+2)}{2(x+1)}$ we take $x\to x-1$ so $\ln x\leq\frac{x^2-1}{2x}$ after then $x=\frac{b}{a}$ so we obtain $\frac{2ab}{a+b}\leq\frac{b-a}{\ln b-\ln a}$ or $H(a,b)\leq L(a,b)$
6). $(b-a)I(a,b)=\int\limits_a^b\ln x dx\leq\int\limits_a^b\frac{x^2-1}{2x}dx=\frac{b^2-a^2}{4}-\frac{1}{2}\ln\frac{b}{a}$
Theorem 18. If $x>0$, then

$$(x+1)^4\geq\left(x^2-x+1\right)(x+1)^2+x$$

Proof. The inequality is equivalent with

$$x\left(3x^2+5x+3\right)\geq 0$$

Corollary 18.1. If $x_k>0$ $(k=1,2,...,n)$, then
1). $\sum\limits_{k=1}^{n}\frac{(x_k+1)^4}{\left(x_k^2-x_k+1\right)(x_k+1)^2+x_k}\geq n$
2). $\sum\limits_{cyclic}\frac{(x_1+1)^4}{\left(x_2^2-x_2+1\right)(x_2+1)^2+x_2}\geq n$

Theorem 19. If $x \in (0,1)$, then

$$x \geq \left(1-x^2\right)\arcsin x$$

Proof. If $f(x) = \frac{x}{1-x^2} - \arcsin x$, then

$$f'(x) = \frac{1+x^2-\left(1-x^2\right)\sqrt{1-x^2}}{\left(1-x^2\right)^2} = \frac{(1-t)\left(t^2+2t+2\right)}{t^4} \geq 0$$

where $t = \sqrt{1-x^2}$.
Therefore $f(x) \geq f(0) = 0$.

Corollary 19.1. If $x_k \in (0,1)$ $(k = 1,2,...,n)$, then
1). $\sum\limits_{k=1}^{n} \frac{x_k}{\left(1-x_k^2\right)\arcsin x_k} \geq n$
2). $\sum\limits_{cyclic} \frac{x_1}{\left(1-x_2^2\right)\arcsin x_3} \geq n$

Theorem 20. If $x \in \left(0, \sqrt{\frac{\pi}{2}}\right)$, then

$$\sin\left(x^2\right) \geq \sin^2 x$$

Proof. If $f(x) = x^2 - \arcsin\left(\sin^2 x\right)$, then
$f'(x) = 2\sin x\left(\frac{x}{\sin x} - \frac{\cos x}{\sqrt{1+\sin^2 x}}\right) \geq 0$ because $\frac{x}{\sin x} > 1$ and $\frac{\cos x}{\sqrt{1+\sin^2 x}} \leq 1$ so $f(x) \geq f(0) = 0$ or $x^2 \geq \arcsin\left(\sin^2 x\right)$ and finally $\sin\left(x^2\right) \geq \sin^2 x$.

Corollary 20.1. If $x_k \in \left(0, \sqrt{\frac{\pi}{2}}\right)$ $(k = 1,2,...,n)$, then
1). $\sum\limits_{k=1}^{n} \frac{\sin\left(x_k^2\right)}{\sin^2(x_k)} \geq n$
2). $\sum\limits_{cyclic} \frac{\sin\left(x_1^2\right)}{\sin^2 x_2} \geq n$
3). $1 \geq (\sin x)^4 + \left(\sin\left(\frac{\pi}{2} - x^2\right)\right)^4$ for all $x \in \left[0, \sqrt{\frac{\pi}{2}}\right]$
4). $\sqrt{\sin\left(2x^2\right)} \geq \sqrt{2}\sin x \sin\sqrt{\frac{\pi}{2} - x^2}$
Proof. 3). If in $\sin\left(x^2\right) \geq \sin^2 x$ we take $x \to \sqrt{\frac{\pi}{2} - x^2}$, then we get $\cos\left(x^2\right) \geq \left(\sin\sqrt{\frac{\pi}{2} - x^2}\right)$, therefore

$$1 = \left(\sin\left(x^2\right)\right)^2 + \left(\cos\left(x^2\right)\right)^2 \geq (\sin x)^4 + \left(\sin\sqrt{\frac{\pi}{2} - x^2}\right)^4$$

Theorem 21. If $x \in \left(0, \frac{\pi}{2}\right)$, then

$$\cos(\sin x)\cos(\cos x) \geq \sin x \cos x \geq \sin(\sin x)\sin(\cos x)$$

Proof. We have $\sin x \leq x$, $\cos(\sin x) \geq \cos x$, and $\sin(\cos x) \leq \cos x$, therefore $\sin(\cos x) \leq \cos x \leq \cos(\sin x)$.
If $x \to \frac{\pi}{2} - x$, then we get

$$\sin(\sin x) \leq \sin x \leq \cos(\cos x)$$

therefore

$$\sin(\sin x)\sin(\cos x) \leq \sin x \cos x \leq \cos(\sin x)\cos(\cos x)$$

Corollary 21.1. If $x_k \in \left(0, \frac{\pi}{2}\right)$ $(k = 1, 2, ..., n)$, then
1). $\sum\limits_{k=1}^{n} \frac{\cos(\sin x_k)\cos(\cos x_k)}{\sin x_k \cos x_k} \geq n$
2). $\sum\limits_{cyclic} \frac{\cos(\sin x_1)\cos(\cos x_2)}{\sin x_3 \cos x_4} \geq n$
3). $\sum\limits_{k=1}^{n} \frac{\sin x_k \cos x_k}{\sin(\sin x_k)\sin(\cos x_k)} \geq n$
4). $\sum\limits_{cyclic} \frac{\sin x_1 \cos x_2}{\sin(\sin x_3)\sin(\cos x_4)} \geq n$
5). $\sum\limits_{k=1}^{n} \frac{\cos(\sin x_k)\cos(\cos x_k)}{\sin(\sin x_k)\sin(\cos x_k)} \geq n$
6). $\sum\limits_{cyclic} \frac{\cos(\sin x_1)\cos(\cos x_2)}{\sin(\sin x_3)\sin(\cos x_4)} \geq n$
7). In all acute triangle ABC holds
$\sum \cos(\sin A)\cos(\cos A) \geq \frac{sr}{R^2} \geq \sum \sin(\sin A)\sin(\cos A)$

Theorem 22. If $x \geq 1$, then

$$x - 1 \geq \sqrt{x}\ln x$$

Proof. If $t \geq 1$ and $f(t) = t - \frac{1}{t} - 2\ln t$, then $f'(t) = \frac{(t-1)^2}{t^2} \geq 0$ so $f(t) \geq f(1) = 0$ in which we take $t = \sqrt{x}$.

Corollary 22.1. If $x_k > 1$ $(k = 1, 2, ..., n)$, then
1). $\sum\limits_{k=1}^{n} \frac{x_k - 1}{\sqrt{x_k}\ln x_k} \geq n$
2). $\sum\limits_{cyclic} \frac{x_1 - 1}{\sqrt{x_2}\ln x_2} \geq n$

3). If $G(a,b)=\sqrt{ab}$, then $G(a,b)\leq L(a,b)$

4). $I(a,b)\leq \exp\left(\frac{2\left(a+\sqrt{ab}+b\right)-6}{3\left(\sqrt{a}+\sqrt{b}\right)}\right)$

Proof. 3). In $\ln x\leq \sqrt{x}-\frac{1}{\sqrt{x}}$ we take $x=\frac{b}{a}$ so we obtain

$$G(a,b)=\sqrt{ab}\leq \frac{b-a}{\ln b-\ln a}=L(a,b)$$

4). $(b-a)\ln I(a,b)=\int\limits_a^b \ln x dx\leq \int\limits_a^b\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)dx=$
$=\frac{2}{3}\left(b\sqrt{b}-a\sqrt{a}\right)-2\left(\sqrt{b}-\sqrt{a}\right)$

Theorem 23. If $x\geq 1$, then

$$x^x e^{1-x}\geq 1$$

Proof. If $f(x)=1+x\ln x-x$, then $f'(x)=\ln x\geq 0$ so $f(x)\geq f(1)=0$.

Corollary 23.1. If $x_k\geq 1$ $(k=1,2,...,n)$, then

1). $\sum\limits_{k=1}^n x_k^{x_k}e^{1-x_k}\geq n$

2). $\sum\limits_{cyclic} x_1^{x_1}e^{1-x_2}\geq n$

Theorem 24. If $x\geq 0$, then

$$2e^x\geq (x+1)^2+1 \text{ and } 6e^x\geq (x+1)^3+3x-1$$

Proof. If $f(x)=2e^x-(x+1)^2-1$, then $f'(x)=2(e^x-x-1)\geq 0$ so $f(x)\geq f(0)=0$. If $g(x)=6e^x-(x+1)^3-3x+1$, then $g'(x)=f(x)\geq 0$ etc.

Corollary 24.1. If $x_k>0$ $(k=1,2,...,n)$, then

1). $\sum\limits_{k=1}^n \frac{e^{x_k}}{(x_k+1)^2+1}\geq \frac{n}{2}$

2). $\sum\limits_{cyclic} \frac{e^{x_1}}{(x_2+1)^2+1}\geq \frac{n}{2}$

3). $\sum\limits_{k=1}^n \left((x_k+1)^3+3x_k-1\right)e^{-x_k}\leq 6n$

4). $\sum\limits_{cyclic} \left((x_1+1)^3+3x_1-1\right)e^{-x_2}\leq 6n$

Theorem 25. If $x \in \left(0, \frac{\pi}{2}\right)$, then

$$x\left(tgx - x\right) \geq \ln^2\left(\cos x\right)$$

Proof. If $f\left(x\right) = \sqrt{x\left(tgx - x\right)} + \ln\left(\cos x\right)$, then
$f'\left(x\right) = \frac{\left(\sqrt{x}tgx - \sqrt{tgx - x}\right)^2}{2\sqrt{x(tgx-x)}} \geq 0$ so $f\left(x\right) \geq f\left(0\right) = 0.$

Corollary 25.1. If $x_k \in \left(0, \frac{\pi}{2}\right)$ $\left(k = 1, 2, ..., n\right)$, then
1). $\sum\limits_{k=1}^{n} \frac{x_k(tgx_k - x_k)}{\ln^2(\cos x_k)} \geq n$
2). $\sum\limits_{cyclic} \frac{x_1(tgx_2 - x_2)}{\ln^2(\cos x_3)} \geq n$
3). In all acute triangle ABC holds $\pi \geq \sum \frac{\ln^2(\cos A)}{tgA - A}$

Theorem 26. If $x \in [-1, 1]$, then

$$\frac{2\sqrt{2} - \sqrt{2 - x}}{\sqrt{2 + x}} \geq 1$$

Corollary 25.1. If $x_k \in [-1, 1]$ $\left(k = 1, 2, ..., n\right)$, then
1). $\sum\limits_{k=1}^{n} \frac{2\sqrt{2} - \sqrt{2 - x_k}}{\sqrt{2 + x_k}} \geq n$
2). $\sum\limits_{cyclic} \frac{2\sqrt{2} - \sqrt{2 - x_1}}{\sqrt{2 + x_2}} \geq n$

REFERENCES

[1] Bătineţu-Giurgiu, D.M., Batinetu-Giurgiu, M., Birchiu, D., Semenescu, A., *Analiza Matematica, Probleme pentru clasa a XI-a,* (in Romanian), Editura Matrix, Bucuresti, 2003

[2] Bencze, M., *About a family of inequalities*, Octogon Mathematical Magazine, Vol. 8,Nr. 1, April 2000, pp. 130-140.

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On Hardy-type integral inequalities involving many functions

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In memory of Professor C. O. Imoru

ABSTRACT. In this paper, we use Jensen's inequality, and a modification of an inequality involving some constants, to obtain a Hardy-type integral inequality involving many functions. Our inequality features a refinement term and is sharper than the inequality of Cheung, Hanjš and Pečarić(2000) in the segment $(1, \infty)$ of the real line.

1. INTRODUCTION AND PRELIMINARIES

The classical Hardy's inequality (1920) states that, for any $p > 1$ and any integrable function f(x)≥0 on (0,∞), if $F(x) = \int_0^x f(x)dx$, then

$$\int_0^\infty \left[\frac{F(x)}{x}\right]^p dx < \left[\frac{p}{p-1}\right]^p \int_0^\infty f(x)^p dx \tag{1.1}$$

Unless f ≡ 0, where the constant here is best possible.
In view of of the usefulness of the inequality 1.1 in analysis and its applications, it has received considerable attention and a number of papers have appeared in [1-12], which deal with its various improvements, extensions, generalizations and applications.
Of particular importance, relevance and great motivation for this research, is the following work of Cheung, Hanjš and Pečarić [6][Theorem 1]:
For any $i = 1, ..., n$, let $f_i : (0, \infty) \to (0, \infty)$ be absolutely continuous, let $g_i : (0, \infty) \to (0, \infty)$ be integrable, and $p_i > q_i > 0$, $m_i > q_i$ be real numbers such that $\sum q_i = 1$ and

$$1 + (\frac{p_i}{m_i - q_i})\frac{xf_i'(x)}{f_i(x)} \geq \frac{1}{\gamma_i} \quad a.e$$

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for some constants $\gamma_i > 0$. If we denote $p = \sum p_i$, $m = \sum m_i$ and

$$\eta_i(x) = \frac{1}{f_i(x)} \int_0^x \frac{f_i(t)g_i(t)}{t} dt \qquad x \in (0, \infty),$$

then

$$\int_0^\infty x^{-m} \prod_i \left[\eta_i{}^{p_i}(x)\right] dx \le$$

$$\le \left(\prod_j C_j^{-pj}\right) \sum_i q_i C_i^{pi/qi} \left[\frac{p_i\gamma_i}{m_i - q_i}\right] \int_0^\infty x^{-}(m_i/q_i) g_i^{p_i/q_i}(x) dx. \qquad (1.2)$$

In establishing their result, Cheung et al made use of Holder's inequality. In this work, our main tool is the inequality of Jensen for convex functions.

2. MAIN RESULTS

For our main result we shall need the following lemma

Lemma 2.1. Let p $\geq$ q $>$ 0 and r $\neq$ 1 be real numbers. Let $f : [a, b] \to (0, \infty)$ be absolutely continuous and let $g : [a, b] \to [0, \infty)$ be integrable with $0 < a \le b < \infty$. Let

$$\varphi_a(x) = \int_a^x \frac{f(t)g(t)}{f(x)t} dt, \varphi_b(x) = \int_x^b \frac{f(t)g(t)}{f(x)t} dt, \delta = \frac{q}{p}(1 - r) for\ r \neq 1,$$

$$1 + \left(\frac{p}{q}\right)^2 \left(\frac{1}{r-1}\right) \frac{xf'(x)}{f(x)} \ge \frac{1}{\lambda} > 0 \qquad \text{a.e for r} > 1 \qquad (2.1)$$

and

$$1 - \left(\frac{p}{q}\right)^2 \left(\frac{1}{1-r}\right) \frac{xf'(x)}{f(x)} \ge \frac{1}{\lambda} > 0 \qquad \text{a.e for r} > 1 \qquad (2.2)$$

for some constant $\lambda > 0$.

Then if r $>$ 1

$$\int_a^b x^{-r} \varphi_a^{p/q}(x) dx + \left(\frac{p}{q}\right) \frac{\lambda}{r-1} b^{1-r} \varphi_a^{p/q}(b) \le$$

$$\leq \lambda \left(\left(\frac{p}{q} \right) \frac{1}{r-1} \right)^{p/q} \int_a^b x^{-r} g^{p/q}(x) dx \tag{2.3}$$

and for $r < 1$

$$\int_a^b x^{-r} \varphi_b^{p/q}(x) dx + \left(\frac{p}{q} \right) \frac{\lambda}{1-r} a^{1-r} \varphi_b^{p/q}(a) \leq$$

$$\leq \lambda \left(\left(\frac{p}{q} \right) \frac{1}{1-r} \right)^{p/q} \int_a^b x^{-r} g^{p/q}(x) dx \tag{2.4}$$

Proof. The following adaptations of Jensen's inequality for convex functions [1 -4] will be used in the proof of the Lemma 2.1

$$\left[\int_a^x d\lambda(t) \right]^{1-\tau} \left[\int_a^x h(x,t)^{\frac{1}{\tau}} d\lambda(t) \right]^{\tau} \leq \int_a^x h(x,t) d\lambda(t) \tag{2.5}$$

$$\left[\int_x^b d\lambda(t) \right]^{1-\tau} \left[\int_x^b h(x,t)^{\frac{1}{\tau}} d\lambda(t) \right]^{\tau} \leq \int_x^b h(x,t) d\lambda(t) \tag{2.6}$$

where $h(x,t) \geq 0$ for x $\geq 0, t \geq 0, \lambda$ is non-decreasing and $\tau \geq 1$.

Let $h(x,t) = x^{\delta} t^{\tau(1+\delta)} \left[\frac{f(t)g(t)}{f(x)t} \right]^{\tau}$,$d\lambda(t) = t^{-(1+\delta)} dt$ and $\delta = \frac{1-r}{\tau}$.

Using the above definitions of h, λ and δ in 2.5 and 2.6, we obtain

$$\left[-\delta^{-1} \right]^{1-\tau} \left[x^{-\delta} - a^{-\delta} \right]^{1-\tau} x^{\delta} \varphi_a^{\tau}(x) \leq x^{\delta} \theta_a(x) \quad \forall x \in [a,b]. \tag{2.7}$$

$$\left[\delta^{-1} \right]^{1-\tau} \left[x^{-\delta} - b^{-\delta} \right]^{1-\tau} x^{\delta} \varphi_b^{\tau}(x) \leq x^{\delta} \theta_b(x) \quad \forall x \in [a,b]. \tag{2.8}$$

Where $\theta_a(x) = \int_a^x t^{(\tau-1)(1+\delta)} \left[\frac{f(t)g(t)}{f(x)t} \right]^{\tau} dt, \theta_b(x) = \int_x^b t^{(\tau-1)(1+\delta)} \left[\frac{f(t)g(t)}{f(x)t} \right]^{\tau} dt.$

Multiply through 2.7 and 2.8 by x^{-1} and then integrate with respect to x on [a, b] to get

$$\left[-\delta^{-1} \right]^{1-\tau} \int_a^b \left[x^{-\delta} - a^{-\delta} \right]^{1-\tau} x^{\delta-1} \varphi_a^{\tau}(x) dx \leq \int_a^b x^{\delta-1} \theta_a(x) dx \tag{2.9}$$

$$\left[\delta^{-1} \right]^{1-\tau} \int_a^b \left[x^{-\delta} - b^{-\delta} \right]^{1-\tau} x^{\delta-1} \varphi_b^{\tau}(x) dx \leq \int_a^b x^{\delta-1} \theta_b(x) dx. \tag{2.10}$$

Integrate the RHS of the 2.9 by parts and then factorise to obtain

$$\int_a^b x^{\delta-1}\theta_a(x)\left(1+\frac{\tau^2}{r-1}\frac{xf'(x)}{f(x)}\right)dx = \delta^{-1}b^\delta\theta_a(b)-\delta^{-1}\int_a^b f^\tau(x)x^{\tau\delta-1}dx$$

Suppose that for some constant $\lambda > 0$,

$$\left(1+\frac{\tau^2}{r-1}\frac{xf'(x)}{f(x)}\right) > \frac{1}{\lambda} \qquad a.e. \tag{2.11}$$

then

$$\int_a^b x^{\delta-1}\theta_a(x)\left(\frac{1}{\lambda}\right)dx \leq \int_a^b x^{\delta-1}\theta_a(x)\left(1+\frac{\tau^2}{r-1}\frac{xf'(x)}{f(x)}\right)dx =$$

$$= \delta^{-1}b^\delta\theta_a(b)-\delta^{-1}\int_a^b f^\tau(x)x^{\tau\delta-1}dx. \tag{2.12}$$

Whence on arranging

$$\int_a^b x^{\delta-1}\theta_a(x)dx+\lambda\left(-\delta^{-1}\right)b^\delta\theta_a(b) \leq \lambda\left(-\delta^{-1}\right)\int_a^b f^\tau(x)x^{\tau\delta-1}dx. \tag{2.13}$$

Now it follows from 2.7 and the fact that $\lambda\ \left[-\delta^{-1}\right] > 0 \quad for\ \ r > 1$ that

$$\lambda\left(-\delta^{-1}\right)^{2-\tau}\left[b^{-\delta}-a^{-\delta}\right]^{1-\tau}b^\delta\varphi_a^\tau(b) \leq \lambda\left(-\delta^{-1}\right)b^\delta\theta_a(b), \qquad for\ \ b\in[a,b]. \tag{2.14}$$

Combining 2.13, 2.14 and 2.9 yields

$$\left[-\delta^{-1}\right]^{1-\tau}\int_a^b\left[x^{-\delta}-a^{-\delta}\right]^{1-\tau}x^{\delta-1}\varphi_a^\tau(x)dx+$$

$$+\lambda\left(-\delta^{-1}\right)^{2-\tau}\left[b^{-\delta}-a^{-\delta}\right]^{1-\tau}b^\delta\varphi_a^\tau(b) \leq \lambda\left(-\delta^{-1}\right)\int_a^b f^\tau(x)x^{\tau\delta-1}dx. \tag{2.15}$$

Use the fact that $\left[x^{-\delta}-a^{-\delta}\right]^{1-\tau} \geq \left[x^{-\delta}\right]^{1-\tau} \quad \forall\ x\in[a,b]\ \ and\ \ \left[-\delta^{-1}\right]^{1-\tau} > 0$ for $\tau \geq 1$ and $r > 1$ to reduce 2.15 to

$$\int_a^b x^{\delta\tau-1}\varphi_a^\tau(x)dx+\lambda\left(-\delta^{-1}\right)b^\delta\varphi_a^\tau(b) \leq \lambda\left(-\delta^{-1}\right)^\tau\int_a^b f^\tau(x)x^{\tau\delta-1}dx. \tag{2.16}$$

Similarly, for the case $r<1$, start with inequality 2.10 and follow the same arguements with slight modifications to the conditions. Specifically, use

$$\left(1-\frac{\tau^2}{1-r}\frac{xf'(x)}{f(x)}\right)>\frac{1}{\lambda}\quad \text{a.e.} \quad \text{for some constant } \lambda>0, \tag{2.17}$$

with $\left[\delta^{-1}\right]>0$, $\left[x^{-\delta}-b^{-\delta}\right]^{1-\tau}\geq\left[x^{-\delta}\right]^{1-\tau}$ $\forall\, x\in[a,b]$ to yield

$$\int_a^b x^{\delta\tau-1}\varphi_b^{\tau}(x)dx+\lambda\left(-\delta^{-1}\right)a^{\delta}\varphi_b^{\tau}(a)\leq\lambda\left(-\delta^{-1}\right)^{\tau}\int_a^b f^{\tau}(x)x^{\tau\delta-1}dx. \tag{2.18}$$

Finally, observe that if $p\geq q>0$, then $p/q\geq 1$. Thus. Inequalities 2.1, 2.2, 2.3 and 2.4 follow immediately from 2.11, 2.17, 2.16 and 2.18 respectively by recalling that $\delta=(1-r)/\tau$ and letting $\tau=\frac{p}{q}$.

The following Theorem is an improvement over the result of Cheung, Hanjš and Pečarić[6,Theorems 1 and 2].

Theorem 2.1. For any i =1, ...,n, let $f_i:[a,b]\to(0,\infty)$ be absolutely continuous, let $g_i:[a,b]\to[0,\infty)$ be integrable with $0<a\leq b<\infty$. Let $p_i\geq q_i>0, m_i\neq q_i$ be real numbers such that $\sum q_i=1$,

$$1+\left(\frac{p_i^2}{q_i(m_i-q_i)}\right)\frac{xf^{'}(x)}{f_i(x)}\geq\frac{1}{\lambda_i}>0 \qquad \text{a.e} \tag{2.19}$$

and

$$1-\left(\frac{p_i^2}{q_i(q_i-m_i)}\right)\frac{fu^{'}(x)}{f_i(x)}\geq\frac{1}{\lambda_i}>0 \qquad \text{a.e} \tag{2.20}$$

for some constant $\lambda_i>0$. If we denote $p=\sum p_i, m=\sum m_i$, and

$$\varphi_{a,i}(x)=\int_a^x\frac{f_i(t)g_i(t)}{f_i(x)t}dt, \varphi_{b,i}(x)=\int_x^b\frac{f_i(t)g_i(t)}{f_i(x)t}dt, \qquad x\in(0,\infty),$$

then for $m_i>q_i$

$$\int_a^b\prod_{i=1}^n\left[x^{-m_i}\varphi_{a,i}^{p_i}(x)\right]dx+\prod_{j=1}^n C_j^{-p_j}\sum_{i=1}^n q_iC_i^{p_i/q_i}\frac{\lambda p_i}{m_i-q_i}b^{1-m_i/q_i}\varphi_{a,i}^{p_i/q_i}(b)\leq$$

$$\le \prod_{j=1}^{n} C_j^{-p_j} \sum_{i=1}^{n} q_i C_i^{p_i/q_i} \lambda_i \left(\frac{p_i}{m_i - q_i}\right)^{p_i/q_i} \int_0^\infty x^{-m_i/q_i} g_i(x)^{p_i/q_i} dx \tag{2.21}$$

and for $m_i < q_i$

$$\int_a^b \prod_{i=1}^{n} \left[x^{-m_i} \varphi_{b,i}^{p_i}(x)\right] dx + \prod_{j=1}^{n} C_j^{-p_j} \sum_{i=1}^{n} q_i C_i^{p_i/q_i} \frac{\lambda p_i}{q_i - m_i} b^{1-m_i/q_i} \varphi_{b,i}^{p_i/q_i}(a) \le$$

$$\le \prod_{j=1}^{n} C_j^{-p_j} \sum_{i=1}^{n} q_i C_i^{p_i/q_i} \lambda_i \left(\frac{p_i}{q_i - m_i}\right)^{p_i/q_i} \int_0^\infty x^{-m_i/q_i} g_i(x)^{p_i/q_i} dx \tag{2.22}$$

*P*roof. Firstly, observe that $m_i > q_i$ implies that $m_i/q_i > 1$. Consequently, it follows from Lemma 2 for the case $r = m_i/q_i > 1$ that

$$\int_a^b x^{-m_i/q_i} \varphi_{a,i}^{p_i/q_i}(x) dx + \frac{\lambda_i p_i}{m_i - q_i} b^{1-m_i/q_i} \varphi_{a,i}^{p_i/q_i}(b) \le$$

$$\le \lambda_i \left(\frac{p_i}{m_i - q_i}\right)^{p_i/q_i} \int_a^b x^{-m_i/q_i} g_i^{p_i/q_i}(x) dx \tag{2.23}$$

Now for any $C_i > 0$, we have by the arithmetic-geometric inequality [6], that

$$\prod_{i=1}^{n} \left[x^{-m_i} \varphi_{a,i}^{p_i}(x)\right] = \prod_{i=1}^{n} \left(\left[\left(x^{-(m_i/p_i)} C_i \varphi_{a,i}(x)\right)^{p_i/q_i}\right]^{q_i} C_i^{-p_i}\right) =$$

$$= \prod_{j=1}^{n} C_j^{-p_j} \prod_{i=1}^{n} \left[\left(x^{-(m_i/p_i)} C_i \varphi_{a,i}(x)\right)^{p_i/q_i}\right]^{q_i} \le$$

$$\le \prod_{j=1}^{n} C_j^{-p_j} \sum_{i=1}^{n} q_i C_i^{p_i q_i} x^{-m_i/q_i} \varphi_{a,i}^{p_i/q_i}(x) \tag{2.24}$$

Integrate both sides of 2.24 with respect to x on [a, b], to obtain

$$\int_a^b \prod_{i=1}^{n} \left[x^{-m_i} \varphi_{a,i}^{p_i}(x)\right] dx \le$$

$$\le \prod_{j=1}^{n} C_j^{-p_j} \sum_{i=1}^{n} q_i C_i^{p_i/q_i} \int_a^b x^{-m_i/q_i} \varphi_{a,i}^{p_i/q_i}(x) dx. \tag{2.25}$$

We combine inequalities 2.25 and 2.23, expanding and rearranging(letting $m = \sum m_i$) to obtain

$$\int_a^b x^{-m} \prod_{i=1}^{n} \left[\varphi_{a,i}^{p_i}(x)\right] dx + \prod_{j=1}^{n} C_j^{-p_j} \sum_{i=1}^{n} q_i C_i^{p_i/q_i} \frac{\lambda_i p_i}{m_i - q_i} b^{1-m_i/q_i} \varphi_{a,i}^{p_i/q_i}(b) \leq$$

$$\leq \prod_{j=1}^{n} C_j^{-p_j} \sum_{i=1}^{n} q_i C_i^{p_i/q_i} \lambda_i \left(\frac{p_i}{m_i - q_i}\right)^{p_i/q_i} \int_0^\infty x^{-m_i/q_i} g_i(x)^{p_i/q_i} dx. \qquad (2.26)$$

For the case when $m_i < q_i$, it follows from Lemma 2.1 by using silimar arguments to those in the proof for the case $m_i > q_i$.

Remark 2.2. *If we let* $a \to 0^+$ and $b \to \infty$ *then 2.26 reduces to*

$$\int_0^\infty x^{-m} \prod_{i=1}^{n} \left[\varphi_{a,i}^{p_i}(x)\right] dx \leq$$

$$\leq \prod_{j=1}^{n} C_j^{-p_j} \sum_{i=1}^{n} q_i C_i^{p_i/q_i} \lambda_i \left(\frac{p_i}{m_i - q_i}\right)^{p_i/q_i} \int_0^\infty x^{-m_i/q_i} g_i(x)^{p_i/q_i} dx. \qquad (2.27)$$

We claim that under certain conditions, the above inequality is sharper than the one by Cheung, Hanjš and Pečarić[6, Theorem 1]:

$$\int_0^\infty x^{-m} \prod_i \left[\varphi_i^{p_i}(x)\right] dx \leq$$

$$\leq \left(\prod_j C_j^{-pj}\right) \sum_i q_i C_i^{pi/qi} \left[\frac{p_i \gamma_i}{m_i - q_i}\right] \int_0^\infty x^{-}(m_i/q_i) g_i^{p_i/q_i}(x) dx. \qquad (2.28)$$

Precisely, if $\gamma_i \in (1, \infty)$

To justify our claim, we recall the following conditions:

$$1 + \frac{p_i}{q_i}\left(\frac{p_i}{(m_i - q_i)}\right) \frac{x f'(x)}{f_i(x)} \geq \frac{1}{\lambda_i} > 0 \qquad \text{a.e}$$

and

$$1+\left(\frac{p_i}{m_i-q_i}\right)\frac{xf'(x)}{f_i(x)}\geq\frac{1}{\gamma_i}>0 \qquad \text{a.e.}$$

By comaparison, it is obvious that

$$1+\frac{p_i}{q_i}\left(\frac{p_i}{(m_i-q_i)}\right)\frac{xf'(x)}{f_i(x)}\geq 1+\left(\frac{p_i}{m_i-q_i}\right)\frac{xf'(x)}{f_i(x)}$$

and $\frac{1}{\lambda_i}\geq\frac{1}{\gamma_i}>0$. Thus

$$0\leq\lambda_i\leq\gamma_i.$$

Finally, we now note that if $\gamma_i\in(1,\infty)$, then

$$0\leq\lambda_i\leq\gamma_i\leq(\gamma_i)^{p_i/q_i}.$$

for $p_i/q_i\geq 1$.

We make similar claim for 2.22.

REFERENCES

[1] Adeagbo-Sheikh, A. G. and Imoru,C.O., *An Integral Inequality of the Hardy,s type*, Kragujevac j. Math. 29 (2006) 57-61.
[2] Imoru,C.O., *On Some Integral Inequalities Related to Hard's*, Canadian Mathematical Bulletin, Vol. **20** (3)(1977), 307-312.
[3] Imoru,C.O. and Adeagbo-Sheikh, A. G., *On Some Weighted Mixed Norm Hardy-type Integral Inequality*, J. Inequal. Pure Appl. Math. **8**(4)(2007), Art. 101. 5pp.
[4] Imoru,C.O. and Adeagbo-Sheikh, A. G., *On an integral of The Hardy-type*, Australian Journal of Mathematical Analysis and Applications.Vol. **4**, No.**2**, Art.**2**, pp. 1-5, 2007.
[5] Boas, R.P., *Integrability Theorems for Trigonometric Transforms*, ergebnisse der Mathematik und ihrer Grenzgeblete Vol. **38**, (1967).
[6] Cheung, W. S., Hanjš, Z. and Pečarić, J. E., *Some hardy-Type Ineqalities*, Journal of Mathematical Analysis abd Application **250**, 621-634 (2000).
[7] Imoru,C.O., *On some Integral Inequalities Related to Hardy's*, Can. Math. Bull Vol. 20(3) (1977), 307-312.
[8] Levinson, N., *Generalizations of an inequality of Hardy*, Duke math. J.'**31** (1964), 389-394.
[9] Pachpatte, B. G. , *On a new class of Hardy type inequalities*, Proc. R. soc. Edin. **105A** (1987),265-274.

[10] Hardy, G. H., *Notes on a theorem of Hilbert*, Math. Z. **6** (1920), 314-317.
[11] Hardy, G. H., *Notes on some points in the integralcalculus*, Messenger Math. **57** (1928), 12-16.
[12] Izumi, M. and Izumi, S+., *On some inequalities related for Fourier series*, J. Anal. Math. J. *21* (1968), 277-291.

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A method to generate new inequalities in triangle

Mihály Bencze[13]

ABSTRACT. In this paper we present a method which generate new inequalities in triangle.

MAIN RESULTS

Theorem 1. Let ABC be a triangle, I the center of incircle, $A_1 = \frac{A}{2}$, $B_1 = \frac{B}{2}$, $I_1 = \frac{\pi + C}{2}$, $AB = c$, $AI = 4R \sin \frac{B}{2} \sin \frac{C}{2}$, $BI = 4R \sin \frac{C}{2} \sin \frac{A}{2}$, denote R_1, r_1, s_1 the circumradius, inradius, semiperimeter of triangle AIB. Then we have the following relations.

$$R_1 = R \sin \frac{C}{2},$$

$$r_1 = 4R \sin \frac{A}{4} \sin \frac{B}{4} \sin \frac{\pi + C}{4} \sin \frac{C}{2},$$

$$s_1 = 2R \sin \frac{C}{2} \left(\sin \frac{A}{2} + \sin \frac{B}{2} + \cos \frac{C}{2} \right)$$

Proof. In triangle AIB we have

$$\frac{AB}{\sin (AIB)} = 2R_1 \text{ or } \frac{2R \sin \frac{C}{2} \cos \frac{C}{2}}{\sin \frac{\pi + C}{2}} = 2R_1$$

finally $R_1 = R \sin \frac{C}{2}$. In same way we have:

$$r_1 = 4R_1 \sin \frac{A}{4} sin \frac{B}{4} \sin \frac{\pi + C}{4} = 4R \sin \frac{A}{4} \sin \frac{B}{4} \sin \frac{\pi + C}{4} \sin \frac{C}{2}$$

$$2S_1 = AB + AI + BI = c + 4R \left(\sin \frac{B}{2} \sin \frac{C}{2} + \sin \frac{C}{2} \sin \frac{A}{2} \right) =$$

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$$= 2R\sin C + 4R\sin\frac{C}{2}\left(\sin\frac{A}{2}+\sin\frac{B}{2}\right) = 4R\sin\frac{C}{2}\cos\frac{C}{2} +$$

$$+4R\sin\frac{C}{2}\left(\sin\frac{A}{2}+\sin\frac{B}{2}\right) = 4R\sin\frac{C}{2}\left(\sin\frac{A}{2}+\sin\frac{B}{2}+\cos\frac{C}{2}\right)$$

Corollary 1. In all triangle ABC holds

$$\cos\frac{A}{4}+\cos\frac{B}{4}+\cos\frac{\pi+C}{4} \geq 2\left(\sin\frac{A}{2}+\sin\frac{B}{2}+\cos\frac{C}{2}\right)$$

and his permutations.

Proof. First we show that in all triangle ABC holds

$$\sum\cos\frac{A}{2} \geq \sum\sin A = \frac{s}{R}$$

We have

$$2\sum\sin A = \sum(\sin B+\sin C) = 2\sum\sin\frac{B+C}{2}\cos\frac{B-C}{2} \leq 2\sum\cos\frac{A}{2}$$

Using this relation in triangle AIB we obtain

$$\cos\frac{A}{4}+\cos\frac{B}{4}+\cos\frac{\pi+C}{4} \geq \frac{s_1}{R_1} = 2\left(\sin\frac{A}{2}+\sin\frac{B}{2}+\cos\frac{C}{2}\right)$$

Corollary 2. In all triangle ABC holds

$$\sin\frac{A}{4}+\sin\frac{B}{4}+\sin\frac{\pi+C}{4} \geq 1+4\sin\frac{A}{4}\sin\frac{B}{4}\sin\frac{\pi+C}{4}$$

Proof. In all triangle ABC holds

$$\sum\sin\frac{A}{2} \geq \sum\cos A = 1+\frac{r}{R}$$

We have

$$2\sum\cos A = \sum(\cos B+\cos C) = 2\sum\cos\frac{B+C}{2}\cos\frac{B-C}{2} \leq 2\sum\sin\frac{A}{2}$$

Using this relation in triangle AIB we have

$$\sin\frac{A}{4}+\sin\frac{B}{4}+\sin\frac{\pi+C}{4} \geq 1+\frac{r_1}{R_1} = 1+4\sin\frac{A}{4}\sin\frac{B}{4}\sin\frac{\pi+C}{4}$$

Corollary 3. In all triangle ABC holds

$$\sin^2\frac{2\pi-A}{8}+\sin^2\frac{2\pi-B}{8}+\sin^2\frac{\pi-C}{8}\le 1-2\sin\frac{A}{4}\sin\frac{B}{4}\sin\frac{\pi+C}{4}$$

and his permutations.
Proof. In all acute triangle ABC we have

$$\sum\sin^2\frac{A+B}{4}\le\sum\sin^2\frac{A}{2}=1-\frac{r}{2R}.$$

If $f:R\to R$ is convex, then $\sum f(A)\ge\sum f\left(\frac{A+B}{2}\right)$ because

$$\sum f(A)=\sum\frac{f(A)+f(B)}{2}\ge\sum f\left(\frac{A+B}{2}\right)$$

If we take $f(x)=\sin^2\frac{x}{2}$, then we obtain the affirmation. Using this for the triangle AIB we obtain

$$\sin^2\frac{\frac{A}{2}+\frac{B}{2}}{4}+\sin^2\frac{\frac{B}{2}+\frac{\pi+C}{2}}{4}+\sin^2\frac{\frac{C}{2}+\frac{\pi+A}{2}}{4}\le 1-\frac{r_1}{2R_1}=$$

$$=1-2\sin\frac{A}{4}\sin\frac{B}{4}\sin\frac{\pi+C}{4}$$

Corollary 4. In all triangle ABC holds

$$\cos^2\frac{2\pi-A}{8}+\cos^2\frac{2\pi-B}{8}+\cos^2\frac{\pi-C}{8}\ge 2\left(1+\sin\frac{A}{4}\sin\frac{B}{4}\sin\frac{\pi+C}{4}\right)$$

and his permutations.
Proof. In all acute triangle ABC we have

$$\sum\cos^2\frac{A+B}{4}\ge\sum\cos^2\frac{A}{2}=2+\frac{r}{2R}$$

We using the function $f(x)=\cos^2\frac{x}{2}$ which is concave. Using this in triangle AIB we obtain

$$\cos^2\frac{\frac{A}{2}+\frac{B}{2}}{4}+\cos^2\frac{\frac{B}{2}+\frac{\pi+C}{2}}{4}+\cos^2\frac{\frac{C}{2}+\frac{\pi+A}{2}}{4}\ge 2+\frac{r_1}{2R_1}=$$

$$=2\left(1+\sin\frac{A}{4}\sin\frac{B}{4}\sin\frac{\pi+C}{4}\right)$$

Corollary 5. In all triangle ABC holds

$$tg\frac{2\pi-A}{8}+tg\frac{2\pi-B}{8}+tg\frac{\pi-C}{8}\leq\frac{2\left(1+\sin\frac{A}{4}\sin\frac{B}{4}\sin\frac{\pi+C}{4}\right)}{\sin\frac{A}{2}+\sin\frac{B}{2}+\cos\frac{C}{2}}$$

and his permutations.

Proof. The function $f\left(x\right)=tg\frac{x}{2}$, $x\in\left(0,\pi\right)$ is convex, therefore in any triangle ABC we have

$$\sum tg\frac{A+B}{4}\leq\sum tg\frac{A}{2}=\frac{4R+r}{s}$$

Using this in triangle AIB we get

$$tg\frac{\frac{A}{2}+\frac{B}{2}}{4}+tg\frac{\frac{B}{2}+\frac{\pi+C}{2}}{4}+tg\frac{\frac{C+\pi}{2}+\frac{A}{2}}{2}\leq\frac{4R_1+r_1}{s_1}=\frac{2\left(1+\sin\frac{A}{2}\sin\frac{B}{4}\sin\frac{\pi+C}{4}\right)}{\sin\frac{A}{2}+\sin\frac{B}{2}+\cos\frac{C}{2}}$$

Corollary 6. In all triangle ABC holds

$$ctg\frac{2\pi-A}{8}+ctg\frac{2\pi-B}{8}+ctg\frac{\pi-C}{8}\leq\frac{\sin\frac{A}{2}+\sin\frac{B}{2}+\cos\frac{C}{2}}{2\sin\frac{A}{4}\sin\frac{B}{4}\sin\frac{\pi+C}{4}}$$

and his permutations.

Proof. The function $f\left(x\right)=ctg\frac{x}{2}$, $x\in\left(0,\pi\right)$ is convex, therefore in any triangle ABC we have

$$\sum ctg\frac{A+B}{4}\leq\sum ctg\frac{A}{2}=\frac{s}{r}$$

Using this in triangle AIB we get

$$ctg\frac{\frac{A}{2}+\frac{B}{2}}{4}+ctg\frac{\frac{B}{2}+\frac{\pi+C}{2}}{4}+ctg\frac{\frac{C+\pi}{2}+\frac{A}{4}}{4}\leq\frac{s_1}{r_1}=\frac{\sin\frac{A}{2}+\sin\frac{B}{2}+\cos\frac{C}{2}}{2\sin\frac{A}{4}\sin\frac{B}{4}\sin\frac{+C}{4}}$$

Corollary 7. In all triangle ABC holds

$$tg^2\frac{2\pi-A}{8}+tg^2\frac{2\pi-B}{8}+tg^2\frac{\pi-C}{8}\leq4\left(\frac{1+\sin\frac{A}{4}\sin\frac{B}{4}\sin\frac{\pi+C}{4}}{\sin\frac{A}{2}+\sin\frac{B}{2}+\cos\frac{C}{2}}\right)^2-2$$

and his permutations.

Proof. The function $f\left(x\right)=tg^2\frac{x}{2}$, $x\in\left(0,\pi\right)$ is convex, therefore in any triangle ABC holds

$$\sum tg^2\frac{A+B}{4}\leq\sum tg^2\frac{A}{2}=\left(\frac{4R+r}{s}\right)^2-2$$

Using this in triangle AIB we have

$$tg^2\frac{\frac{A}{2}+\frac{B}{2}}{4}+tg^2\frac{\frac{B}{2}+\frac{\pi+C}{2}}{4}+tg^2\frac{\frac{C+\pi}{2}+\frac{A}{2}}{2}\leq\left(\frac{4R_1+r_1}{s_1}\right)^2-2=$$

$$=4\left(\frac{1+\sin\frac{A}{4}\sin\frac{B}{4}\sin\frac{\pi+C}{4}}{\sin\frac{A}{2}+\sin\frac{B}{2}+\cos\frac{C}{2}}\right)^2-2$$

Corollary 8. In all triangle ABC holds

$$ctg^2\frac{2\pi-A}{8}+ctg^2\frac{2\pi-B}{8}+ctg^2\frac{\pi-C}{8}\leq$$

$$\leq\frac{1}{4}\left(\frac{\sin\frac{A}{2}+\sin\frac{B}{2}+\cos\frac{C}{2}}{\sin\frac{A}{4}\sin\frac{B}{4}\sin\frac{\pi+C}{4}}\right)^2-\frac{2}{\sin\frac{A}{4}\sin\frac{B}{4}\sin\frac{\pi+C}{4}}-2$$

and his permutations.
Proof. The function $f(x)=ctg^2\frac{x}{2}$, $x\in(0,\pi)$ is convex, therefore in any triangle ABC holds

$$\sum ctg^2\frac{A+B}{4}\leq\sum ctg^2\frac{A}{2}=\left(\frac{s}{r}\right)^2-\frac{8R}{r}-2$$

Using this in triangle AIB we have

$$ctg^2\frac{\frac{A}{2}+\frac{B}{2}}{4}+ctg^2\frac{\frac{B}{2}+\frac{\pi+C}{2}}{4}+ctg^2\frac{\frac{C+\pi}{2}+\frac{A}{2}}{2}\leq\left(\frac{s_1}{r_1}\right)^2-\frac{8R_1}{r_1}-2=$$

$$=\frac{1}{4}\left(\frac{\sin\frac{A}{2}+\sin\frac{B}{2}+\cos\frac{C}{2}}{\sin\frac{A}{4}\sin\frac{B}{4}\sin\frac{\pi+C}{4}}\right)^2=\frac{2}{\sin\frac{A}{4}\sin\frac{B}{4}\sin\frac{\pi+C}{4}}-2$$

Corollary 9. In all triangle ABC holds

$$\sin\frac{2\pi-A}{8}\sin\frac{2\pi-B}{8}\sin\frac{\pi-C}{8}\geq\sin\frac{A}{4}\sin\frac{B}{4}\sin\frac{\pi+C}{4}$$

and his permutations.
Proof. The function $f(x)=\ln\left(\sin\frac{x}{2}\right)$, $x\in(0,\pi)$ is concave, therefore in any triangle ABC holds

$$\sum \ln\left(\sin\frac{A}{2}\right) \geq \sum \ln\left(\sin\frac{A+B}{4}\right) \text{ or}$$

$$\prod \sin\frac{A+B}{4} \geq \prod \sin\frac{A}{2} = \frac{r}{4R}$$

Using this in triangle AIB we get

$$\sin\frac{\frac{A}{2}+\frac{B}{2}}{4}\sin\frac{\frac{B}{2}+\frac{\pi+C}{2}}{4}\sin\frac{\frac{\pi+C}{2}+\frac{A}{2}}{4} \geq \frac{r_1}{4R_1} = \sin\frac{A}{4}\sin\frac{B}{4}\sin\frac{\pi+C}{4}$$

Corollary 10. In all triangle ABC holds

$$\cos\frac{2\pi-A}{8}\cos\frac{2\pi-B}{8}\cos\frac{\pi-C}{8} \geq \frac{1}{2}\left(\sin\frac{A}{2}+\sin\frac{B}{2}+\cos\frac{C}{2}\right)$$

and his permutations.
Proof. The function $f(x) = \ln\left(\cos\frac{x}{2}\right)$, $x \in (0, \pi)$ is concave, therefore in any triangle ABC holds

$$\sum \ln\left(\cos\frac{A}{2}\right) \leq \sum \ln\left(\sin\frac{A+B}{4}\right) \text{ or}$$

$$\prod \cos\frac{A+B}{4} \geq \prod \cos\frac{A}{2} = \frac{s}{4R}$$

Using this in triangle AIB we get

$$\cos\frac{\frac{A}{2}+\frac{B}{2}}{4}\cos\frac{\frac{B}{2}+\frac{\pi+C}{2}}{4}\cos\frac{\frac{\pi+C}{2}+\frac{A}{2}}{4} \geq \frac{s_1}{4R_1} = \frac{1}{2}\left(\sin\frac{A}{2}+\sin\frac{B}{2}+\cos\frac{C}{2}\right)$$

Corollary 11. In all triangle ABC holds

$$\left(1+\cos\frac{A}{4}\right)\left(1+\cos\frac{B}{4}\right)\left(1+\cos\frac{\pi+C}{4}\right) \geq 1+2\left(\sin\frac{A}{2}+\sin\frac{B}{2}+\cos\frac{C}{2}\right)+$$

$$+4\sin\frac{A}{4}\sin\frac{B}{4}\sin\frac{\pi+C}{4}+\left(\sin\frac{A}{2}+\sin\frac{B}{2}+\cos\frac{C}{2}\right)^2+$$

$$+4\sin^2\frac{A}{4}\sin^2\frac{B}{4}\sin^2\frac{\pi+C}{4}+4\left(\sin\frac{A}{2}+\sin\frac{B}{2}+\cos\frac{C}{2}\right)\sin\frac{A}{4}\sin\frac{B}{4}\sin\frac{\pi+C}{4}$$

and his permutations.
Proof. The function $f\left(x\right)=\ln\left(1+\sin x\right),\ x\in\left(0,\pi\right)$ is concave, therefore in any triangle ABC holds

$$\sum\ln\left(1+\sin A\right)\leq\sum\ln\left(1+\sin\frac{A+B}{2}\right)\text{ or}$$

$$\prod\left(1+\cos\frac{A}{2}\right)\geq\prod\left(1+\sin A\right)=1+\frac{s}{R}+\left(\frac{s}{2R}\right)^2+\left(\frac{r}{2R}\right)^2+\frac{r}{R}+\frac{sr}{2R^2}$$

Using this in triangle AIB we get

$$\left(1+\cos\frac{A}{4}\right)\left(1+\cos\frac{B}{4}\right)\left(1+\cos\frac{\pi+C}{4}\right)\geq$$

$$\geq1+\frac{s_1}{R_1}+\frac{r_1}{R_1}+\left(\frac{s_1}{2R_1}\right)^2+\left(\frac{r_1}{2R_1}\right)^2+\frac{s_1r_1}{2R_1^2}$$

Corollary 12. In all triangle ABC holds

$$\left(1+\sin\frac{A}{4}\right)\left(1+\sin\frac{B}{4}\right)\left(1+\sin\frac{\pi+C}{4}\right)\geq$$

$$\geq5\left(\sin\frac{A}{2}+\sin\frac{B}{2}+\cos\frac{C}{2}\right)^2+12\sin^2\frac{A}{4}\sin^2\frac{B}{4}\sin^2\frac{\pi+C}{4}-3$$

and his permutations.
Proof. The function $f\left(x\right)=\ln\left(1+\cos x\right),\ x\in\left(0,\pi\right)$ is concave, therefore in any triangle ABC we have

$$\sum\ln\left(1+\cos A\right)\leq\sum\ln\left(1+\cos\frac{A+B}{2}\right)\text{ or}$$

$$\prod\left(1+\sin\frac{A}{2}\right)\geq\prod\left(1+\cos A\right)=\frac{5}{4}\left(\frac{s}{R}\right)^2+\frac{3}{4}\left(\frac{r}{R}\right)^2-3$$

Using this for the triangle AIB we get

$$\left(1+\sin\frac{A}{4}\right)\left(1+\sin\frac{B}{4}\right)\left(1+\sin\frac{\pi+C}{4}\right)\geq\frac{5}{4}\left(\frac{s_1}{R_1}\right)^2+\frac{3}{4}\left(\frac{r_1}{R_1}\right)^2-3$$

Corollary 13. In all triangle ABC holds

$$\left(1-\cos\frac{A}{4}\right)\left(1-\cos\frac{B}{4}\right)\left(1-\cos\frac{\pi+C}{4}\right)\geq$$

$$\geq 1-2\left(\sin\frac{A}{2}+\sin\frac{B}{2}+\cos\frac{C}{2}\right)+4\sin\frac{A}{4}\sin\frac{B}{4}\sin\frac{\pi+C}{4}+$$

$$+4\left(\sin\frac{A}{2}+\sin\frac{B}{2}+\cos\frac{C}{2}\right)^2+4\sin^2\frac{A}{4}\sin^2\frac{B}{4}\sin^2\frac{\pi+C}{4}-$$

$$-4\left(\sin\frac{A}{2}+\sin\frac{B}{2}+\cos\frac{C}{2}\right)\sin\frac{A}{4}\sin\frac{B}{4}\sin\frac{\pi+C}{4}$$

and his permutations.
Proof. The function $f(x)=\ln(1-\sin x)$, $x\in(0,\pi)$ is concave, therefore in all triangle ABC holds

$$\sum\ln(1-\sin A)\leq\sum\ln\left(1-\sin\frac{A+B}{2}\right)\text{ or}$$

$$\prod\left(1-\cos\frac{A}{2}\right)\geq\prod(1-\sin A)=1-\frac{s}{R}+\frac{r}{R}+\left(\frac{s}{2R}\right)^2+\left(\frac{r}{2R}\right)^2-\frac{sr}{2R^2}$$

Using this for the triangle AIB we get

$$\left(1-\cos\frac{A}{4}\right)\left(1-\cos\frac{B}{4}\right)\left(1-\cos\frac{\pi+C}{4}\right)\geq$$

$$\geq 1-\frac{s_1}{R_1}+\frac{r_1}{R_1}+\left(\frac{s_1}{2R_1}\right)^2+\left(\frac{r_1}{2R_1}\right)^2-\frac{s_1r_1}{2R_1^2}$$

Corollary 14. In all triangle ABC holds

$$\left(1-\sin\frac{A}{4}\right)\left(1-\sin\frac{B}{4}\right)\left(1-\sin\frac{\pi+C}{4}\right)\geq 3\left(\sin\frac{A}{2}+\sin\frac{B}{2}+\cos\frac{C}{2}\right)^2+$$

$$+20\sin^2\frac{A}{4}\sin^2\frac{B}{4}\sin^2\frac{\pi+C}{4}-3$$

and his permutations.
Proof. The function $f(x)=\ln(1-\cos x)$, $x\in(0,\pi)$ is concave, therefore in all triangle ABC holds

$$\sum \ln(1-\cos x) \leq \sum \ln\left(1-\cos\frac{A+B}{2}\right) \text{ or}$$

$$\prod\left(1-\sin\frac{A}{2}\right) \geq \prod(1-\cos A) = \frac{3}{4}\left(\frac{s}{R}\right)^2 + \frac{5}{4}\left(\frac{r}{R}\right)^2 - 3$$

Using this in triangle AIB we get

$$\left(1-\sin\frac{A}{4}\right)\left(1-\sin\frac{B}{4}\right)\left(1-\sin\frac{\pi+C}{4}\right) \geq \frac{3}{4}\left(\frac{s_1}{R_1}\right)^2 + \frac{5}{4}\left(\frac{r_1}{R_1}\right)^2 - 3$$

REFERENCE

[1] Bencze, M., Chang-Jian, Z., *Some applications of Popoviciu‘s inequality*, Octogon Mathematical Magazine, Vol. 16, No. 2, October 2008, pp. 846-854.

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Some Related Results to CBS Inequality

José Luis Díaz-Barrero and Eusebi Jarauta-Bragulat [14]

ABSTRACT. In this paper elementary numerical inequalities are used to obtain some additive inequalities related to the classical Cauchy-Bunyakowsky-Schwarz inequality.

1. INTRODUCTION

Cauchy-Bunyakowsky-Schwarz inequality, for short CBS inequality, plays a very important role in some branches of Mathematics such as Real and Complex Analysis, Probability and Statistics, Hilbert Spaces Theory, Numerical Analysis and Differential Equations. Many discrete inequalities are connected in some way with CBS inequality as it has been extensively documented by Mitrinovic ([]1), [2] and more recently by Dragomir [3] among others. In this paper we derive some real additive inequalities, related to classical CBS, using elementary numerical inequalities similar the ones obtained in ([4], [5]). Furthermore, their complex companions are also given.

2. MAIN RESULTS

In the sequel we present some additive counterparts to CBS inequality that will be derived using elementary numerical inequalities. We begin with a generalization of CBS inequality extending the one appeared in [4].

Theorem 1. Let $a_1, a_2, \ldots, a_n$; $b_1, b_2, \ldots, b_n$; $c_1, c_2, \ldots, c_n$ and $d_1, d_2, \ldots, d_n$ be positive real numbers and let $r_1, r_2, \ldots, r_n$ and $s_1, s_2, \ldots, s_n$ be nonnegative numbers. Then, for all integer p, holds:

$$\frac{1}{2}\left(\sum_{k=1}^{n} r_k a_k^p \sum_{k=1}^{n} s_k b_k^p + \sum_{k=1}^{n} r_k c_k^p \sum_{k=1}^{n} s_k d_k^p\right) \geq$$

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$$\geq \left(\sum_{k=1}^{n} r_k a_k^{p/2} c_k^{p/2}\right) \left(\sum_{k=1}^{n} s_k b_k^{p/2} d_k^{p/2}\right)$$

Proof. Applying mean inequalities to positive numbers a and b, we have

$$a^p + b^p \geq 2a^{p/2} b^{p/2}$$

valid for all integer p. Therefore, for $1 \leq i, j \leq n$, we have

$$a_i^p b_j^p + c_i^p d_j^p \geq 2a_i^{p/2} b_j^{p/2} c_i^{p/2} d_j^{p/2}$$

Multiplying up by $r_i s_j \geq 0, (1 \leq i, j \leq n)$, both sides of the preceding inequalities yields

$$r_i s_j a_i^p b_j^p + r_i s_j c_i^p d_j^p \geq 2r_i s_j a_i^{p/2} b_j^{p/2} c_i^{p/2} d_j^{p/2}$$

Adding up the above inequalities, we obtain:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \left(r_i s_j a_i^p b_j^p + r_i s_j c_i^p d_j^p\right) = \sum_{k=1}^{n} r_k a_k^p \sum_{k=1}^{n} s_k b_k^p + \sum_{k=1}^{n} r_k c_k^p \sum_{k=1}^{n} s_k d_k^p \geq$$

$$\geq \sum_{i=1}^{n} \sum_{j=1}^{n} \left(2r_i s_j a_i^{p/2} b_j^{p/2} c_i^{p/2} d_j^{p/2}\right) = 2\left(\sum_{k=1}^{n} r_k a_k^{p/2} c_k^{p/2}\right) \left(\sum_{k=1}^{n} s_k b_k^{p/2} d_k^{p/2}\right)$$

and this completes the proof.

Notice that when $p = 2$, $r_k = s_k = 1$ and $c_k = b_k, d_k = a_k, (1 \leq k \leq n)$, we get CBS inequality.

In what follows the same idea is used to obtain some related results to CBS inequality. We start with

Theorem 2. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be positive numbers and let $c_1, c_2, \ldots, c_n$ and $d_1, d_2, \ldots, d_n$ be nonnegative numbers. Then, for all integer p, holds:

$$\frac{1}{2}\left(\sum_{k=1}^{n} d_k \sum_{k=1}^{n} c_k a_k^{p/2} + \sum_{k=1}^{n} c_k \sum_{k=1}^{n} d_k b_k^{p/2}\right) \geq \left(\sum_{k=1}^{n} c_k a_k^{p/2}\right) \left(\sum_{k=1}^{n} d_k b_k^{p/2}\right)$$

Proof. Applying mean inequalities to positive numbers a and b, we have

$$a^p + b^p \geq 2a^{p/2}b^{p/2}$$

valid for all positive integer p. Therefore, for $1 \leq i, j \leq n$, we have

$$a_i^p + b_j^p \geq 2a_i^{p/2}b_j^{p/2}$$

Multiplying both sides by $c_i d_j \geq 0$, $(1 \leq i, j \leq n)$, we obtain

$$c_i d_j a_i^p + c_i d_j b_j^p \geq 2c_i d_j a_i^{p/2} b_j^{p/2}$$

Adding up the preceding inequalities, yields

$$\sum_{i=1}^n \sum_{j=1}^n \left(c_i d_j a_i^p + c_i d_j b_j^p \right) = \sum_{k=1}^n d_k \sum_{k=1}^n c_k a_k^p + \sum_{k=1}^n c_k \sum_{k=1}^n d_k b_k^p \geq$$

$$\geq 2 \sum_{i=1}^n \sum_{j=1}^n \left(c_i d_j a_i^{p/2} b_j^{p/2} \right) = 2 \left(\sum_{k=1}^n c_k a_k^{p/2} \right) \left(\sum_{k=1}^n d_k b_k^{p/2} \right)$$

and this completes the proof.
The complex version of the preceding result is stated in the following

Corollary 1. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be complex numbers and let $c_1, c_2, \ldots, c_n$ and $d_1, d_2, \ldots, d_n$ be nonnegative numbers. Then, for all integer p, holds:

$$\frac{1}{2} \left(\sum_{k=1}^n d_k \sum_{k=1}^n c_k |a_k|^{p/2} + \sum_{k=1}^n c_k \sum_{k=1}^n d_k |b_k|^{p/2} \right) \geq$$

$$\geq \left(\sum_{k=1}^n c_k |a_k|^{p/2} \right) \left(\sum_{k=1}^n d_k |b_k|^{p/2} \right)$$

Now, we state and proof our second main result.

Theorem 3. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be positive numbers and let $c_1, c_2, \ldots, c_n$ and $d_1, d_2, \ldots, d_n$ be nonnegative numbers. Then, for all integer $p \geq 1$, holds:

$$\sum_{k=1}^n c_k \sum_{k=1}^n d_k b_k^p + \sum_{k=1}^n d_k \sum_{k=1}^n c_k a_k^p \geq$$

$$\geq \left(\sum_{k=1}^n c_k a_k^{p-1} \sum_{k=1}^n d_k b_k \right) + \left(\sum_{k=1}^n c_k a_k \sum_{k=1}^n d_k b_k^{p-1} \right)$$

Proof. To prove the preceding inequality we need the following
Lemma 1. Let a, b be positive real numbers. Then, for every integer $p \geq 1$, holds:

$$a^p + b^p \geq a^{p-1}b + ab^{p-1}$$

Proof. We will argue by mathematical induction. The cases when $p = 1$ and $p = 2$ trivially hold. Suppose that the given inequality holds for $p - 1$, that is, it holds that $a^{p-1} + b^{p-1} \geq a^{p-2}b + ab^{p-2}$. Writting now

$$a^p + b^p = a\left(a^{p-1} + b^{p-1}\right) + b^p - ab^{p-1}$$

and taking into account the inductive hypotheses, we get

$$a^p + b^p \geq a\left(a^{p-2}b + ab^{p-2}\right) + b^p - ab^{p-1} = a^{p-1}b + a^2b^{p-2} + b^p - ab^{p-1}$$

Since $a^2b^{p-2} + b^p - ab^{p-1} = b^{p-2}\left(a^2 + b^2 - ab\right) \geq b^{p-2}(ab) = ab^{p-1}$, then $a^p + b^p \geq a^{p-1}b + ab^{p-1}$ as desired. We observe that equality holds if, and only if, $a = b$ and the proof is complete.

From the previous lemma, we have for $1 \leq i, j \leq n$,

$$a_i^p + b_j^p \geq a_i^{p-1}b_j + a_ib_j^{p-1}$$

Multiplying both sides by $c_id_j \geq 0$, $(1 \leq i, j \leq n)$, we obtain

$$c_id_ja_i^p + c_id_jb_j^p \geq c_id_ja_i^{p-1}b_j + c_id_ja_ib_j^{p-1}$$

Adding up the preceding inequalities yields

$$\sum_{k=1}^n d_k \sum_{k=1}^n c_ka_k^p + \sum_{k=1}^n c_k \sum_{k=1}^n d_kb_k^p = \sum_{i=1}^n \sum_{j=1}^n \left(c_id_ja_i^p + c_id_jb_j^p\right) \geq$$

$$\geq \sum_{i=1}^n \sum_{j=1}^n \left(c_id_ja_i^{p-1}b_j + ca_id_ja_ib_j^{p-1}\right) =$$

$$= \left(\sum_{k=1}^n c_ka_k^{p-1} \sum_{k=1}^n d_kb_k\right) + \left(\sum_{k=1}^n c_ka_k \sum_{k=1}^n d_kb_k^{p-1}\right)$$

as claimed, and the proof is complete.

The complex counterpart of the previous result is given in the next

Corollary 2. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be complex numbers and let $c_1, c_2, \ldots, c_n$ and $d_1, d_2, \ldots, d_n$ be nonnegative numbers. Then, for all integer $p \geq 1$, holds:

$$\sum_{k=1}^{n} c_k \sum_{k=1}^{n} d_k |b_k|^p + \sum_{k=1}^{n} d_k \sum_{k=1}^{n} c_k |a_k|^p \geq$$

$$\geq \left(\sum_{k=1}^{n} c_k |a_k|^{p-1} \sum_{k=1}^{n} d_k |b_k| \right) + \left(\sum_{k=1}^{n} c_k |a_k| \sum_{k=1}^{n} d_k |b_k|^{p-1} \right)$$

Finally, we will use a constrained elementary inequality to obtain the following result.

Theorem 4. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be positive numbers and let $c_1, c_2, \ldots, c_n$ and $d_1, d_2, \ldots, d_n$ be nonnegative numbers. If α, β are positive numbers such that $\alpha = 1 + \beta$, then

$$\frac{1}{\alpha} \left(\sum_{k=1}^{n} d_k \sum_{k=1}^{n} c_k a_k^{\alpha} + \beta \sum_{k=1}^{n} c_k \sum_{k=1}^{n} d_k b_k \right) \geq \left(\sum_{k=1}^{n} c_k a_k \right) \left(\sum_{k=1}^{n} d_k b_k^{\beta} \right)$$

Proof. We begin with a Lemma.

Lemma 2. Let a, b, α and β be real numbers such that $a \geq 0$, $b, \alpha, \beta > 0$ and $\alpha = 1 + \beta$. Then,

$$a^{\alpha} + \beta b^{\alpha} \geq \alpha a b^{\beta}$$

with equality if, and only if, $a = b$.

Proof. The inequality claimed can be written in the equivalent form

$$b^{\beta}(\alpha a - \beta b) \leq a^{\alpha}$$

When $a = 0$ the inequality is strict, and when $a = b$ the inequality becomes equality. Hence, we can assume that $a > 0$ and $a \neq b$. Set $\lambda = a/b$. Then, the inequality is equivalent to $\alpha\lambda - \beta < \lambda^{\alpha}$ for $\lambda \neq 1$.

Therefore, we have to prove that holds $\lambda^{\alpha} - \alpha\lambda + \alpha - 1 > 0$ for any $0 < \lambda \neq 1$. Indeed, let f be the function defined by $f(\lambda) = \lambda^{\alpha} - \alpha\lambda + \alpha - 1$. It is easy to see that $f'(1) = 0$, $f'(\lambda) < 0$ in $(0, 1)$ and $f'(\lambda) > 0$ in $(1, +\infty)$. This implies that $f(\lambda) > f(1) = 0$ if $\lambda \neq 1$ and this completes the proof.

Now carrying out the same procedure as in the previous results, we can write for $1 \leq i, j \leq n$,

$$a_i^{\alpha} + \beta b_j^{\alpha} \geq \alpha a_i b_j^{\beta}$$

Multiplying up both sides for $c_i d_j > 0, 1 \leq i, j \leq n$, yields

$$c_i d_j a_i^{\alpha} + \beta c_i d_j b_j^{\alpha} \geq \alpha c_i d_j a_i b_j^{\beta}$$

Adding up those inequalities, we get

$$\sum_{k=1}^{n} d_k \sum_{k=1}^{n} c_k a_k^{\alpha} + \beta \sum_{k=1}^{n} c_k \sum_{k=1}^{n} d_k b_k = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(c_i d_j a_i^{\alpha} + \beta c_i d_j b_j^{\alpha} \right) \geq$$

$$\geq \alpha \sum_{i=1}^{n} \sum_{j=1}^{n} c_i d_j a_i b_j^{\beta} = \alpha \left(\sum_{k=1}^{n} c_k a_k \right) \left(\sum_{k=1}^{n} d_k b_k^{\beta} \right)$$

from which the result immediately follows and the proof is complete.

Likewise, the complex version of the about inequality is presented in

Corollary 3. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be complex numbers and let $c_1, c_2, \ldots, c_n$ and $d_1, d_2, \ldots, d_n$ be nonnegative numbers. If α, β are positive numbers such that $\alpha = 1 + \beta$, then

$$\frac{1}{\alpha} \left(\sum_{k=1}^{n} c_k \sum_{k=1}^{n} c_k |a_k|^{\alpha} + \beta \sum_{k=1}^{n} c_k \sum_{k=1}^{n} d_k |b_k| \right) \geq \left(\sum_{k=1}^{n} c_k |a_k| \right) \left(\sum_{k=1}^{n} d_k |b_k|^{\beta} \right)$$

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REFERENCES

[1] Mitrinovic, D. S. *Analytic Inequalities*, Springer-Verlag, Berlin/ Heidelberg/ New York, 1970.

[2] Mitrinovic, D. S., Pecčarić and A. M. Fink *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrech/Boston/London, 1993.

[3] Dragomir, S. S. *A Survey on Cauchy-Buniakowsky-Schwarz Type Discrete Inequalities,* RGMIA Monographs, Victoria University, 2000. (ONLINE: http://rgmia.vu.edu.au/monographs/).

[4] Dragomir, S. S. “On some inequalities.” *Caite Metodico Ştiinţifice,* No. 13, (1984), pp. 20. Faculty of Matematics, Timişoara University, Romania.

[5] Dragomir, S. S. “On Cauchy-Buniakowski-Schwarz’s Inequality for Real Numbers.” *Caiete Metodico Ştiinţifice,* No. 57, (1989), pp. 24. Faculty of Matematics, Timişoara University, Romania.

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One some new type inequalities in triangle

Mihály Bencze and Wei-Dong Jiang[15]

ABSTRACT. In this paper we present some new type inequalitie in triangle, giving a generalization of D. Milosevic inequality.

MAIN RESULTS

Theorem 1. Let ABC be a triangle, and $g_1 : R \to (0, +\infty)$ a log-convex, $g_2 : R \to (0, +\infty)$ a log-concave and $f, h : R \to (0, +\infty)$, then for all $x, y, z > 0$ we have

$$1).\ \sum \frac{y+z}{x} \cdot \frac{f(a)\, g_1(A)}{h(a)} \geq 6 g_1\left(\frac{\pi}{3}\right) \sqrt[3]{\prod \frac{f(a)}{h(a)}}$$

$$2).\ \sum \frac{y+z}{x} \cdot \frac{f(a)}{g_2(A)\, h(a)} \geq \frac{6}{g_2\left(\frac{\pi}{3}\right)} \sqrt[3]{\prod \frac{f(a)}{h(a)}}$$

Proof. Using the AM-GM inequality we have:

$$\frac{x}{y} \cdot \frac{f(b)\, g_1(B)}{h(b)} + \frac{y}{x} \cdot \frac{f(a)\, g_1(A)}{h(a)} \geq 2\sqrt{\frac{g_1(A)\, g_1(B)\, f(a)\, f(b)}{h(a)\, h(b)}},$$

therefore

$$\sum \frac{y+z}{x} \cdot \frac{f(a)\, g_1(A)}{h(a)} \geq 2 \sum \sqrt{\frac{g_1(A)\, g_1(B)\, f(a)\, f(b)}{h(a)\, h(b)}} \geq$$

$$\geq 6 \sqrt[3]{\prod \frac{g_1(A)\, f(a)}{h(a)}} \geq 6 g_1\left(\frac{\pi}{3}\right) \sqrt[3]{\prod \frac{f(a)}{h(a)}}$$

because

$$\prod g_1(A) \geq g_1^3\left(\frac{1}{3} \sum A\right) = g_1^3\left(\frac{\pi}{3}\right)$$

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In same way we get:

$$\sum \frac{y+z}{x} \cdot \frac{f(a)}{g_2(A)\,h(a)} \geq 6\sqrt[3]{\prod \frac{f(a)}{g_2(A)\,h(a)}} \geq \frac{6}{g_2\left(\frac{\pi}{3}\right)}\sqrt[3]{\prod \frac{f(a)}{h(a)}}$$

because

$$\prod g_2(A) \leq g_2^3\left(\frac{1}{3}\sum A\right) = g_2^3\left(\frac{\pi}{3}\right)$$

Corollary 1.1. In all triangle ABC for all $\lambda, x, y, z > 0$ holds
1). $\sum \frac{y+z}{x} \cdot \frac{a}{A^\lambda (s-a)} \geq 6\left(\frac{3}{\pi}\right)^\lambda \sqrt[3]{\frac{4R}{r}}$ a refinement of [1] and [2]
2). $\sum \frac{y+z}{x} \cdot \frac{r_a}{A^\lambda (r_b+r_c)} \geq 6\left(\frac{3}{\pi}\right)^\lambda \sqrt[3]{\frac{r}{4R}}$
3). $\sum \frac{y+z}{x} \cdot \frac{h_a}{A^\lambda (h_b+h_c)} \geq 6\left(\frac{3}{\pi}\right)^\lambda \sqrt[3]{\frac{2Rr}{s^2+r^2+2Rr}}$
4). $\sum \frac{y+z}{x} \cdot \frac{\sin^2\frac{A}{2}}{A^\lambda\left(\sin^2\frac{B}{2}+\sin^2\frac{C}{2}\right)} \geq 6\left(\frac{3}{\pi}\right)^\lambda \sqrt[3]{\frac{2Rr^2}{(2R-r)(s^2+r^2-8Rr)-2Rr^2}}$
5). $\sum \frac{y+z}{x} \cdot \frac{\cos^2\frac{A}{2}}{A^\lambda\left(\cos^2\frac{B}{2}+\cos^2\frac{C}{2}\right)} \geq 6\left(\frac{3}{\pi}\right)^\lambda \sqrt[3]{\frac{2Rr^2}{(4R+r)^3+(2R+r)s^2}}$
6). $\sum \frac{y+z}{x} \cdot \frac{a}{A^\lambda (r_b+r_c)} \geq 6\left(\frac{3}{\pi}\right)^\lambda \sqrt[3]{\frac{r}{s}}$
7). $\sum \frac{y+z}{x} \cdot \frac{r_a}{A^\lambda (h_b+h_c)} \geq 6\left(\frac{3}{\pi}\right)^\lambda \sqrt[3]{\frac{R^2}{s^2+r^2+2Rr}}$

Proof. In Theorem 1 we take $g_2(x) = x^\lambda$ and
1). $f(a) = a, h(a) = s - a$
2). $f(a) = r_a, h(a) = r_b + r_c$
3). $f(a) = h_a, h(a) = h_b + h_c$
4). $f(a) = \sin^2\frac{A}{2}, h(a) = \sin^2\frac{A}{2} + \sin^2\frac{C}{2}$
5). $f(a) = \cos^2\frac{A}{2}, h(a) = \cos^2\frac{A}{2} + \cos^2\frac{C}{2}$
6). $f(a) = a, h(a) = r_b + r_c$
7). $f(a) = r_a, h(a) = h_b + h_c$

Corollary 1.2. Let ABC be a triangle, then for all $x, y, z, \lambda > 0$ we have:
1). $\sum \frac{y+z}{x} \cdot \frac{a}{(s-a)\sin^\lambda A} \geq 6\left(\frac{2}{\sqrt{3}}\right)^\lambda \sqrt{\frac{4R}{r}}$
2). $\sum \frac{y+z}{x} \cdot \frac{r_a}{(r_b+r_c)\sin^\lambda A} \geq 6\left(\frac{2}{\sqrt{3}}\right)^\lambda \sqrt{\frac{r}{4R}}$
3). $\sum \frac{y+z}{x} \cdot \frac{h_a}{(h_b+h_c)\sin^\lambda A} \geq 6\left(\frac{2}{\sqrt{3}}\right)^\lambda \sqrt[3]{\frac{2R}{s^2+r^2+2Rr}}$
4). $\sum \frac{y+z}{x} \cdot \frac{\sin^2\frac{A}{2}}{\left(\sin^2\frac{B}{2}+\sin^2\frac{C}{2}\right)\sin^\lambda A} \geq 6\left(\frac{2}{\sqrt{3}}\right)^\lambda \sqrt{\frac{2Rr^2}{(2R-r)(s^2+r^2-8Rr)-2Rr^2}}$
5). $\sum \frac{y+z}{x} \cdot \frac{\cos^2\frac{A}{2}}{\left(\cos^2\frac{B}{2}+\cos^2\frac{C}{2}\right)\sin^\lambda A} \geq 6\left(\frac{2}{\sqrt{3}}\right)^\lambda \sqrt[3]{\frac{2Rr^2}{(4R+r)^3+(2R+r)s^2}}$

6). $\sum \frac{y+z}{x} \cdot \frac{a}{(r_b+r_c)\sin^{\lambda} A} \geq 6\left(\frac{2}{\sqrt{3}}\right)^{\lambda} \sqrt[3]{\frac{r}{s}}$

7). $\sum \frac{y+z}{x} \cdot \frac{a}{(h_b+h_c)\sin^{\lambda} A} \geq 6\left(\frac{2}{\sqrt{3}}\right)^{\lambda} \sqrt[3]{\frac{R^2}{s^2+r^2+2Rr}}$

Proof. In Theorem 1 we take $g_2(x) = \sin^{\lambda} x$ and use the situations from Corollary 1.1.

Corollary 1.3. Let ABC be a triangle, then for all $\lambda, x, y, z > 0$ we have:

1). $\sum \frac{y+z}{x} \cdot \frac{a \cdot ch^{\lambda} A}{s-a} \geq 6\left(ch\frac{\pi}{3}\right)^{\lambda} \sqrt[3]{\frac{4R}{r}}$

2). $\sum \frac{y+z}{x} \cdot \frac{r_a ch^{\lambda} A}{r_b+r_c} \geq 6\left(ch\frac{\pi}{3}\right)^{\lambda} \sqrt[3]{\frac{r}{4R}}$

3). $\sum \frac{y+z}{x} \cdot \frac{h_a ch^{\lambda} A}{h_b+h_c} \geq 6\left(ch\frac{\pi}{3}\right)^{\lambda} \sqrt[3]{\frac{2Rr}{s^2+r^2+2Rr}}$

4). $\sum \frac{y+z}{x} \cdot \frac{\sin^2\frac{A}{2} ch^{\lambda} A}{\sin^2\frac{B}{2}+\sin^2\frac{C}{2}} \geq 6\left(ch\frac{\pi}{3}\right)^{\lambda} \sqrt[3]{\frac{2Rr^2}{(2R-r)(s^2+r^2-8Rr)-2Rr^2}}$

5). $\sum \frac{y+z}{x} \cdot \frac{\cos^2\frac{A}{2} ch^{\lambda} A}{\cos^2\frac{B}{2}+\cos^2\frac{C}{2}} \geq 6\left(ch\frac{\pi}{3}\right)^{\lambda} \sqrt[3]{\frac{2Rr^2}{(4R+r)^3+(2R+r)s^2}}$

6). $\sum \frac{y+z}{x} \cdot \frac{a \cdot ch^{\lambda} A}{r_b+r_c} \geq 6\left(ch\frac{\pi}{3}\right)^{\lambda} \sqrt[3]{\frac{r}{s}}$

6). $\sum \frac{y+z}{x} \cdot \frac{a \cdot ch^{\lambda} A}{r_b+r_c} \geq 6\left(ch\frac{\pi}{3}\right)^{\lambda} \sqrt[3]{\frac{r}{s}}$

7). $\sum \frac{y+z}{x} \cdot \frac{r_a ch^{\lambda} A}{h_b+h_c} \geq 6\left(ch\frac{\pi}{3}\right)^{\lambda} \sqrt[3]{\frac{R^2}{s^2+r^2+2Rr}}$

Proof. In Theorem 1 we take $g_1(x) = ch^{\lambda} x$ and we use the situations from Corollary 1.1.

Corollary 1.4. Let ABC be a triangle, then for all $\lambda, x, y, z > 0$ we have:

1). $\sum \frac{y+z}{x} \cdot \frac{a\left(1+e^{A}\right)^{\lambda}}{s-a} \geq 6\left(1+e^{\frac{\pi}{3}}\right)^{\lambda} \sqrt[3]{\frac{4R}{r}}$

2). $\sum \frac{y+z}{x} \cdot \frac{r_a\left(1+e^{A}\right)^{\lambda}}{r_b+r_c} \geq 6\left(1+e^{\frac{\pi}{3}}\right)^{\lambda} \sqrt[3]{\frac{r}{4R}}$

3). $\sum \frac{y+z}{x} \cdot \frac{h_a\left(1+e^{A}\right)^{\lambda}}{h_b+h_c} \geq 6\left(1+e^{\frac{\pi}{3}}\right)^{\lambda} \sqrt[3]{\frac{2Rr}{s^2+r^2+2Rr}}$

4). $\sum \frac{y+z}{x} \cdot \frac{\left(1+e^{A}\right)^{\lambda}\sin^2\frac{A}{2}}{\sin^2\frac{B}{2}+\sin^2\frac{C}{2}} \geq 6\left(1+e^{\frac{\pi}{3}}\right)^{\lambda} \sqrt[3]{\frac{2Rr^2}{(2R-r)(s^2+r^2-8Rr)-2Rr^2}}$

5). $\sum \frac{y+z}{x} \cdot \frac{\left(1+e^{A}\right)^{\lambda}\cos^2\frac{A}{2}}{\cos^2\frac{B}{2}+\cos^2\frac{C}{2}} \geq 6\left(1+e^{\frac{\pi}{3}}\right)^{\lambda} \sqrt[3]{\frac{2Rr^2}{(4R+r)+(2R+r)s^2}}$

6). $\sum \frac{y+z}{x} \cdot \frac{a\left(1+e^{A}\right)^{\lambda}}{r_b+r_c} \geq 6\left(1+e^{\frac{\pi}{3}}\right)^{\lambda} \sqrt[3]{\frac{r}{s}}$

7). $\sum \frac{y+z}{x} \cdot \frac{r_a\left(1+e^{A}\right)^{\lambda}}{h_b+h_c} \geq 6\left(1+e^{\frac{\pi}{3}}\right)^{\lambda} \sqrt[3]{\frac{R^2}{s^2+r^2+2Rr}}$

Proof. In Theorem 1 we take $g_1(x) = (1+e^{x})^{\lambda}$ and we use the situations from Corollary 1.1.

REFERENCES

[1] Milosevic, D., *Problem 76**, Univ. Beograd, Publ. Elec. Fac. Ser 17/2006.
[2] Jiang, W.D., and Bencze, .M, *O problema a lui D.M. Milosevic,* (in Romanian)Revista de Matematica din Valea Jiului, Nr.1, martie 2009, pp. 8.

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Two geometric inequalities involved two triangles

Yu-Lin Wu[16]

ABSTRACT. In this short note, we prove two geometric inequality conjectures involved two triangles posed by Liu [9].

1. INTRODUCTION AND MAIN RESULTS

For $\triangle ABC$, let a, b, c be the side-lengths, $\triangle$ the area, s the semi-perimeter, R the circumradius and r the inradius, respectively. Denote by w_a, w_b and w_c the interior bisectors of angles, and h_a, h_b, h_c the altitudes, respectively.

It's has been a long time since the scholar studied the inequality involved two triangles. The very famous one is the following Neuberg–Pedoe's inequality [11].

$$a'^2(b^2+c^2-a^2)+b'^2(c^2+a^2-b^2)+c'^2(a^2+b^2-c^2)\geq 16\triangle\triangle'$$

Recently, Chinese scholar studied some geometric inequalities involved two triangles. For example, Zhang [14] and Gao [4] proved the inequalities as follows, respectively.

$$a^2a'^2+b^2b'^2+c^2c'^2\geq 16\triangle\triangle'$$

$$a'(b+c-a)+b'(c+a-b)+c'(a+b-c)\geq\sqrt{48\triangle\triangle'}$$

An [1] obtained several inequalities as follows.

$$aa'+bb'+cc'\geq 4\sqrt{\left(\frac{aa'}{bb'}+\frac{bb'}{cc'}+\frac{cc'}{aa'}\right)\triangle\triangle'}$$

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$$\frac{1}{aa'bb'}+\frac{1}{bb'cc'}+\frac{1}{cc'aa'}\leq\frac{9}{16\triangle\triangle'}$$

$$a'(s-a)+b'(s-b)+c'(s-c)\geq 4\sqrt{\left(\sum\sin^2\frac{A}{2}\cdot\sum\sin^2\frac{A'}{2}\right)\triangle\triangle'}$$

Wu [12] proved the following two inequalities.

$$h_ah'_a+h_bh'_b+h_ch'_c\leq\frac{3}{4}(aa'+bb'+cc')$$

$$\frac{1}{h_ah'_a}+\frac{1}{h_bh'_b}+\frac{1}{h_ch'_c}\geq\frac{12}{aa'+bb'+cc'}$$

Leng [8] showed the proof of the inequality as follows.

$$w_aw'_a+w_bw'_b+w_cw'_c\leq\frac{3}{4}(aa'+bb'+cc')$$

Jiang [6] proved the following inequality.

$$\frac{\sin A'}{a}+\frac{\sin B'}{b}+\frac{\sin C'}{c}\leq\frac{1}{2}\left(\frac{1}{R}+\frac{1}{r}\right)$$

J. Liu [9] posed the following two interesting geometric inequality conjectures in 2008.

Conjecture 1. For $\triangle ABC$ and $\triangle A'B'C'$, prove or disprove

$$(b+c)\cot\frac{A'}{2}+(c+a)\cot\frac{B'}{2}+(a+b)\cot\frac{C'}{2}\geq 4(w_a+w_b+w_c). \tag{1.1}$$

Conjecture 2. For $\triangle ABC$ and $\triangle A'B'C'$, and real numbers x,y,z, prove or disprove

$$x^2aa'+y^2bb'+z^2cc'\geq\frac{4}{3}(yzw_aw'_a+zxw_bw'_b+xyw_cw'_c). \tag{1.2}$$

We prove the two conjectures in this paper.

2. PRELIMINARY RESULTS

Lemma 1. ([3, 5]) For real numbers $x_1, x_2, x_3, y_1, y_2, y_3$ such that

$$x_1x_2 + x_2x_3 + x_3x_1 \geq 0$$

and

$$y_1y_2 + y_2y_3 + y_3y_1 \geq 0,$$

the following inequality holds.

$$(y_2 + y_3)x_1 + (y_3 + y_1)x_2 + (y_1 + y_2)x_3 \geq$$

$$\geq 2\sqrt{(x_1x_2 + x_2x_3 + x_3x_1)(y_1y_2 + y_2y_3 + y_3y_1)} \tag{2.1}$$

With equality holds if and only if $\frac{x_1}{y_1} = \frac{x_2}{y_2} = \frac{x_3}{y_3}$.

Lemma 2. ([2]) In $\triangle ABC$, we have

$$\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2} \geq 3\sqrt{3}. \tag{2.2}$$

Lemma 3. (10, 13) In $\triangle ABC$, we have

$$w_aw_b + w_bw_c + w_cw_a \leq 3r(4R + r) \tag{2.3}$$

Lemma 4. In $\triangle ABC$, we have

$$w_a + w_b + w_c \leq \frac{3}{2}\sqrt{ab + bc + ca}. \tag{2.4}$$

Proof. With well-known inequalities [2] $w_a \leq \sqrt{s(s-a)}$, etc. We have

$$w_a^2 + w_b^2 + w_c^2 \leq s(s-a) + s(s-b) + s(s-c) = s^2. \tag{2.5}$$

From inequality (2.5) and Lemma 3, we obtain

$$(w_a + w_b + w_c)^2 \leq s^2 + 24Rr + 6r^2. \tag{2.6}$$

With known identity $ab + bc + ca = s^2 + 4Rr + r^2$, we get

$$\frac{9}{4}(ab+bc+ca) - (s^2 + 24Rr + 6r^2) = \frac{5}{4}[s^2 - 16Rr + 5r^2 + 4r(R - 2r)]. \tag{2.7}$$

From identity (2.7), Gerretssen's inequality $s^2 \geq 16Rr - 5r^2$ and Euler's inequality $R \geq 2r$, we can conclude that

$$\frac{9}{4}(ab+bc+ca) - (s^2+24Rr+6r^2) \geq 0 \Longleftrightarrow s^2+24Rr+6r^2 \leq$$

$$\leq \frac{3}{2}\sqrt{ab+bc+ca}. \tag{2.8}$$

Inequality (2.4) follows from inequality (2.6) and (2.8) immediately. Thus, we complete the proof of Lemma 4.

Lemma 5. (Wolstenholme's inequality, see [7]) For $\triangle ABC$ and real numbers x, y, z, we have

$$x^2+y^2+z^2 \geq 2yz\cos A + 2zx\cos B + 2xy\cos C, \tag{2.9}$$

with equality holds if and only if $x : y : z = \sin A : \sin B : \sin C$.

3. THE PROOF OF CONJECTURE 1

By Lemma 1, we get

$$(b+c)\cot\frac{A'}{2} + (c+a)\cot\frac{B'}{2} + (a+b)\cot\frac{C'}{2} \geq$$

$$\geq 2\sqrt{(ab+bc+ca)\left(\cot\frac{A'}{2}\cot\frac{B'}{2} + \cot\frac{B'}{2}\cot\frac{C'}{2} + \cot\frac{C'}{2}\cot\frac{A'}{2}\right)} \tag{3.1}$$

By Lemma 2 and $AM-GM$ inequality, we obtain

$$\cot\frac{A'}{2}\cot\frac{B'}{2} + \cot\frac{B'}{2}\cot\frac{C'}{2} + \cot\frac{C'}{2}\cot\frac{A'}{2} \geq$$

$$\geq 3\left(\cot\frac{A'}{2}\cot\frac{B'}{2}\cot\frac{C'}{2}\right)^{\frac{2}{3}} = 9. \tag{3.2}$$

With inequality (3.1)-(3.2), together with Lemma 4, we can conclude that inequality (1.1) holds. The proof of conjecture 1 is complete.

3. THE PROOF OF CONJECTURE 2

With known inequalities $w_a \leq \sqrt{bc}\cos\frac{A}{2}$, etc, we get

$$\frac{4}{3}(yzw_aw'_a + zxw_bw'_b + xyw_cw'_c) \leq$$

$$\leq \frac{4}{3}\left(yz\sqrt{bcb'c'}\cos\frac{A}{2}\cos\frac{A'}{2} + zx\sqrt{cac'a'}\cos\frac{B}{2}\cos\frac{B'}{2} + xy\sqrt{aba'b'}\cos\frac{C}{2}\cos\frac{C'}{2}\right) =$$

$$= \frac{4}{3}\left[yz\sqrt{bcb'c'}\left(\cos\frac{A+A'}{2} + \cos\frac{A-A'}{2}\right) + zx\sqrt{cac'a'}\left(\cos\frac{B+B'}{2} + \right.\right.$$

$$\left.\left. + \cos\frac{B-B'}{2}\right) + xy\sqrt{aba'b'}\left(\cos\frac{C+C'}{2} + \cos\frac{C-C'}{2}\right)\right] \qquad (4.1)$$

For $\frac{A+A'}{2} + \frac{B+B'}{2} + \frac{C+C'}{2} = \pi$, then by Lemma 5, we have

$$yz\sqrt{bcb'c'}\cos\frac{A+A'}{2} + zx\sqrt{cac'a'}\cos\frac{B+B'}{2} + xy\sqrt{aba'b'}\cos\frac{C+C'}{2} \leq$$

$$\leq \frac{1}{2}(x^2aa' + y^2bb' + z^2cc'). \qquad (4.2)$$

From $\cos\frac{A-A'}{2} \leq 1, \cos\frac{B-B'}{2} \leq 1, \cos\frac{C-C'}{2} \leq 1$, and $x^2 + y^2 + z^2 \geq xy + yz + zx$, we obtain

$$yz\sqrt{bcb'c'}\cos\frac{A-A'}{2} + zx\sqrt{cac'a'}\cos\frac{B-B'}{2} + xy\sqrt{aba'b'}\cos\frac{C-C'}{2} \leq$$

$$\leq yz\sqrt{bcb'c'} + zx\sqrt{cac'a'} + xy\sqrt{aba'b'} \leq x^2aa' + y^2bb' + z^2cc'. \qquad (4.2)$$

Inequality (1.2) follows from inequalities (4.1)-(4.3) immediately. Thus, we complete the proof of Conjecture 2.

REFERENCES

[1] An, Z.-P., *Discussing about an Important Embedding Inequality in Triangle*, Maths Teaching in Middle Schools, (5)(1994), 15–16. (in Chinese)
[2] Bottema, O., R. Ž. Djordević, R. R. Janić, D. S. Mitrinović and P.M.Vasić, *Geometric Inequality*, Wolters-Noordhoff Publishing, Groningen, The Netherlands, 1969.
[3] Pham Huu Duc, *An Unexpectedly Useful Inequality*, Mathematical Reflections, 3(1)(2008).
[4] Gao, L., *A New Inequality involved Two Triangles*, Xiamen Mathematical Communications, (3)(1983), 9. (in Chinese)
[5] Tran Quang Hung, *On Some Geometric Inequalities*, Mathematical Reflections, 3(3)(2008).
[6] Jiang, W.-D., *The Proof Of CIQ.131*, Communications in Studies of Inequalities, 12(1)(2005), 98–99. (in Chinese)
[7] Kuang, J.-C., *Chángyòng Bùděngshì* (*Applied Inequalities*), 3rd ed., Shandong Science and Technology Press, Jinan City, Shandong Province, China, 2004, 229. (in Chinese)
[8] Leng, G.-S., *Geometric Inequalities*, East China Normal University Press, Shanghai City, China, 2005, 68–69. (in Chinese)
[9] Liu, J., *Nine Sine Inequality*, manuscript, 2008, 77. (in Chinese)
[10] Liu, J., *Inequalities involved Interior Bisectors of Angles in Triangle*, Forward Position of Elementary Mathematics, 1996, 90–96. (in Chinese)
[11] Pedoe, D., *An Inequality for two triangles*, Proc Cambridge Philos. Soc., 38(1942), 397–398.
[12] Wu, Y.-S., *The Proof of Whc143*, High-School Mathematics Monthly, 20(9)(1997), 40. (in Chinese)
[13] Yang, X.-Z., *Research in Inequalities*, Tibet People's Press, Lhasa, 2000, 574. (in Chinese)
[14] Zhang, Z.-M., *Pedoe's Inequality and Another Inequality involved Two Triangles*, Bulletin of Maths, (1)(1980), 28. (in Chinese)
[15] Octogon Mathematical Magazine (1993-2009)

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Some applications of certain inequalities

Mihály Bencze and Nicuşor Minculete[17]

ABSTRACT. The purpose of this paper is to show several geometric inequalities. Which are based on the algebric inequalities.

1. INTRODUCTION

Bellow, we present three lemmas which will help us find several geometric inequalities for the triangle, for the bicentric quadrilateral and for the polygon.
Note that in [2] M. Dinca proved the inequality

$$\left(\frac{x+y+z}{3}\right)^3 \geq \left(\sqrt{\frac{xy+yz+zx}{3}}\right)^2 \cdot \sqrt{\frac{x^2+y^2+z^2}{3}} \tag{1.1}$$

where $x, y, z > 0$. From this inequality we will deduce two inequalities for which we will establish several geometric inequalities.

2. MAIN RESULTS

Lemma 1. If x and y are positive real numbers, then

$$\left(\frac{x+y}{2}\right)^4 \geq xy\left(\frac{x^2+y^2}{2}\right) \geq x^2y^2 \tag{2.1}$$

Proof. For $y = xt$, the first inequality of the statement becomes

$$\left(\frac{1+t}{2}\right)^4 \geq t\left(\frac{1+t^2}{2}\right)$$

which means that

$$(t-1)^2 \geq 0,$$

which is true.
Since $\frac{x^2+y^2}{2} \geq xy$, it is easy to see that $xy\left(\frac{x^2+y^2}{2}\right) \geq x^2y^2$.

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Lemma 2. In any triangle ABC, the inequality

$$2R\cos\frac{A}{2}\geq\frac{b+c}{2}, \tag{2.2}$$

holds.

Proof. Since

$$b+c=2R\left(\sin B+\sin C\right)=4R\sin\frac{B+C}{2}\cos\frac{B-C}{2}=$$

$$=4R\cos\frac{A}{2}\cos\frac{B-C}{2}\leq 4R\cos\frac{A}{2},$$

it follows that

$$b+c\leq 4R\cos\frac{A}{2},$$

which implies the statement required.

Lemma 3. In any triangle ABC, we have the inequality

$$R^2\left(b+c\right)^2\cos^2\frac{A}{2}\geq bc\left(\frac{b^2+c^2}{2}\right)\geq b^2c^2, \tag{2.3}$$

and it‘s permutations.

Proof. Making the substitutions $x=a$, $y=b$, $z=c$ in Lemma 1, we deduce that

$$\left(\frac{b+c}{2}\right)^4\geq bc\left(\frac{b^2+c^2}{2}\right)\geq b^2c^2$$

But, by using Lemma 2, we have

$$R^2\cos^2\frac{A}{2}\left(b+c\right)^2=4R^2\cos^2\frac{A}{2}\left(\frac{b+c}{2}\right)^2\geq\left(\frac{b+c}{2}\right)^2\left(\frac{b+c}{2}\right)^2=$$

$$=\left(\frac{b+c}{2}\right)^2\geq bc\left(\frac{b^2+c^2}{2}\right)\geq b^2c^2$$

Theorem 4. There are the following inequalities:

$$\frac{R}{4r}\geq\frac{\sqrt{2\left(a^2+b^2\right)\left(b^2+c^2\right)\left(c^2+a^2\right)}}{\left(a+b\right)\left(b+c\right)\left(c+a\right)}\geq\frac{1}{2}, \tag{2.4}$$

$$\left(\frac{R}{r}\right)^2 \geq \frac{\sqrt{2\left(a^2+b^2\right)\left(b^2+c^2\right)\left(c^2+a^2\right)}}{abc} \geq \frac{1}{4} \tag{2.5}$$

$$4R^2\left(4R+r\right) \geq \sum \sqrt{2bc\left(b^2+c^2\right)} \geq 2\left(p^2+r^2+4Rr\right) \tag{2.6}$$

$$\frac{R}{2r} \geq \sqrt[4]{\frac{m_a m_b m_c}{s_a s_b s_c}} \geq 1 \tag{2.7}$$

$$32R^4 \sum_{cyclic} \cos^4\frac{A}{2} + a^4 + b^4 + c^4 \geq \left(a^3+b^3+c^3\right)\left(a+b+c\right) \tag{2.8}$$

or

$$2R^2(4R+r)^2+2(s^2-r^2-4Rr)^2+16s^2Rr \geq R^2s^2+(s^2+r^2+4Rr)^2+2s^2(s^2-3r^2-6Rr)$$

and

$$\sum_{cyclic} a\cos^4\frac{A}{2} \geq \frac{\triangle}{4R^3}\left(a^2+b^2+c^2\right) \tag{2.9}$$

where s_a, s_b, s_c are the lengths of the symmedians of the triangle ABC.
Proof. Making the product of inequality (2.3) and its permutations, we obtain

$$R^6\left(\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\right)^2\left[\left(a+b\right)\left(b+c\right)\left(c+a\right)\right]^2 \geq$$

$$\geq \frac{1}{8}\left(abc\right)^2\left(a^2+b^2\right)\left(b^2+c^2\right)\left(c^2+a^2\right)$$

But, we know the equalities

$$\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = \frac{s}{4R} \text{ and } abc = 4R\triangle = 4Rsr,$$

which means that the above inequality becomes

$$\frac{R^4s^2}{16}\left[\left(a+b\right)\left(b+c\right)\left(c+a\right)\right]^2 \geq 2R^2s^2r^2\left(a^2+b^2\right)\left(b^2+c^2\right)\left(c^2+a^2\right)$$

Hence we deduce that

$$\frac{R^2}{16r^2} \geq \frac{2\left(a^2+b^2\right)\left(b^2+c^2\right)\left(c^2+a^2\right)}{\left(a+b\right)\left(b+c\right)\left(c+a\right)}$$

Therefore, inequality (2.4) holds.
By using Andrica's inequality [3]

$$\frac{4R}{r} \geq \frac{(a+b)(b+c)(c+a)}{abc}$$

and inequality (2.4), we obtain

$$\left(\frac{R}{r}\right)^2 \geq \frac{\sqrt{2(a^2+b^2)(b^2+c^2)(c^2+a^2)}}{abc}$$

From Lemma 1 and Lemma 2, we have

$$4R^2\cos^2\frac{A}{2} \geq \left(\frac{b+c}{2}\right)^2 \geq \sqrt{bc}\sqrt{\frac{b^2+c^2}{2}} \geq bc,$$

so, making the cyclic sum of these, we obtain

$$4R^2\sum_{cyclic}\cos^2\frac{A}{2} \geq \sum_{cyclic}\sqrt{\frac{bc(b^2+c^2)}{2}} \geq \sum_{cyclic} bc$$

Therefore

$$4R^2\left(4+\frac{r}{R}\right) \geq \frac{1}{2}\sum_{cyclic}\sqrt{2bc(b^2+c^2)} \geq 2\left(p^2+r^2+4Rr\right)$$

it follows that inequality (2.6) holds.
We used the equalities

$$\sum_{cyclic}\cos^2\frac{A}{2} = \frac{1}{2}\left(4+\frac{r}{R}\right) \text{ and } \sum_{cyclic} bc = s^2+r^2+4Rr$$

From [13], we know the equality

$$s_a = \frac{2bcm_a}{b^2+c^2}$$

It follows that,

$$(bc)^2\frac{w_a}{s_a} = bc\left(\frac{b^2+c^2}{2}\right)$$

From Lemma 1, we find the relation

$$\left(\frac{b+c}{2}\right)^4 \geq (bc)^2\frac{m_a}{s_a}, \text{ so}$$

$$\left[\frac{(a+b)(b+c)(c+a)}{8}\right]^4 \geq (abc)^4 \frac{m_a m_b m_c}{s_a s_b s_c} \geq (abc)^4$$

Therefore

$$\left[\frac{(a+b)(b+c)(c+a)}{8abc}\right]^4 \geq \frac{m_a m_b m_c}{s_a s_b s_c} \geq 1$$

By using Andrica's inequality, we deduce the following inequality:

$$\left(\frac{R}{2r}\right)^4 \geq \frac{m_a m_b m_c}{s_a s_b s_c} \geq 1$$

Consequently, inequality (2.7) is true. It is easy to see that

$$16R^4 \cos^4 \frac{A}{2} \geq \left(\frac{b+c}{2}\right)^4 \geq bc\left(\frac{b^2+c^2}{2}\right) \tag{2.10}$$

Hence

$$16R^4 \sum_{cyclic} \cos^4 \frac{A}{2} \geq \frac{1}{2}\left(b^3c + bc^3 + a^3c + ca^3 + a^3b + b^3a\right) =$$

$$= \frac{1}{2}\left[\left(a^3+b^3+c^3\right)(a+b+c) - \left(a^4+b^4+c^4\right)\right],$$

which means that

$$32R^4 \sum_{cyclic} \cos^4 \frac{A}{2} + a^4 + b^4 + c^4 \geq \left(a^3+b^3+c^3\right)(a+b+c)$$

From inequality (2.10), by multiplying with a and making the cyclic sum of these, we deduce that

$$16R^4 \sum a \cos^4 \frac{A}{2} \geq abc\left(a^2+b^2+c^2\right) = 4 \triangle R\left(a^2+b^2+c^2\right),$$

so, we obtain inequality (2.9).

Proposition 5. If $x_1, x_2, ..., x_n$ are real numbers with $x_k > 0$, for all $k \in \{1, 2, ..., n\}$, then there are the following inequalities:

$$\prod_{cyclic} \frac{(x_1+x_2)^4}{x_1^2+x_2^2} \geq 2^{3n}\left(\prod_{k=1}^{n} x_k\right)^2 \tag{2.11}$$

and

$$\sum_{cyclic}\frac{(x_1+x_2)^4}{x_1x_2}\geq 16\sum_{k=1}^{n}x_k^2 \tag{2.12}$$

Proof. By using Lemma 1, it is easy to see that inequalities (2.11) and (2.12) hold.

Proposition 6. In any cyclic polygon $A_1A_2...A_n$ with the lengths of sider $A_1A_2=a_1, A_2A_3=a_2, ..., A_{n-1}A_n=a_{n-1}$ and $A_nA_1=a_n$, we have the inequality

$$2^{\frac{n}{2}}R^n\sin^n\frac{\pi}{n}\geq (a_1a_2...a_n)\frac{\sqrt{\left(a_1^2+a_2^2\right)\left(a_2^2+a_3^2\right)...\left(a_n^2+a_1^2\right)}}{\left(a_1+a_2\right)\left(a_2+a_3\right)...\left(a_n+a_1\right)} \tag{2.13}$$

Proof. In the triangle $A_{k-1}A_kA_{k+1}$, we apply Lemma 3 and we have the inequality

$$2R^2\cos^2\frac{A_k}{2}\left(a_{k-1}+a_k\right)^2\geq a_{k-1}a_k\left(a_{k-1}^2+a_k^2\right),$$

for $k\in\{1,...,n\}$.

Making the cyclic product of these inequalities, we obtain

$$2^nR^{2n}\left(\prod_{cyclic}\cos\frac{A_k}{2}\right)^2\prod_{cyclic}\left(a_{k-1}+a_k\right)^2\geq\left(\prod_{cyclic}a_k\right)^2\left[\prod_{cyclic}\left(a_{k-1}^2+a_k^2\right)\right]$$

From Jensen's inequality, we have the relation

$$\cos\frac{A_1}{2}\cos\frac{A_2}{2}...\cos\frac{A_k}{2}\leq\sin^n\frac{\pi}{n} \tag{2.14}$$

Consequently, we obtain

$$2^{\frac{n}{2}}R^n\sin^n\frac{\pi}{n}\prod_{cyclic}\left(a_{k-1}+a_k\right)\geq\prod_{cyclic}a_k\sqrt{\prod_{cyclic}\left(a_{k-1}^2+a_k^2\right)},$$

from where we deduce the statement.

Corollary 7. In any bicentric quadrilaterals $ABCD$ with the lengths of sides a, b, c and d, the following inequalities:

$$\left(\frac{R^2}{\triangle}\right)^2\geq\frac{\sqrt{\left(a^2+b^2\right)\left(b^2+c^2\right)\left(c^2+d^2\right)\left(d^2+a^2\right)}}{\left(a+b\right)\left(b+c\right)\left(c+d\right)\left(d+a\right)} \tag{2.15}$$

and

$$2R^2 \geq \triangle \tag{2.16}$$

hold, where R is the circumradius and $\triangle$ is the area of the quadrilateral.
Proof. For $n = 4$ in the relation (2.13), we deduce that

$$4R^4 \sin^4 \frac{\pi}{4} \geq abcd \frac{\sqrt{(a^2+b^2)(b^2+c^2)(c^2+d^2)(d^2+a^2)}}{(a+b)(b+c)(c+d)(d+a)}$$

But we know that in any bicentric quadrilateral we have

$$\triangle = \sqrt{abcd}, \text{ so } \triangle^2 = abcd.$$

Therefore

$$R^4 \geq \triangle^2 \frac{\sqrt{(a^2+b^2)(b^2+c^2)(c^2+d^2)(d^2+a^2)}}{(a+b)(b+c)(c+d)(d+a)}$$

it follows that inequality (2.15) holds.
Since $2\left(a^2+b^2\right) \geq (a+b)^2$, we can say that

$$16\left(a^2+b^2\right)\left(b^2+c^2\right)\left(c^2+d^2\right)\left(d^2+a^2\right) \geq [(a+b)(b+c)(c+d)(d+a)]^2,$$

which implies the inequality

$$\frac{\sqrt{(a^2+b^2)(b^2+c^2)(c^2+d^2)(d^2+a^2)}}{(a+b)(b+c)(c+d)(d+a)} \geq \frac{1}{4}.$$

Therefore, using inequality (2.15), we deduce

$$4R^4 \geq \triangle^2 \text{ so } 2R^2 \geq \triangle$$

Corollary 8. In any triangle ABC, there are the inequalities:

$$\frac{s^2\left(s^2+r^2+2Rr\right)^4}{2^9 Rr^2} + 16s^2R^2r^2 \geq$$

$$\geq 2\left(s^2-r^2-4Rr\right)\left[\left(s^2+r^2+4Rr\right)^2 - 16s^2Rr\right] \tag{2.17}$$

and

$$\frac{s^2R^4}{2r^2} + s^2r^2 \geq \left[(4R+r)^2 - 2s^2\right]\left(s^2-8Rr-2r^2\right) \tag{2.18}$$

Proof. For $n = 3$ in inequality (2.11), we deduce

$$[(x_1 + x_2)(x_2 + x_3)(x_3 + x_1)]^4 \geq 2^9 (x_1 x_2 x_3)^2 \left(x_1^2 + x_2^2\right)\left(x_2^2 + x_3^2\right)\left(x_3^2 + x_1^2\right)$$

Taking the substitutions

$$(x_1, x_2, x_3) \in \{(a, b, c), (r_a, r_{b,}, r_c)\}$$

we deduce inequalities (2.17) and (2.18).

Proposition 9. If $x, y, z > 0$, then there is the inequality

$$\left(\frac{x+y+z}{3}\right)^3 \geq (\sqrt[3]{xyz})^2 \sqrt{\frac{x^2+y^2+z^2}{3}} \geq xyz \tag{2.19}$$

Proof. From inequality (1.1) and from the inequalities

$$\sqrt{\frac{xy+yz+zx}{3}} \geq \sqrt[3]{xyz} \text{ and } \sqrt{\frac{x^2+y^2+z^2}{3}} \geq \sqrt[3]{xyz},$$

we deduce the inequality (2.19).

Corollary 8. In any triangle ABC, there are the following inequalities:

$$\left(\frac{2s}{3}\right)^3 \geq \left(\sqrt[3]{4sRr}\right)^2 \sqrt{\frac{2\left(s^2 - r^2 - 4Rr\right)}{3}} \geq 4sRr, \tag{2.20}$$

$$\left(\frac{s}{3}\right)^3 \geq \left(\sqrt[3]{sr^2}\right)^2 \sqrt{\frac{s^2 - 2r^2 - 8Rr}{3}} \geq sr^2, \tag{2.21}$$

$$\left(\frac{s^2+r^2+4Rr}{6R}\right)^3 \geq \left(\sqrt[3]{\frac{2s^2r^2}{R}}\right)^2 \sqrt{\frac{\left(s^2+r^2+4Rr\right)^2 - 16s^2Rr}{3}} \geq \frac{2s^2r^2}{R}, \tag{2.22}$$

$$\left(\frac{4R+r}{3}\right)^3 \geq \left(\sqrt[3]{s^2r}\right)^2 \sqrt{\frac{(4R+r)^2 - 2s^2}{3}} \geq s^2r, \tag{2.23}$$

$$\left(\frac{2R-r}{6R}\right)^3 \geq \left(\sqrt[3]{\frac{r^2}{16R^2}}\right)^2 \sqrt{\frac{8R^2+r^2-s^2}{24R^2}} \geq \frac{r^2}{16R^2} \tag{2.24}$$

and

$$\left(\frac{4R+r}{6R}\right)^3 \geq \left(\sqrt[3]{\frac{s^2}{16R^2}}\right)^2 \sqrt{\frac{(4R+r)^2 - s^2}{24R^2}} \geq \frac{s^2}{16R^2} \tag{2.25}$$

Proof. For $(x, y, z) \in$
$\{(a, b, c), (s-a, s-b, s-c), (h_a, h_b, h_c), (r_a, r_b, r_c), \left(\sin^2\frac{A}{2}, \sin^2\frac{B}{2}, \sin^2\frac{C}{2}\right),$
$\left(\cos^2\frac{A}{2}, \cos^2\frac{B}{2}, \cos^2\frac{C}{2}\right)\}$ in Proposition 9, we obtain the above inequalities.

Proposition 11. If $x, y, z > 0$, then there is the inequality

$$\frac{x+y+z}{3} \geq \sqrt{\frac{xy+yz+zx}{3}} \geq \sqrt[3]{xyz} \tag{2.26}$$

Proof. Since $3\left(x^2+y^2+z^2\right) \geq (xy+yz+zx)^2$, we have

$$\sqrt{\frac{x^2+y^2+z^2}{3}} \geq \frac{xy+yz+zx}{3}$$

and, by using inequality (2.19), we deduce the inequality from the statement.

Corollary 12. In any triangle ABC, there are the following inequalities:

$$\frac{2s}{3} \geq \sqrt{\frac{s^2+r^2+4Rr}{3}} \geq \sqrt[3]{4sRr}, \tag{2.27}$$

$$\frac{s}{3} \geq \sqrt{\frac{s^2-2r^2-8Rr}{3}} \geq \sqrt[3]{sr^2}, \tag{2.28}$$

$$\frac{s^2+r^2+4Rr}{2R} \geq \sqrt{\frac{2s^2r}{3R}} \geq \sqrt[3]{\frac{2s^2r^2}{R}}, \tag{2.29}$$

$$\frac{4R+r}{3} \geq \sqrt{\frac{s^2}{3}} \geq \sqrt[3]{s^2r}, \tag{2.30}$$

$$\frac{2R-r}{6R} \geq \sqrt{\frac{s^2+r^2-8Rr}{48R^2}} \geq \sqrt[3]{\frac{r^2}{16R^2}} \tag{2.31}$$

and

$$\frac{4R+r}{6R} \geq \sqrt{\frac{s^2+(4R+r)^2}{48R^2}} \geq \sqrt[3]{\frac{s^2}{16R^2}} \tag{2.32}$$

Proof. For $(x, y, z) \in$
$\left\{(a, b, c), (s-a, s-b, s-c), (h_a, h_b, h_c), (r_a, r_b, r_c), \left(\sin^2 \frac{A}{2}, \sin^2 \frac{B}{2}, \sin^2 \frac{C}{2}\right),\right.$
$\left.\left(\cos^2 \frac{A}{2}, \cos^2 \frac{B}{2}, \cos^2 \frac{C}{2}\right)\right\}$ in Proposition 11, we obtain the above inequalities.

REFERENCES

[1] Bottema,O., Djordjevic, R.Z., Janic, R.R., Mitrinovic, D.S., and Vasic, P.M., *Geometric inequalities,* Groningen, 1969.
[2] Dincă, M., *O imbunatatire a inegalitatii lui Nesbitt,* Minus, Nr. 1/2009, Targoviste.
[3] Minculete, N., *Egalitati si inegalitati geometrice in triunghi,* Editura Eurocarpatica, Sf. Gheorghe, 2003.
[4] Octogon Mathematical Magazine (1993-2009)

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A refinement of Jensen‘s inequality

Mihály Bencze and Zhao Changjian[18]

ABSTRACT. In this paper we give a refinement and some applications of the inequality $\frac{1+x^n}{2} \geq \left(\frac{1+x}{2}\right)^n$ which is a particular case of the Jensen‘s inequality

MAIN RESULTS

Theorem 1. If $x \geq 0$, $n, k \in N$, $k \leq n$, then
1). $x^k + x^{n-k} \leq 1 + x^n$
2). $n\left(1 + x^{n+1}\right) \geq 2\sum\limits_{k=1}^{n} x^{n+1-k}$
3). $2n\left(1 + x + x^2 + ... + x^{n+1}\right) \geq (n+2)(1+x)\left(1 + x + x^2 + ... + x^n\right)$
4). $\frac{1+x^n}{2} \geq \frac{1+x+x^2+...+x^n}{n+1} \geq \left(\frac{1+x}{2}\right)^n$
Proof.
1). $x^k + x^{n-k} \leq 1 + x^n$ is equivalent with $\left(x^k - 1\right)\left(x^{n-k} - 1\right) \geq 0$, or

$$(x-1)^2\left(x^{k-1} + x^{k-1} + ... + x + 1\right)\left(x^{n-k-1} + x^{n-k-2} + ... + x + 1\right) \geq 0$$

Using 1). we have
2). $n\left(1 + x^{n+1}\right) = \sum\limits_{k=1}^{n}\left(1 + x^{n+1}\right) \geq \sum\limits_{k=1}^{n}\left(x^k + x^{n+1-k}\right) = 2\sum\limits_{k=1}^{n} x^{n+1-k}$
3). The inequality
$2n\left(1 + x + x^2 + ... + x^{n+1}\right) \geq (n+1)(1+x)\left(1 + x + x^2 + ... + x^n\right)$ is equivalent with

$$\frac{1 + x + x^2 + ... + x^{n+1}}{n+2} \geq \left(\frac{1 + x + ... + x^n}{n+1}\right)\left(\frac{1+x}{2}\right).$$

After elementary calculus we get:

$$n\left(1 + x^{n+1}\right) \geq 2\sum_{k=1}^{n} x^{n+1-k}$$

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which result from 2).
Using 1). we get
4). $(n+1)(1+x^n)=\sum\limits_{k=0}^{n}(1+x^n)\ge\sum\limits_{k=0}^{n}\left(x^k+x^{n-k}\right)=2\sum\limits_{k=0}^{n}x^k$ or

$$\frac{1+x^n}{2}\ge\frac{1+x+x^2+...+x^n}{n+1}$$

The inequality $\frac{1+x+x^2+...+x^n}{n+1}\ge\left(\frac{1+x}{2}\right)^n$ we prove by induction.
For $n=1$ we have equality, if $n=2$, then $(x-1)^2\ge 0$, true. We suppose true for n, and we prove for $n+1$. But from 3) we get

$$\frac{1+x+x^2+...+x^n}{n+1}\ge\left(\frac{1+x+...+x^n}{n+1}\right)\left(\frac{1+x}{2}\right)\ge\left(\frac{1+x}{2}\right)^{n+1},$$

therefore is true for all $n\in N^*$.

Remark 1. *The inequality*

$$\frac{1+x^n}{2}\ge\frac{1+x+x^2+...+x^n}{n+1}\ge\left(\frac{1+x}{2}\right)^n$$

is a new refinement of Jensen‘s inequality, for the function $f(x)=x^n$, $n\in N$.
Corollary 1.1. If $x\ge 0$ the

$$\prod_{k=0}^{n}\left(x^k+x^{n-k}\right)\le(1+x^n)^n$$

Proof. From Theorem 1, point 1) we get

$$\prod_{k=0}^{n}\left(x^k+x^{n-k}\right)\le\prod_{k=0}^{n}(1+x^n)=(1+x^n)^n$$

If $x=\frac{a}{b}$ then we have the following.

Remark 2. *If $a,b>0$ then*

$$\prod_{k=0}^{n}\left(a^kb^{n-k}+a^{n-k}b^k\right)\le(a^n+b^n)^n$$

Corollary 1.2. If $a, b > 0$ then

$$\frac{a^n+b^n}{2} \geq \frac{a^n+a^{n-1}b+...+ab^{n-1}+b^n}{n+1} \geq \left(\frac{a+b}{2}\right)^n$$

for all $n \in N$.
Proof. In Theorem 1 we get $x = \frac{a}{b}$.

Remark 3. *If* $n = 2$, *then we obtain a problem of M. Lascu.*

Corollary 1.3. If $f : R \rightarrow R$ is a convex and increasing function and $g : R \rightarrow R$ is a concave and increasing function, then
1). $\frac{f(1)+f(x)+...+f(x^n)}{n+1} \geq f\left(\left(\frac{1+x}{2}\right)^n\right)$
2). $\frac{g(1)+g(x)+...+g(x^n)}{n+1} \leq g\left(\frac{1+x^n}{2}\right)$
for all $x \geq 0$ and $n \in N$.
Proof. From Jensen‘s inequality and from Theorem 1 we get
1). $\frac{f(1)+f(x)+...+f(x^n)}{n+1} \geq f\left(\frac{1+x+...+x^n}{n+1}\right) \geq \left(\left(\frac{1+x}{2}\right)^n\right)$
2). $\frac{g(1)+g(x)+...+g(x^n)}{n+1} \leq g\left(\frac{1+x+...+x^n}{n+1}\right) \leq g\left(\frac{1+x^n}{2}\right)$

Corollary 1.4. If $x, y, z > 0$, then

$$\sum x^2 \geq \frac{2\sum x^2+\sum xy}{3} \geq \frac{\sum x^2+\sum xy}{2} \geq \sum xy$$

Proof. In Corollary 1.2 we take $n = 2$ so we obtain

$$\frac{x^2+y^2}{2} \geq \frac{x^2+xy+y^2}{3} \geq \left(\frac{x+y}{2}\right)^2 = \frac{x^2+2xy+y^2}{4}$$

and

$$\sum x^2 = \sum \frac{x^2+y^2}{2} \geq \sum \frac{x^2+xy+y^2}{3} = \frac{1}{3}\left(2\sum x^2+\sum xy\right) \geq$$

$$\geq \sum \frac{x^2+2xy+y^2}{4} = \frac{\sum x^2+\sum xy}{2} \geq \sum xy$$

Corollary 1.5. In all triangle ABC holds
1). $2\left(s^2-r^2-4Rr\right) \geq \frac{1}{3}\left(5s^2-3r^2-12Rr\right) \geq \frac{1}{2}\left(3s^2-r^2-4Rr\right) \geq$
$\geq s^2+r^2+4Rr$
2). $s^2-2r^2-8Rr \geq \frac{1}{3}\left(2s^2-3r^2-12Rr\right) \geq \frac{1}{2}\left(s^2-r^2-4Rr\right) \geq r(4R+r)$
3). $(4R+r)^2-2s^2 \geq \frac{1}{3}\left(2(4R+r)^2-3s^2\right) \geq \frac{1}{2}\left((4R+r)^2-s^2\right) \geq s^2$

4). $2\left(8R^2+r^2-s^2\right)\geq\frac{1}{3}\left(5r^2-8Rr+40R^2-4s^2\right)\geq$
$\geq\frac{1}{2}\left(3r^2-8Rr+16R^2-2s^2\right)\geq s^2+r^2-8Rr$
5). $2\left((4R+r)^2-s^2\right)\geq\frac{1}{3}\left(5(4R+r)^2-3s^2\right)\geq\frac{1}{2}\left(3(4R+r)^2-s^2\right)\geq$
$\geq s^2+(4R+r)^2$

Proof. In Corollary 1.4 we take
$(x,y,z)\in\left\{(a,b,c),(s-a,s-b,s-c),(r_a,r_b,r_c),\left(\sin^2\frac{A}{2},\sin^2\frac{B}{2},\sin^2\frac{C}{2}\right),\right.$
$\left.\left(\cos^2\frac{A}{2},\cos^2\frac{B}{2},\cos^2\frac{C}{2}\right)\right\}$

Corollary 1.6. If $x,y,z>0$ and $a,b>0$, then
1). $\sum x^2\geq\frac{a\sum x^2+b\sum xy}{a+b}\geq\sum xy$
2). $\sum x^2\geq\left(\left(\sum x^2\right)^a\left(\sum xy\right)^b\right)^{\frac{1}{a+b}}\geq\sum xy$

Corollary 1.7. If $x\geq 0$ and $n\in N, n\geq 2$, then

$$\left(\frac{1+x+x^2+...+x^{n-1}}{n}\right)^{\frac{n}{n-1}}\leq\frac{1+x^n}{2}$$

Proof. The function $f(x)=x^{\frac{n}{n-1}}, x\geq 0$ is convex, therefore from Theorem 1 point 4) we get

$$\left(\frac{1+x+x^2+...+x^{n-1}}{n}\right)^{\frac{n}{n-1}}=f\left(\frac{1+x+x^2+...+x^{n-1}}{n}\right)\leq$$

$$\leq\frac{f(1)+f(x)+...+f(x^n)}{n}=\frac{1+x^{\frac{n}{n-1}}+\left(x^{\frac{n}{n-1}}\right)^2+...+\left(x^{\frac{n}{n-1}}\right)^{n-1}}{n}\leq$$

$$\leq\frac{1+\left(x^{\frac{n}{n-1}}\right)^{n-1}}{2}=\frac{1+x^n}{2}$$

Corollary 1.8. If $f:(0,+\infty)\to R$ where $f(x)=\sqrt[n]{x}$, $n\in N^*$, $n\geq 2$, then exist $c\in\left(\left(\frac{\sqrt[n]{a}+\sqrt[n]{b}}{2}\right);\frac{a+b}{2}\right)$, $0<a\leq b$ such that

$$f(b)-f(a)=(b-a)f'(c)$$

Proof. We have $f(b)-f(a)=(b-a)f'(c)$ or

$$\frac{\sqrt[n]{b}-\sqrt[n]{a}}{b-a}=\frac{1}{n\sqrt[n]{c^{n-1}}}$$

therefore

$$c=\left(\frac{b-a}{n\left(\sqrt[n]{b}-\sqrt[n]{a}\right)}\right)^{\frac{n}{n-1}}=$$

$$=\left(\frac{\sqrt[n]{a^{n-1}-1}+\sqrt[n]{a^{n-2}b}+...+\sqrt[n]{ab^{n-2}}+\sqrt[n]{b^{n-1}}}{n}\right)^{\frac{n}{n-1}}\geq\left(\frac{\sqrt[n]{b}+\sqrt[n]{a}}{2}\right)^{n}$$

which follows from Corollary 1.2 for $a\to\sqrt[n]{a}$, $b\to\sqrt[n]{b}$, $n\to n-1$.
In same way

$$c=\left(\frac{\sqrt[n]{a^{n-1}}+\sqrt[n]{a^{n-2}b}+...+\sqrt[n]{ab^{n-2}}+\sqrt[n]{b^{n-1}}}{n}\right)^{\frac{n}{n-1}}\leq\frac{a+b}{2}$$

which follows from Corollary 1.7 for $x=\sqrt[n]{\frac{b}{a}}$.

Remark 4. *Because* $\left(\frac{\sqrt[n]{a}+\sqrt[n]{b}}{2}\right)^{n}\geq\sqrt{ab}$, *therefore for all* $n\in N$, $n\geq 2$, $c\in\left(\sqrt{ab},\frac{a+b}{2}\right)$.

Open Question 1. If $f:(0,+\infty)\to R$ is convex and $x_k>0$ $(k=1,2,...,n)$, then

1). $\frac{f(x_1)+f(x_2)}{2}\geq\frac{f(x_1)+\sum\limits_{k=1}^{n}f\left(\left(\frac{x_2}{x_1}\right)^{\frac{k}{n}}\right)}{n+1}\geq f\left(\frac{x_1+x_2}{2}\right)$

2). $\frac{1}{n}\sum\limits_{k=1}^{n}f(x_k)\geq\frac{1}{n(n+1)}\left(\sum\limits_{k=1}^{n}f(x_k)+\sum\limits_{cyclic}\sum\limits_{k=1}^{n}f\left(\frac{x_2}{x_1}\right)^{\frac{k}{n}}\right)\geq f\left(\frac{1}{n}\sum\limits_{k=1}^{n}x_k\right)$

Open Question 2. If $f:(0,+\infty)\to R$ is convex and $n-$ time differentiable, then exist $p_k>0$ $(k=1,2,...,n+1)$ such that for all $x\geq 0$ holds

1). $\frac{1+x^n}{2}\geq\frac{\sum\limits_{k=0}^{n}\frac{x^{n-k}}{(n-k)!}}{\sum\limits_{k=0}^{n}\frac{1}{(n-k)!}}\geq\left(\frac{1+x}{2}\right)^{n}$

2). $\frac{f(1)+f(x)}{2}\geq\frac{\sum\limits_{k=1}^{n+1}p_kf^{(k-1)}(x)}{\sum\limits_{k=1}^{n+1}p_k}\geq f\left(\frac{1+x}{2}\right)$

REFERENCES

[1] Octogon Mathematical Magazine, 1999-2009.
[2] Bencze,M., *Inequalities*, (manuscript), 1982.

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Generalizations and analogues of the Nesbitt's inequality

Fuhua Wei and Shanhe Wu[19]

ABSTRACT. The Nesbitt's inequality is generalized by introducing exponent and weight parameters. Several Nesbitt-type inequalities for n variables are provided.

Finally, two analogous forms of Nesbitt's inequality are given.

1. INTRODUCTION

The Nesbitt's inequality states that if x, y, z are positive real numbers, then

$$\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}\geq\frac{3}{2}, \tag{1}$$

the equality occurs if and only if the three variables are equal ([1], see also [2]).
It is well known that this cyclic sum inequality has many applications in the proof of fractional inequalities. In this paper we shall establish some generalizations and analogous forms of the Nesbitt's inequality.

2. GENERALIZATIONS OF THE NESBITT'S INEQUALITY

Theorem 1. Let $x,\ y,\ z,\ k$ be positive real numbers. Then

$$\frac{x}{ky+z}+\frac{y}{kz+x}+\frac{z}{kx+y}\geq\frac{3}{1+k}. \tag{2}$$

Proof. By using the Cauchy-Schwarz inequality (see [3]), we have

$$(kxy+zx+kyz+xy+kxz+yz)\left(\frac{x^2}{kxy+zx}+\frac{y^2}{kyz+xy}+\frac{z^2}{kxz+yz}\right)\geq$$

$$\geq(x+y+z)^2.$$

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Hence

$$\frac{x}{ky+z}+\frac{y}{kz+x}+\frac{z}{kx+y}\geq\frac{(x+y+z)^2}{(1+k)(xy+yz+zx)}=$$

$$=\frac{x^2+y^2+z^2+2xy+2yz+2zx}{(1+k)(xy+yz+zx)}\geq\frac{3}{1+k}.$$

The Theorem 1 is proved.

Theorem 2. Let $x_1, x_2, \ldots, x_n$ be positive real numbers, $n \geq 2$. Then

$$\frac{x_1}{x_2+x_3+\cdots+x_n}+\frac{x_2}{x_1+x_3+x_4+\cdots+x_n}+\cdots+\frac{x_n}{x_1+x_2+\cdots+x_{n-1}}\geq$$

$$\geq\frac{n}{n-1}. \tag{3}$$

Proof. Let $s = x_1 + x_2 + \cdots + x_n$, one has

$$\frac{x_1}{x_2+x_3+\cdots+x_n}+\frac{x_2}{x_1+x_3+x_4+\cdots+x_n}+\cdots+\frac{x_n}{x_1+x_2+\cdots+x_{n-1}}=$$

$$=\frac{x_1}{s-x_1}+\frac{x_2}{s-x_2}+\cdots+\frac{x_n}{s-x_n}.$$

By symmetry, we may assume that $x_1 \geq x_2 \geq \cdots \geq x_n$, then

$$s-x_1\leq s-x_2\leq\cdots\leq s-x_n,\quad \frac{x_1}{s-x_1}\geq\frac{x_2}{s-x_2}\geq\cdots\geq\frac{x_n}{s-x_n}.$$

Using the Chebyshev's inequality (see [3]) gives

$$\frac{x_1}{s-x_1}(s-x_1)+\frac{x_2}{s-x_2}(s-x_2)+\cdots+\frac{x_n}{s-x_n}(s-x_n)$$

$$\leq\frac{1}{n}\left(\frac{x_1}{s-x_1}+\frac{x_2}{s-x_2}+\cdots+\frac{x_n}{s-x_n}\right)[(s-x_1)+(s-x_2)+\cdots+(s-x_n)],$$

or equivalently

$$\frac{x_1}{s-x_1}+\frac{x_2}{s-x_2}+\cdots+\frac{x_n}{s-x_n}\geq\frac{n}{n-1},$$

this is exactly the required inequality.

Theorem 3. Let $x_1, x_2, \ldots, x_n$ be positive real numbers, $n \geq 2,\ k \geq 1$. Then

$$\left(\frac{x_1}{x_2+x_3+\cdots+x_n}\right)^k + \left(\frac{x_2}{x_1+x_3+x_4+\cdots+x_n}\right)^k + \cdots$$

$$+\left(\frac{x_n}{x_1+x_2+\cdots+x_{n-1}}\right)^k \geq \frac{n}{(n-1)^k}. \tag{4}$$

Proof. Using the power mean inequality and the inequality (3), we have

$$\left(\frac{x_1}{x_2+x_3+\cdots+x_n}\right)^k + \left(\frac{x_2}{x_1+x_3+x_4+\cdots+x_n}\right)^k + \cdots$$

$$+\left(\frac{x_n}{x_1+x_2+\cdots+x_{n-1}}\right)^k \geq n^{1-k}\left(\sum_{i=1}^{n}\frac{x_i}{s-x_i}\right)^k \geq \frac{n}{(n-1)^k}.$$

This completes the proof.

Theorem 4. Let $x_1, x_2, \ldots, x_n$ be positive real numbers, and let $\lambda \geq 1,\ r \geq s > 0,\ \sum\limits_{i=1}^{n} x_i^s = p$. Then

$$\sum_{i=1}^{n}\left(\frac{x_i^r}{p-x_i^s}\right)^{\lambda} \geq n^{1-\lambda}\left(\frac{n}{n-1}\right)^{\lambda}\left(\frac{p}{n}\right)^{\lambda(\frac{r}{s}-1)}. \tag{5}$$

Proof. Using the power mean inequality (see [3]), we have

$$\sum_{i=1}^{n}\left(\frac{x_i^r}{p-x_i^s}\right)^{\lambda} \geq n^{1-\lambda}\left(\sum_{i=1}^{n}\frac{x_i^r}{p-x_i^s}\right)^{\lambda}.$$

On the other hand, by symmetry, we may assume that $x_1 \geq x_2 \geq \cdots \geq x_n$, then

$$x_1^s \geq x_2^s \geq \cdots \geq x_n^s > 0, \quad p-x_n^s \geq p-x_{n-1}^s \geq \cdots \geq p-x_1^s > 0.$$

Applying the generalized Radon's inequality (see [4-7])

$$\sum_{i=1}^{n}\frac{a_i^{\alpha}}{b_i} \geq n^{2-\alpha}(\sum_{i=1}^{n}a_i)^{\alpha}/(\sum_{i=1}^{n}b_i)$$

($a_1 \geq a_2 \geq \cdots \geq a_n > 0,\ b_n \geq b_{n-1} \geq \cdots \geq b_1 > 0,\ \alpha \geq 1$), we deduce that

$$\sum_{i=1}^{n}\frac{x_i^r}{p-x_i^s}=\sum_{i=1}^{n}\frac{(x_i^s)^{\frac{r}{s}}}{p-x_i^s}\geq n^{2-\frac{r}{s}}\cdot\frac{(\sum\limits_{i=1}^{n}x_i^s)^{\frac{r}{s}}}{\sum\limits_{i=1}^{n}(p-x_i^s)}=\frac{n}{n-1}\left(\frac{p}{n}\right)^{\frac{r}{s}-1},$$

Therefore

$$\sum_{i=1}^{n}\left(\frac{x_i^r}{p-x_i^s}\right)^{\lambda}\geq n^{1-\lambda}\left(\sum_{i=1}^{n}\frac{x_i^r}{p-x_i^s}\right)^{\lambda}\geq n^{1-\lambda}\left(\frac{n}{n-1}\right)^{\lambda}\left(\frac{p}{n}\right)^{\lambda(\frac{r}{s}-1)}.$$

The proof of Theorem 4 is complete.

In Theorem 4, choosing $\lambda=1$, $s=1$, $n=3$, $x_1=x$, $x_2=y$, $x_3=z$, we get

Theorem 5. Let x, y, z be positive real numbers, and let $x+y+z=p$, $r\geq 1$. Then

$$\frac{x^r}{y+z}+\frac{y^r}{z+x}+\frac{z^r}{x+y}\geq\frac{3}{2}\left(\frac{p}{3}\right)^{r-1}. \tag{6}$$

In particular, when $r=1$, the inequality (6) becomes the Nesbitt's inequality (1).

3. ANALOGOUS FORMS OF THE NESBITT'S INEQUALITY

Theorem 6. Let x,y,z be positive real numbers, Then

$$\sqrt{\frac{x}{x+y}}+\sqrt{\frac{y}{y+z}}+\sqrt{\frac{z}{z+x}}\leq\frac{3\sqrt{2}}{2} \tag{7}$$

Proof. Note that

$$\sqrt{\frac{x}{x+y}}+\sqrt{\frac{y}{y+z}}+\sqrt{\frac{z}{z+x}}=\sqrt{z+x}\sqrt{\frac{x}{(x+y)(z+x)}}+$$
$$+\sqrt{x+y}\sqrt{\frac{y}{(y+z)(x+y)}}+\sqrt{y+z}\sqrt{\frac{z}{(z+x)(y+z)}}.$$

By using the Cauchy-Schwarz inequality, we have

$$\left(\sqrt{z+x}\sqrt{\frac{x}{(x+y)(z+x)}}+\sqrt{x+y}\sqrt{\frac{y}{(y+z)(x+y)}}+\right.$$

$$+\sqrt{y+z}\sqrt{\frac{z}{(z+x)(y+z)}}\Bigg)^2 \leq$$

$$\leq (z+x+x+y+y+z)\left[\frac{x}{(x+y)\,(z+x)}+\frac{y}{(y+z)\,(x+y)}+\frac{z}{(z+x)\,(y+z)}\right].$$

Thus, to prove the inequality (7), it suffices to show that

$$(x+y+z)\left[\frac{x}{(x+y)\,(z+x)}+\frac{y}{(y+z)\,(x+y)}+\frac{z}{(z+x)\,(y+z)}\right]\leq\frac{9}{4}.$$

Direct computation gives

$$(x+y+z)\left[\frac{x}{(x+y)\,(z+x)}+\frac{y}{(y+z)\,(x+y)}+\frac{z}{(z+x)\,(y+z)}\right]-\frac{9}{4}=$$

$$\begin{aligned}
&= \frac{(x+y+z)[x(y+z)+y(z+x)+z(x+y)]}{(x+y)\,(y+z)\,(z+x)}-\frac{9}{4}=\\
&= \frac{4(x+y+z)[x(y+z)+y(z+x)+z(x+y)]-9\,(x+y)\,(y+z)\,(z+x)}{4\,(x+y)\,(y+z)\,(z+x)}=\\
&= \frac{8(x+y+z)(xy+yz+zx)-9\,(x+y)\,(y+z)\,(z+x)}{4\,(x+y)\,(y+z)\,(z+x)}=\\
&= \frac{6xyz-x^2y-x^2z-xy^2-y^2z-xz^2-yz^2}{4\,(x+y)\,(y+z)\,(z+x)}\leq 0,
\end{aligned}$$

where the inequality sign is due to the arithmetic-geometric means inequality. The Theorem 6 is thus proved.

Theorem 7. Let x, y, z be positive real numbers, $\alpha \leq 1/2$, Then

$$\left(\frac{x}{x+y}\right)^{\alpha}+\left(\frac{y}{y+z}\right)^{\alpha}+\left(\frac{z}{z+x}\right)^{\alpha}\leq\frac{3}{2^{\alpha}} \tag{8}$$

Proof. It follows from the power mean inequality that

$$\left(\frac{x}{x+y}\right)^{\alpha}+\left(\frac{y}{y+z}\right)^{\alpha}+\left(\frac{z}{z+x}\right)^{\alpha}$$

$$\leq 3^{1-2\alpha}\left(\sqrt{\frac{x}{x+y}}+\sqrt{\frac{y}{y+z}}+\sqrt{\frac{z}{z+x}}\right)^{2\alpha}\leq 3^{1-2\alpha}\left(\frac{3}{\sqrt{2}}\right)^{2\alpha}=\frac{3}{2^{\alpha}}.$$

The inequality (8) is proved.

Remark. *The inequality (8) is the exponential generalization of inequality (7). As a further generalization of inequality (7), we put forward the following the following conjecture.*

Conjecture. Let $x_1, x_2, \ldots, x_n$ be positive real numbers, $n \geq 2,\ \alpha \leq 1/2$. Then

$$\left(\frac{x_1}{x_1+x_2}\right)^{\alpha}+\left(\frac{x_2}{x_2+x_3}\right)^{\alpha}+\cdots+\left(\frac{x_{n-1}}{x_{n-1}+x_n}\right)^{\alpha}+\left(\frac{x_n}{x_n+x_1}\right)^{\alpha} \leq \frac{n}{2^{\alpha}}. \quad (9)$$

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REFERENCES

[1] Nesbitt, A. M., *Problem 15114*, Educational Times, 3 (1903), 37–38.

[2] Drâmbe, M. O., *Inequalities - Ideas and Methods,* Ed. Gil, Zalău, 2003.

[3] Mitrinović, D. S. and Vasić, P. M., *Analytic Inequalities*, Springer-Verlag, New York, 1970.

[4] Wu, Sh.-H., *An exponential generalization of a Radon inequality,* J. Huaqiao Univ. Nat. Sci. Ed., 24 (1) (2003), 109–112.

[5] Wu, Sh.-H., *A result on extending Radon's inequality and its application*, J. Guizhou Univ. Nat. Sci. Ed., 22 (1) (2004), 1–4.

[6] Wu, Sh.-H., *A new generalization of the Radon inequality*, Math. Practice Theory, 35 (9) (2005), 134–139.

[7] Wu, Sh.-H., *A class of new Radon type inequalities and their applications*, Math. Practice Theory, 36 (3) (2006), 217–224.

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Various proofs of the Cauchy-Schwarz inequality

Hui-Hua Wu and Shanhe Wu[20]

ABSTRACT. In this paper twelve different proofs are given for the classical Cauchy-Schwarz inequality.

1. INTRODUCTION

The Cauchy-Schwarz inequality is an elementary inequality and at the same time a powerful inequality, which can be stated as follows:

Theorem. Let $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ be two sequences of real numbers, then

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \geq \left(\sum_{i=1}^{n} a_i b_i\right)^2, \tag{1}$$

with equality if and only if the sequences $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ are proportional, i.e., there is a constant λ such that $a_k = \lambda b_k$ for each $k \in \{1, 2, \ldots, n\}$.

As is known to us, this classical inequality plays an important role in different branches of modern mathematics including Hilbert spaces theory, probability and statistics, classical real and complex analysis, numerical analysis, qualitative theory of differential equations and their applications (see [1-12]).

In this paper we show some different proofs of the Cauchy-Schwarz inequality.

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2. SOME DIFFERENT PROOFS OF THE CAUCHY-SCHWARZ INEQUALITY

Proof 1. Expanding out the brackets and collecting together identical terms we have

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(a_ib_j-a_jb_i)^2=\sum_{i=1}^{n}a_i^2\sum_{j=1}^{n}b_j^2+\sum_{i=1}^{n}b_i^2\sum_{j=1}^{n}a_j^2-2\sum_{i=1}^{n}a_ib_i\sum_{j=1}^{n}b_ja_j=$$

$$=2\left(\sum_{i=1}^{n}a_i^2\right)\left(\sum_{i=1}^{n}b_i^2\right)-2\left(\sum_{i=1}^{n}a_ib_i\right)^2.$$

Because the left-hand side of the equation is a sum of the squares of real numbers it is greater than or equal to zero, thus

$$\left(\sum_{i=1}^{n}a_i^2\right)\left(\sum_{i=1}^{n}b_i^2\right)\geq\left(\sum_{i=1}^{n}a_ib_i\right)^2.$$

Proof 2. Consider the following quadratic polynomial

$$f(x)=\left(\sum_{i=1}^{n}a_i^2\right)x^2-2\left(\sum_{i=1}^{n}a_ib_i\right)x+\sum_{i=1}^{n}b_i^2=\sum_{i=1}^{n}(a_ix-b_i)^2.$$

Since $f(x)\geq 0$ for any $x\in\mathbb{R}$, it follows that the discriminant of $f(x)$ is negative, i.e.,

$$\left(\sum_{i=1}^{n}a_ib_i\right)^2-\left(\sum_{i=1}^{n}a_i^2\right)\left(\sum_{i=1}^{n}b_i^2\right)\leq 0.$$

The inequality (1) is proved.

Proof 3. When $\sum\limits_{i=1}^{n}a_i^2=0$ or $\sum\limits_{i=1}^{n}b_i^2=0$, (1) is an identity.
We can now assume that

$$A_n=\sum_{i=1}^{n}a_i^2\neq 0,\quad B_n=\sum_{i=1}^{n}b_i^2\neq 0,\quad x_i=\frac{a_i}{\sqrt{A_n}},\quad y_i=\frac{b_i}{\sqrt{B_n}}\quad (i=1,2,\ldots,n),$$

then

$$\sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} y_i^2 = 1.$$

The inequality (1) is equivalent to

$$x_1y_1 + x_2y_2 + \cdots x_ny_n \leq 1,$$

that is

$$2(x_1y_1 + x_2y_2 + \cdots x_ny_n) \leq x_1^2 + x_2^2 + \cdots + x_n^2 + y_1^2 + y_2^2 + \cdots + y_n^2,$$

or equivalently

$$(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2 \geq 0,$$

which is evidently true. The desired conclusion follows.

Proof 4. Let $A = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$, $B = \sqrt{b_1^2 + b_2^2 + \cdots + b_n^2}$. By the arithmetic-geometric means inequality, we have

$$\sum_{i=1}^{n} \frac{a_ib_i}{AB} \leq \sum_{i=1}^{n} \frac{1}{2}\left(\frac{a_i^2}{A^2} + \frac{b_i^2}{B^2}\right) = 1,$$

so that

$$\sum_{i=1}^{n} a_ib_i \leq AB = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}\sqrt{b_1^2 + b_2^2 + \cdots + b_n^2}.$$

Thus

$$\left(\sum_{i=1}^{n} a_ib_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right).$$

Proof 5. Let
$A_n = a_1^2 + a_2^2 + \cdots + a_n^2$, $B_n = a_1b_1 + a_2b_2 + \cdots + a_nb_n$, $C_n = b_1^2 + b_2^2 + \cdots + b_n^2$.
It follows from the arithmetic-geometric means inequality that

$$\frac{A_nC_n}{B_n^2} + 1 = \sum_{i=1}^{n} \frac{a_i^2C_n}{B_n^2} + \sum_{i=1}^{n} \frac{b_i^2}{C_n} = \sum_{i=1}^{n}\left(\frac{a_i^2C_n}{B_n^2} + \frac{b_i^2}{C_n}\right) \geq 2\sum_{i=1}^{n} \frac{a_ib_i}{B_n} = 2,$$

therefore

$$A_nC_n \geq B_n^2,$$

that is

$$(a_1^2+a_2^2+\cdots+a_n^2)(b_1^2+b_2^2+\cdots+b_n^2) \geq (a_1b_1+a_2b_2+\cdots+a_nb_n)^2.$$

Proof 6. Below, we prove the Cauchy-Schwarz inequality by mathematical induction.
Beginning the induction at 1, the $n = 1$ case is trivial.
Note that

$$(a_1b_1+a_2b_2)^2 = a_1^2b_1^2+2a_1b_1a_2b_2+a_2^2b_2^2 \leq a_1^2b_1^2+a_1^2b_2^2+a_2^2b_1^2+a_2^2b_2^2 =$$

$$= (a_1^2+a_2^2)(b_1^2+b_2^2),$$

which implies that the inequality (1) holds for $n = 2$.
Assume that the inequality (1) holds for an arbitrary integer k, i.e.,

$$\left(\sum_{i=1}^{k} a_ib_i\right)^2 \leq \left(\sum_{i=1}^{k} a_i^2\right)\left(\sum_{i=1}^{k} b_i^2\right).$$

Using the induction hypothesis, one has

$$\sqrt{\sum_{i=1}^{k+1} a_i^2}\cdot\sqrt{\sum_{i=1}^{k+1} b_i^2} = \sqrt{\sum_{i=1}^{k} a_i^2 + a_{k+1}^2}\cdot\sqrt{\sum_{i=1}^{k} b_i^2 + b_{k+1}^2} \geq$$

$$\geq \sqrt{\sum_{i=1}^{k} a_i^2}\cdot\sqrt{\sum_{i=1}^{k} b_i^2} + |a_{k+1}b_{k+1}| \geq \sum_{i=1}^{k}|a_ib_i| + |a_{k+1}b_{k+1}| = \sum_{i=1}^{k+1}|a_ib_i|.$$

It means that the inequality (1) holds for $n = k+1$, we thus conclude that the inequality (1) holds for all natural numbers n. This completes the proof of inequality (1).

Proof 7. Let

$$\begin{aligned} A &= \{a_1b_1,\cdots a_1b_n, a_2b_1,\cdots, a_2b_n,\cdots, a_nb_1,\cdots a_nb_n\}\\ B &= \{a_1b_1,\cdots a_1b_n, a_2b_1,\cdots, a_2b_n,\cdots, a_nb_1,\cdots a_nb_n\}\end{aligned}$$

$$C = \{a_1b_1, \cdots a_1b_n, a_2b_1, \cdots, a_2b_n, \cdots, a_nb_1, \cdots a_nb_n\}$$
$$D = \{a_1b_1, \cdots a_nb_1, a_1b_2, \cdots, a_nb_2, \cdots, a_1b_n, \cdots a_nb_n\}$$

It is easy to observe that the set A and B are similarly sorted, while the set C and D are mixed sorted.

Applying the rearrangement inequality, we have

$$(a_1b_1)(a_1b_1) + \cdots + (a_1b_n)(a_1b_n) + (a_2b_1)(a_2b_1) + \cdots + (a_2b_n)(a_2b_n) + \cdots$$

$$+(a_nb_1)(a_nb_1) + \cdots + (a_nb_n)(a_nb_n) \geq (a_1b_1)(a_1b_1) + \cdots + (a_1b_n)(a_nb_1) +$$

$$+(a_2b_1)(a_1b_2) + \cdots + (a_2b_n)(a_nb_2) + \cdots + (a_nb_1)(a_1b_n) + \cdots + (a_nb_n)(a_nb_n),$$

which can be simplified to the inequality

$$(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) \geq (a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2$$

as desired.

Proof 8. By the arithmetic-geometric means inequality, one has for $\lambda > 0$,

$$|a_ib_i| \leq \frac{1}{2}\left(\lambda a_i^2 + \frac{b_i^2}{\lambda}\right).$$

Choosing $\lambda = \sqrt{\sum\limits_{i=1}^{n} b_i^2 \Big/ \sum\limits_{i=1}^{n} a_i^2}$ in the above inequality gives

$$|a_ib_i| \leq \left[\sqrt{\frac{\sum\limits_{i=1}^{n} b_i^2}{\sum\limits_{i=1}^{n} a_i^2}}\, a_i^2 + \sqrt{\frac{\sum\limits_{i=1}^{n} a_i^2}{\sum\limits_{i=1}^{n} b_i^2}}\, b_i^2\right].$$

Hence

$$\sum_{i=1}^{n} |a_ib_i| \leq \frac{1}{2}\left[\sqrt{\frac{\sum\limits_{i=1}^{n} b_i^2}{\sum\limits_{i=1}^{n} a_i^2}} \sum_{i=1}^{n} a_i^2 + \sqrt{\frac{\sum\limits_{i=1}^{n} a_i^2}{\sum\limits_{i=1}^{n} b_i^2}} \sum_{i=1}^{n} b_i^2\right],$$

or equivalently

$$\sum_{i=1}^{n}|a_ib_i| \le \frac{1}{2}\left(\sqrt{\sum_{i=1}^{n}b_i^2\sum_{i=1}^{n}a_i^2}+\sqrt{\sum_{i=1}^{n}a_i^2\sum_{i=1}^{n}b_i^2}\right) = \sqrt{\sum_{i=1}^{n}a_i^2}\cdot\sqrt{\sum_{i=1}^{n}b_i^2}.$$

The desired conclusion follows.

Proof 9. Construct the vectors $\alpha = (a_1, a_2, \cdots, a_n)$, $\beta = (b_1, b_2, \cdots, b_n)$. Then for arbitrary real numbers t, one has the following identities for scalar product:

$$(\alpha + t\beta)\cdot(\alpha + t\beta) = \alpha\cdot\alpha + 2(\alpha\cdot\beta)t + (\beta\cdot\beta)t^2 \Longleftrightarrow |\alpha|^2 + 2(\alpha\cdot\beta)t + |\beta|^2 t^2 =$$

$$= |\alpha + t\beta|^2 \ge 0.$$

Thus

$$(\alpha\cdot\beta)^2 - |\alpha|^2|\beta|^2 \le 0.$$

Using the expressions

$$\alpha\cdot\beta = a_1b_1 + a_2b_2 + \cdots + a_nb_n, \quad |\alpha|^2 = \sum_{i=1}^{n}a_i^2, \quad |\beta|^2 = \sum_{i=1}^{n}b_i^2,$$

we obtain

$$\left(\sum_{i=1}^{n}a_ib_i\right)^2 - \left(\sum_{i=1}^{n}a_i^2\right)\left(\sum_{i=1}^{n}b_i^2\right) \le 0.$$

Proof 10. Construct the vectors $\alpha = (a_1, a_2, \cdots, a_n)$, $\beta = (b_1, b_2, \cdots, b_n)$. From the formula for scalar product:

$$\alpha\cdot\beta = |\alpha||\beta|\cos(\alpha,\beta),$$

we deduce that

$$\alpha\cdot\beta \le |\alpha||\beta|.$$

Using the expressions

$$\alpha \cdot \beta = a_1b_1 + a_2b_2 + \cdots + a_nb_n, \quad |\alpha|^2 = \sum_{i=1}^{n} a_i^2, \quad |\beta|^2 = \sum_{i=1}^{n} b_i^2,$$

we get the desired inequality (1).

Proof 11. Since the function $f(x) = x^2$ is convex on $(-\infty, +\infty)$, it follows from the Jensen's inequality that

$$(p_1x_1 + p_2x_2 + \cdots + p_nx_n)^2 \leq p_1x_1^2 + p_2x_2^2 + \cdots + p_nx_n^2, \tag{2}$$

where $x_i \in R$, $p_i > 0$ $(i = 1, 2, \ldots, n)$, $p_1 + p_2 + \cdots + p_n = 1$.

Case I. If $b_i \neq 0$ for $i = 1, 2, \ldots, n$, we apply $x_i = a_i/b_i$ and $p_i = b_i^2/(b_1^2 + b_2^2 + \cdots + b_n^2)$ to the inequality (2) to obtain that

$$\left(\frac{a_1b_1 + a_2b_2 + \cdots + a_nb_n}{b_1^2 + b_2^2 + \cdots + b_n^2}\right)^2 \leq \frac{a_1^2 + a_2^2 + \cdots + a_n^2}{b_1^2 + b_2^2 + \cdots + b_n^2},$$

that is

$$(a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2 \leq (a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2).$$

Case II. If there exists $b_{i_1} = b_{i_2} = \cdots = b_{i_k} = 0$, one has

$$\left(\sum_{i=1}^{n} a_ib_i\right)^2 = \left(\sum_{i \neq i_1, \ldots, i_k, 1 \leq i \leq n} a_ib_i\right)^2 \leq$$

$$\leq \left(\sum_{i \neq i_1, \ldots, i_k, 1 \leq i \leq n} a_i^2\right)\left(\sum_{i \neq i_1, \ldots, i_k, 1 \leq i \leq n} b_i^2\right) \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right).$$

This completes the proof of inequality (1).

Proof 12. Define a sequence $\{S_n\}$ by

$$S_n = (a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2 - \left(a_1^2 + a_2^2 + \cdots + a_n^2\right)\left(b_1^2 + b_2^2 + \cdots + b_n^2\right).$$

Then

$$S_{n+1}-S_n=\left(a_1b_1+a_2b_2+\cdots+a_{n+1}b_{n+1}\right)^2-\left(a_1^2+a_2^2+\cdots+a_{n+1}^2\right)\cdot$$

$$\cdot\left(b_1^2+b_2^2+\cdots+b_{n+1}^2\right)-\left(a_1b_1+a_2b_2+\cdots+a_nb_n\right)^2+\left(a_1^2+a_2^2+\cdots+a_n^2\right)\cdot$$

$$\cdot\left(b_1^2+b_2^2+\cdots+b_n^2\right),$$

which can be simplified to

$$S_{n+1}-S_n=$$

$$=-\left[\left(a_1b_{n+1}-b_1a_{n+1}\right)^2+\left(a_2b_{n+1}-b_2a_{n+1}\right)^2+\cdots+\left(a_nb_{n+1}-b_na_{n+1}\right)^2\right],$$

so

$$S_{n+1}\le S_n\quad(n\in N).$$

We thus have

$$S_n\le S_{n-1}\le\cdots\le S_1=0,$$

which implies the inequality (1).

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REFERENCES

[1] Dragomir, S. S., *Discrete inequalities of the Cauchy-Bunyakovsky-Schwarz type*, Nova Science Publishers, Inc., Hauppauge, NY, 2004.

[2] Mitrinović, D. S., Pečarić, J. E., Fink, A. M., *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht, 1993.

[3] Masjed-Jamei, M., Dragomir, S. S., Srivastava, H.M., *Some generalizations of the Cauchy-Schwarz and the Cauchy-Bunyakovsky inequalities involving four free parameters and their applications*, RGMIA Res. Rep. Coll., 11 (3) (2008), Article 3, pp.1–12 (electronic).

[4] Barnett, N. S., Dragomir, S. S., *An additive reverse of the Cauchy–Bunyakovsky–Schwarz integral inequality*, Appl. Math. Lett., 21 (4) (2008), 388–393.
[5] Lee, E. Y., *A matrix reverse Cauchy–Schwarz inequality*, Linear Algeb. Appl., 430 (2) (2009), 805–810.
[6] Dragomir, S. S., *A survey on Cauchy–Bunyakovsky–Schwarz type discrete inequalities*, J. Inequal. Pure Appl. Math., 4 (3) (2003), Article 63, pp.1–142 (electronic).
[7] Dragomir, S. S., *On the Cauchy–Buniakowsky–Schwarz inequality for sequences in inner product spaces,* Math. Inequal. Appl., 3 (2000), 385–398.
[8] De Rossi, A., Rodino, L., *Strengthened Cauchy–Schwarz inequality for biorthogonal wavelets in Sobolev spaces*, J. Math. Anal. Appl., 299 (1) (2004), 49–60.
[9] Liu, Z., *Remark on a Refinement of the Cauchy–Schwarz inequality*, J. Math. Anal. Appl., 218 (1) (1998), 13–21.
[10] Alzer, H., *On the Cauchy-Schwarz inequality*, J. Math. Anal. Appl., 234 (1) (1999), 6–14.
[11] Alzer, H., *A refinement of the Cauchy-Schwartz inequality,* J. Math. Anal. Appl., 168 (2) (1992), 596–604.
[12] Steiger, W. L., *On a generalization of the Cauchy–Schwarz inequality*, Amer. Math. Monthly, 76 (1969), 815–816.

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About a trigonometrical inequality

Mihály Bencze[21]

ABSTRACT. In this paper we present a trigonometrical inequality and after then we give some applications.

MAIN RESULTS

Theorem 1. If $y_k \in \left(0, \frac{\pi}{2}\right]$ $(k = 1, 2, ..., n)$, then

$$\sum_{k=1}^{n} \frac{1}{\sqrt{1 + tg^2 y_k}} \leq \frac{n}{\sqrt{1 + tg^2\left(\frac{1}{n}\sum\limits_{k=1}^{n} y_k\right)}}$$

Proof. The function $f(x) = \cos x$ is concave for $x \in \left(0, \frac{\pi}{2}\right)$, therefore from Jensen's inequality we get:

$$\sum_{k=1}^{n} \cos y_k \leq n \cos\left(\frac{1}{n}\sum_{k=1}^{n} y_k\right) \text{ or}$$

$$\sum_{k=1}^{n} \frac{1}{\sqrt{1 + tg^2 y_k}} \leq \frac{n}{\sqrt{1 + tg^2\left(\frac{1}{n}\sum\limits_{k=1}^{n} y_k\right)}}$$

Corollary 1.1. If $x_k > 0$ $(k = 1, 2, ..., n)$, then

$$\sum_{k=1}^{n} \frac{1}{\sqrt{1 + x_k^2}} \leq \frac{n}{\sqrt{1 + tg^2\left(\frac{1}{n}\sum\limits_{k=1}^{n} arctg x_k\right)}}$$

Proof. In Theorem 1 we take $y_k = arctg x_k$ $(k = 1, 2, ..., n)$.

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Corollary 1.2. If $x_k \in (0,1]$ $(k=1,2,...,n)$, then

$$\sum_{k=1}^{n} \frac{1}{\sqrt{1+x_k^2}} \le \frac{n}{\sqrt{1+\left(\prod\limits_{k=1}^{n} x_k\right)^{\frac{2}{n}}}}$$

Proof. If $x_k \in (0,1]$, then $y_k \in \left(0,\frac{\pi}{4}\right]$ $(k=1,2,...,n)$ and $g(x)=\ln tgx$ is concave, therefore from Jensen's inequality we get

$$\prod_{k=1}^{n} tgy_k \le tg^n\left(\frac{1}{n}\sum_{k=1}^{n} y_k\right)$$

and from Theorem 1 we get

$$\sum_{k=1}^{n} \frac{1}{\sqrt{1+tg^2 y_k}} \le \frac{n}{\sqrt{1+tg^2\left(\frac{1}{n}\sum\limits_{k=1}^{n} y_k\right)}} \le \frac{n}{\sqrt{1+\left(\prod\limits_{k=1}^{n} tgy_k\right)^{\frac{2}{n}}}}$$

after then yields $y_k = arctgx_k$ $(k=1,2,...,n)$ and we are finish the proof.

Corollary 1.3. If $x,y>0$ and $xy \le 1$, then

$$\frac{1}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+y^2}} \le \frac{2}{\sqrt{1+xy}}$$

Proof. If $y_1+y_2 \le \frac{\pi}{2}$, then $tgy_1 tgy_2 \le tg^2\left(\frac{y_1+y_2}{2}\right)$, therefore from Corollary 1.2 we obtain the result.

Corollary 1.4. If $x_k \in [0,1]$ $(k=1,2,...,n)$, then

$$\sum_{k=1}^{n} \frac{1}{\sqrt{1+x_k^2}} \le \sum_{cyclic} \frac{1}{\sqrt{1+x_1x_2}}$$

Proof. Using the Corollary 1.3 we obtain

$$\begin{cases} \frac{1}{\sqrt{1+x_1^2}} + \frac{1}{\sqrt{1+x_2^2}} \le \frac{2}{\sqrt{1+x_1x_2}} \\ ------------- \\ \frac{1}{\sqrt{1+x_2^2}} + \frac{1}{\sqrt{1+x_3^2}} \le \frac{2}{\sqrt{1+x_2x_3}} \\ \frac{1}{\sqrt{1+x_n^2}} + \frac{1}{\sqrt{1+x_1^2}} \le \frac{2}{\sqrt{1+x_nx_1}} \end{cases}$$

After addition we finish the proof.

Corollary 1.5. If $x_k \in [0,1]$ $(k = 1, 2, ..., n)$, then

$$\sum_{k=1}^{2^n} \frac{1}{\sqrt{1+x_k^2}} \leq \frac{2^n}{\sqrt{1+\left(\prod\limits_{k=1}^{2^n} x_k\right)^{2^{1-n}}}}$$

Proof. Using iterative the Corollary 1.3 we get

$$\sum_{k=1}^{2^n} \frac{1}{\sqrt{1+x_k^2}} = \left(\frac{1}{\sqrt{1+x_1^2}} + \frac{1}{\sqrt{1+x_2^2}}\right) + \left(\frac{1}{\sqrt{1+x_3^2}} + \frac{1}{\sqrt{1+x_4^2}}\right) + ...$$

$$\leq \frac{2}{\sqrt{1+x_1x_2}} + \frac{2}{\sqrt{1+x_3x_4}} + ... \leq$$

$$\leq \frac{4}{\sqrt{1+\sqrt{x_1x_2x_3x_4}}} + ... \leq \frac{2^n}{\sqrt{1+\left(\prod\limits_{k=1}^{2^n} x_k\right)^{2^{1-n}}}}$$

Corollary 1.6. In all triangle ABC holds

$$\sum \cos\frac{A}{2} \leq \sum \frac{1}{\sqrt{1+tg\frac{A}{2}tg\frac{B}{2}}}$$

Proof. In Corollary 1.4 we take $n = 3$, $x_1 = tg\frac{A}{2}$, $x_2 = tg\frac{B}{2}$, $x_3 = tg\frac{C}{2}$.

Corollary 1.7. If $x_k \in \left(0, \ln\left(1+\sqrt{2}\right)\right]$ $(k = 1, 2, ..., n)$, then

$$\sum_{k=1}^{n} \frac{1}{chx_k} \leq \sum_{cyclic} \frac{1}{\sqrt{1+shx_1shx_2}}$$

Proof. In Corollary 1.4 we take $x_k \to shx_k$ $(k = 1, 2, ..., n)$.

Theorem 2. If $x, y, z > 0$ and $xyz \leq 1$, then

$$\max\left\{\frac{2}{\sqrt{1+xy}} + \frac{1}{\sqrt{1+z^2}}; \frac{2}{\sqrt{1+yz}} + \frac{1}{\sqrt{1+x^2}}; \frac{2}{\sqrt{1+zx}} + \frac{1}{\sqrt{1+y^2}}\right\} \leq$$

$$\leq \frac{3}{\sqrt{1+\left(\sqrt[3]{xyz}\right)^2}}$$

Proof. If $u = \sqrt[3]{xyz}$ and $f\left(z\right) = \frac{2}{\sqrt{1+xy}} + \frac{1}{\sqrt{1+z^2}} = \frac{1}{\sqrt{1+z^2}} + \frac{2\sqrt{z}}{\sqrt{z+u^3}}$, then $f\left({}'z\right) = \left(z-u\right)\left(\left(1-u^2\right)z + r\right) = 0$ if and only if $z = u$ but $\lim\limits_{z\searrow 0} f\left(x\right) = 1$ and $f\left(u\right) = \frac{3}{\sqrt{1+u^2}} > 2$, therefore $f\left(z\right) \le f\left(u\right) = \frac{3}{\sqrt{1+u^2}}$ or
$\frac{2}{\sqrt{1+xy}} + \frac{1}{\sqrt{1+z^2}} \le \frac{3}{\sqrt{1+\left(\sqrt[3]{xyz}\right)^2}}$.

Corollary 2.1. In all triangle ABC holds:

$$\max\left\{\frac{2}{\sqrt{1+tg\frac{A}{2}tg\frac{B}{2}}} + \cos\frac{C}{2}; \frac{2}{\sqrt{1+tg\frac{B}{2}tg\frac{C}{2}}} + \cos\frac{A}{2}; \frac{2}{\sqrt{1+tg\frac{C}{2}tg\frac{A}{2}}} + \cos\frac{B}{2}\right\} \le$$

$$\le \frac{3\sqrt[3]{s}}{\sqrt{\sqrt[3]{s^2} + \sqrt[3]{r^2}}}$$

Proof. In Theorem 2 we take $x = tg\frac{A}{2}, y = tg\frac{B}{2}, z = tg\frac{C}{2}$ etc.

Corollary 2.2. If $x, y, z > 0$ and $xyz \le 1$, then

$$\frac{1}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+y^2}} + \frac{1}{\sqrt{1+z^2}} \le \frac{3}{\sqrt{1+\left(\sqrt[3]{xyz}\right)^2}}$$

(see [1])
Proof. From Theorem 1 and Theorem 2 we have:

$$\frac{1}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+y^2}} + \frac{1}{\sqrt{1+z^2}} \le \frac{2}{\sqrt{1+xy}} + \frac{1}{\sqrt{1+z^2}} \le \frac{3}{\sqrt{1+\left(\sqrt[3]{xyz}\right)^2}}$$

Corollary 2.3. If $x, y, z > 0$ and $xyz \le 1$, then

$$\frac{1}{\sqrt{1+x^6}} + \frac{1}{\sqrt{1+y^6}} + \frac{1}{\sqrt{1+z^6}} \le \frac{3}{\sqrt{1+x^2y^2z^2}}$$

Proof. In Corollary 2.2 we take $x \to x^3, y \to y^3, z \to z^3$.

Corollary 2.4. In all triangle ABC holds
1). $\sum \frac{1}{\sqrt{1+\sin^2 A}} \le \frac{3\sqrt[3]{2R^2}}{\sqrt{\sqrt[3]{sr} + \sqrt[3]{2R^2}}}$ 2). $\sum \cos\frac{A}{2} \le \frac{3\sqrt[3]{s}}{\sqrt{\sqrt[3]{s^2} + \sqrt[3]{r^2}}}$
3). $\sum \frac{1}{\sqrt{1+\sin^4\frac{A}{2}}} \le \frac{3\sqrt[3]{16R^2}}{\sqrt{\sqrt[3]{r^2} + \sqrt[3]{16R^2}}}$ 4). $\sum \frac{1}{\sqrt{1+\cos^4\frac{A}{2}}} \le \frac{3\sqrt[3]{16R^2}}{\sqrt{\sqrt[3]{s^2} + \sqrt[3]{16R^2}}}$

Proof. In Corollary 2.2 we take
$(x,y,z)\in\left\{(\sin A,\sin B,\sin C)\,;\left(tg\frac{A}{2},tg\frac{B}{2},tg\frac{C}{2}\right);\left(\sin^2\frac{A}{2},\sin^2\frac{B}{2},\sin^2\frac{C}{2}\right);\right.$
$\left.\left(\cos^2\frac{A}{2},\cos^2\frac{B}{2},\cos^2\frac{C}{2}\right)\right\}.$

Corollary 2.5. In all triangle ABC holds
1). $\sum\frac{1}{\sqrt{1+\sin^6 A}}\le\frac{6R^2}{\sqrt{s^2r^2+4R^4}}$ 2). $\sum\frac{1}{\sqrt{1+tg^6\frac{A}{2}}}\le\frac{6s^2}{\sqrt{s^2+r^2}}$
3). $\sum\frac{1}{\sqrt{1+\sin^{12}\frac{A}{2}}}\le\frac{48R^2}{\sqrt{r^4+256R^4}}$ 4). $\sum\frac{1}{\sqrt{1+\cos^{12}\frac{A}{2}}}\le\frac{48R^2}{\sqrt{s^4+256R^4}}$

Proof. In Corollary 2.3 we take
$(x,y,z)\in\left\{(\sin A,\sin B,\sin C)\,;\left(tg\frac{A}{2},tg\frac{B}{2},tg\frac{C}{2}\right);\left(\sin^2\frac{A}{2},\sin^2\frac{B}{2},\sin^2\frac{C}{2}\right);\right.$
$\left.\left(\cos^2\frac{A}{2},\cos^2\frac{B}{2},\cos^2\frac{C}{2}\right)\right\}$

Corollary 2.6. If $x,y,z>0$ and $xyz\le 1$, then
1). $2\sum\frac{1}{\sqrt{1+xy}}+\sum\frac{1}{\sqrt{1+x^2}}\le\frac{9}{\sqrt{1+\left(\sqrt[3]{xyz}\right)^2}}$
2). $2\sum\frac{1}{\sqrt{1+x^3y^3}}+\sum\frac{1}{\sqrt{1+x^6}}\le\frac{9}{\sqrt{1+x^2y^2z^2}}$

Proof. 1). Using the Theorem 2 we get:

$$2\sum\frac{1}{\sqrt{1+xy}}+\sum\frac{1}{\sqrt{1+x^2}}=\sum\left(\frac{2}{\sqrt{1+xy}}+\frac{1}{\sqrt{1+z^2}}\right)\le$$
$$\le\sum\frac{3}{\sqrt{1+\left(\sqrt[3]{xyz}\right)^2}}=\frac{9}{\sqrt{1+\left(\sqrt[3]{xyz}\right)^2}}$$

2). In 1) we take $x\to x^3,y\to y^3,z\to z^3$

Corollary 2.7. In all triangle ABC holds
1). $2\sum\frac{1}{\sqrt{1+\sin A\sin B}}+\sum\frac{1}{\sqrt{1+\sin^2 A}}\le\frac{9\sqrt[3]{2R^2}}{\sqrt{\sqrt[3]{sr}+\sqrt[3]{2R^2}}}$
2). $2\sum\frac{1}{\sqrt{1+tg\frac{A}{2}tg\frac{B}{2}}}+\sum\cos\frac{A}{2}\le\frac{9\sqrt[3]{s}}{\sqrt{\sqrt[3]{s^2}+\sqrt[3]{r^2}}}$
3). $2\sum\frac{1}{\sqrt{1+\sin^2\frac{A}{2}\sin^2\frac{B}{2}}}+\sum\frac{1}{\sqrt{1+\sin^4\frac{A}{2}}}\le\frac{9\sqrt[3]{16R^2}}{\sqrt{\sqrt[3]{r^2}+\sqrt[3]{16R^2}}}$
4). $2\sum\frac{1}{\sqrt{1+\cos^2\frac{A}{2}\cos^2\frac{B}{2}}}+\sum\frac{1}{\sqrt{1+\cos^4\frac{A}{2}}}\le\frac{9\sqrt[3]{16R^2}}{\sqrt{\sqrt[3]{s^2}+\sqrt[3]{16R^2}}}$

Proof. In Corollary 2.6 1). we take
$(x,y,z)\in\left\{(\sin A,\sin B,\sin C)\,;\left(tg\frac{A}{2},tg\frac{B}{2},tg\frac{C}{2}\right);\left(\sin^2\frac{A}{2},\sin^2\frac{B}{2},\sin^2\frac{C}{2}\right);\right.$
$\left.\left(\cos^2\frac{A}{2},\cos^2\frac{B}{2},\cos^2\frac{C}{2}\right)\right\}$

Corollary 2.8. In all triangle ABC holds
1). $2\sum\frac{1}{\sqrt{1+\sin^3 A\sin^3 B}}+\sum\frac{1}{\sqrt{1+\sin^6 A}}\le\frac{18R^2}{\sqrt{s^2r^2+4R^4}}$

2). $2\sum\frac{1}{\sqrt{1+tg^3\frac{A}{2}tg^3\frac{B}{2}}}+\sum\frac{1}{\sqrt{1+tg^6\frac{A}{2}}}\le\frac{18s^2}{\sqrt{s^2+r^2}}$

3). $2\sum\frac{1}{\sqrt{1+\sin^6\frac{A}{2}\sin^6\frac{B}{2}}}+\sum\frac{1}{\sqrt{1+\sin^{12}\frac{A}{2}}}\le\frac{144R^2}{\sqrt{r^4+256R^4}}$

4). $2\sum\frac{1}{\sqrt{1+\cos^6\frac{A}{2}\cos^6\frac{B}{2}}}+\sum\frac{1}{\sqrt{1+\cos^{12}\frac{A}{2}}}\le\frac{144R^2}{\sqrt{s^4+256R^4}}$

Proof. In Corollary 2.6 1). we take
$(x,y,z)\in\left\{(\sin A,\sin B,\sin C)\,;\left(tg\frac{A}{2},tg\frac{B}{2},tg\frac{C}{2}\right);\left(\sin^2\frac{A}{2},\sin^2\frac{B}{2},\sin^2\frac{C}{2}\right);\right.$
$\left.\left(\cos^2\frac{A}{2},\cos^2\frac{B}{2},\cos^2\frac{C}{2}\right)\right\}$

Theorem 3. If $y_k\in(0,\pi)$ $(k=1,2,...,n)$, then

$$\sum_{k=1}^{n}\frac{tgy_k}{\sqrt{1+tg^2y_k}}\le\frac{ntg\left(\frac{1}{n}\sum\limits_{k=1}^{n}y_k\right)}{\sqrt{1+tg^2\left(\frac{1}{n}\sum\limits_{k=1}^{n}y_k\right)}}$$

Proof. The function $f(x)=\sin x$ is concave, therefore from Jensen's inequality we have:

$$\sum_{k=1}^{n}\sin y_k\le n\sin\left(\frac{1}{n}\sum_{k=1}^{n}y_k\right)\quad\text{or}$$

$$\sum_{k=1}^{n}\frac{tgy_k}{\sqrt{1+tg^2y_k}}\le\frac{ntg\left(\frac{1}{n}\sum\limits_{k=1}^{n}y_k\right)}{\sqrt{1+tg^2\left(\frac{1}{n}\sum\limits_{k=1}^{n}y_k\right)}}$$

Corollary 3.1. If $y_k\in\left(0,\frac{\pi}{2}\right)$ $(k=1,2,...,n)$, then

$$\sum_{k=1}^{n}\frac{1+tgy_k}{\sqrt{1+tg^2y_k}}\le\frac{n\left(1+tg\left(\frac{1}{n}\sum\limits_{k=1}^{n}y_k\right)\right)}{\sqrt{1+tg^2\left(\frac{1}{n}\sum\limits_{k=1}^{n}y_k\right)}}$$

Proof. We adding the inequalities from Theorem 1 and Theorem 3.

Corollary 3.2. If $x_k>0$ $(k=1,2,...,n)$, then

$$\sum_{k=1}^{n}\frac{x_k}{\sqrt{1+x_k^2}}\leq\frac{ntg\left(\frac{1}{n}\sum\limits_{k=1}^{n}arctgx_k\right)}{\sqrt{1+tg^2\left(\frac{1}{n}\sum\limits_{k=1}^{n}arctgx_k\right)}}$$

Proof. In Theorem 3 we take $y_k = arctgx_k \ (k = 1, 2, ..., n)$.

Corollary 3.3. If $x_k > 0$ $(k = 1, 2, ..., n)$, then

$$\sum_{k=1}^{n}\frac{1+x_k}{\sqrt{1+x_k^2}}\leq\frac{n\left(1+tg\left(\frac{1}{n}\sum\limits_{k=1}^{n}arctgx_k\right)\right)}{\sqrt{1+tg^2\left(\frac{1}{n}\sum\limits_{k=1}^{n}arctgx_k\right)}}$$

Proof. In Corollary 3.1 we take $y_k = arctgx_k \ (k = 1, 2, ..., n)$.

Open Question. If $0 \leq x_k \leq a$ $(k = 1, 2, ..., n)$, then determine all $a > 0$ and $\alpha \in R$ such that

$$\sum_{k=1}^{n}\left(\frac{1}{1+x_k^{\alpha}}\right)^{\frac{1}{\alpha}}\leq n\left(\frac{1}{1+\left(\prod\limits_{k=1}^{n}x_k\right)^{\frac{\alpha}{n}}}\right)^{\alpha}$$

REFERENCES

[1] Arkady Alt, *Problem 3329,* Crux Mathematicoru 3/2009, pp. 180-181.
[2] Octogon Mathematical Magazine (1993-2009)

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On Bergström's inequality involving six numbers

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ABSTRACT. In this paper, we discuss the Bergstöm inequality involving six numbers, four refinements and two reverse inequalities are proved. Finally, two conjectures are put forward, one of them is actually generalization of Schur's inequality.

1. INTRODUCTION

Let x, y, z be real numbers and let u, v, w be positive numbers, then

$$\frac{x^2}{u}+\frac{y^2}{v}+\frac{z^2}{w}\geq\frac{(x+y+z)^2}{u+v+w}, \tag{1.1}$$

with equality if and only if $x:y:z=u:v:w$.
Inequality (1.1) is a special case of the Bergström inequality (see [1]-[5]). Also it is a corollary of Cauchy-Buniakowsky-Schwarz inequality. In this paper, we will prove its four refinements and two reverse inequalities.
In the proofs of the following theorems, we denote cyclic sum on x, y, z, u, v, w by $\sum$, for instance,

$$\sum u=u+v+w, \sum\frac{x^2}{u}=\frac{x^2}{u}+\frac{y^2}{v}+\frac{z^2}{w}, \sum(vz-wy)^2=$$

$$=(vz-wy)^2+(wx-uz)^2+(uy-vx)^2.$$

2. FOUR REFINEMENTS

Firstly, we give the following refinement of Bergström inequality (1.1):

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Theorem 1. Let x, y, z be real numbers and let u, v, w be positive numbers. Then

$$\frac{x^2}{u}+\frac{y^2}{v}+\frac{z^2}{w} \geq \frac{(x+y+z)^2}{u+v+w}+\frac{(vz-wy)^2}{2vw(v+w)}+\frac{(wx-uz)^2}{2wu(w+u)}+\frac{(uy-vx)^2}{2uv(u+v)}, \quad (2.1)$$

with equality if and only if $x : y : z = u : v : w$.
Proof. It is easy to check that

$$\sum \frac{(yw-zv)}{vw\,(v+w)} = 2\sum \frac{x^2}{u} - \sum \frac{(y+z)^2}{v+w}, \quad (2.2)$$

Using the identity, we see that inequality (2) is equivalent to

$$\frac{1}{2}\sum \frac{(y+z)^2}{v+w} \geq \frac{(\sum x)^2}{\sum u},$$

which follows from (1.1) by replacing $x \rightarrow y+z$, $u \rightarrow v+w$ etc., Thus inequality (2) is proved.
Clearly, equality in (2.1) holds if and only if
$(y+z) : (z+x) : (x+y) = (v+w) : (w+u) : (u+v)$, namely
$x : y : z = u : v : w$ and this completes the proof of Theorem 1.
Applying inequality (1.1) to (2.1) we get

$$\frac{x^2}{u}+\frac{y^2}{v}+\frac{z^2}{w} \geq \frac{(x+y+z)^2}{u+v+w}+\frac{[(v-w)x+(w-u)y+(u-v)z]^2}{2[vw(v+w)+wu(w+u)+uv(u+v)]}. \quad (2.3)$$

Further, we find the following stronger result:

Theorem 2. Let x, y, z be real numbers and let u, v, w be positive numbers. Then

$$\frac{x^2}{u}+\frac{y^2}{v}+\frac{z^2}{w} \geq \frac{(x+y+z)^2}{u+v+w}+\frac{[(v-w)x+(w-u)y+(u-v)z]^2}{vw(v+w)+wu(w+u)+uv(u+v)}, \quad (2.4)$$

with equality if and only if $x : y : z = u : v : w$.
*P*roof. We set

$$F_1 = \sum \frac{x^2}{u} - \frac{(\sum x)^2}{\sum u} - \frac{\sum (vz-wy)^2}{\sum vw(v+w)}.$$

It is easy to get the following identity after some computations:

$$F_1 = \frac{a_1x^2+b_1x+c_1}{uvw\sum u\sum vw(v+w)}, \tag{2.5}$$

where

$$\begin{aligned}
a_1 &= v^2w^2(4u^2+v^2+w^2+2vw+2wu+2uv),\\
b_1 &= -2uvw\left[wy(2u^2+2v^2-w^2+vw+wu+uv)+vz(2w^2+2u^2-v^2\right.\\
&\qquad \left.+vw+wu+uv)\right],\\
c_1 &= u^2\left[(4v^2+w^2+u^2+2vw+2wu+2uv)w^2y^2-2vwyz(2v^2+2w^2-u^2\right.\\
&\qquad \left.+vw+wu+uv)+(4w^2+u^2+v^2+2vw+2wu+2uv)v^2z^2\right].
\end{aligned}$$

Now, we shall show first that $c_1 > 0$. Note that

$$\begin{aligned}
\Delta_1(y) &\equiv \left[-2vwz(2v^2+2w^2-u^2+vw+wu+uv)\right]^2-4\left[(4v^2+w^2+u^2\right.\\
&\qquad \left.+2vw+2wu+2uv)w^2\right]\left[(4w^2+u^2+v^2+2vw+2wu+2uv)v^2z^2\right]\\
&= -24v^2w^2z^2\sum u\sum vw(u+v) < 0,
\end{aligned}$$

hence the following inequality:

$$\begin{aligned}
&(4v^2+w^2+u^2+2vw+2wu+2uv)w^2y^2-\\
&-2vwyz(2v^2+2w^2-u^2+vw+wu+uv)\\
&+(4w^2+u^2+v^2+2vw+2wu+2uv)v^2z^2 > 0
\end{aligned}$$

holds for arbitrary real numbers y, z and positive numbers u, v, w. Therefor $c_1 > 0$ is true. Since $a_1 > 0, c_1 > 0$ and

$$\Delta_1(x) \equiv b_1^2 - 4a_1c_1 = -24u^2v^2w^2(vz-wy)^2\sum u\sum vw(v+w) \le 0.$$

We conclude that

$$a_1x^2+b_1x+c_1 \ge 0 \tag{2.6}$$

holds for arbitrary real numbers x. Thus $F_1 \ge 0$ and (2.4) are proved. Equality in (2.6) occurs if and only if $b_1^2 - 4a_1c_1 = 0$, hence the equality in (2.4) occurs iff $yw - zv = 0$. Because of the symmetry, we know that equality in (2.4) also occurs if and only if $zu - xw = 0, xv - uy = 0$. Combining the

above argumentations, the case of equality in (2.4) holds iff $x:y:z=u:v:w$. This completes the proof of Theorem 2.
From the identity:

$$\left(\sum u\right)^3-4\sum vw(v+w)=\sum u(u-v)(u-w)+3uvw \tag{2.7}$$

and the special case of Schur's inequality (see [6]):

$$\sum u(u-v)(u-w)\geq 0, \tag{2.8}$$

we see that

$$\sum vw(v+w)<\frac{1}{4}\left(\sum u\right)^3. \tag{2.9}$$

Therefore, we obtain the following conclusion from Theorem 2:

Corollary 2.1. For any real numbers x,y,z and positive numbers u,v,w, we have

$$\frac{x^2}{u}+\frac{y^2}{v}+\frac{z^2}{w}\geq\frac{(x+y+z)^2}{u+v+w}+\frac{4[(v-w)x+(w-u)y+(u-v)z]^2}{(u+v+w)^3}. \tag{2.10}$$

with equality if and only if $x:y:z=u:v:w$.

Remark 2.1 *T*he constant coefficient 4 in (2.10) is the best possible(We omit the proof).
In addition, it is easy to prove that

$$2(u^3+v^3+w^3)\geq vw(v+w)+wu(w+u)+uv(u+v) \tag{2.11}$$

from this and inequality (2.4) we get again

$$\frac{x^2}{u}+\frac{y^2}{v}+\frac{z^2}{w}\geq\frac{(x+y+z)^2}{u+v+w}+\frac{[(v-w)x+(w-u)y+(u-v)z]^2}{2(u^3+v^3+w^3)}. \tag{2.12}$$

We further find this inequality can be improved as following:

Theorem 3. Let x,y,z be real numbers and let u,v,w be positive numbers. Then

$$\frac{x^2}{u}+\frac{y^2}{v}+\frac{z^2}{w}\geq\frac{(x+y+z)^2}{u+v+w}+\frac{[(v-w)x+(w-u)y+(u-v)z]^2}{u^3+v^3+w^3}, \quad (2.13)$$

with equality if and only if $x:y:z=u:v:w$.
*P*roof. Letting

$$F_2=\frac{x^2}{u}+\frac{y^2}{v}+\frac{z^2}{w}-\frac{(x+y+z)^2}{u+v+w}-\frac{[(v-w)x+(w-u)y+(u-v)z]^2}{u^3+v^3+w^3}.$$

By some calculations we get

$$F_2=\frac{a_2x^2+b_2x+c_2}{uvw\sum u\sum u^3}, \quad (2.14)$$

where
$a_2=vw\{v[(w+u)w^2+(2u^2+uv+v^2)w+(u+v)(u-v)^2]+w(u+w)(w-u)^2\}$,
$b_2=$
$-2uvw\{[(u^2+uv+v^2)w+(u+v)(u-v)^2]y+[v(w^2+wu+u^2)+(w+u)(w-u)^2]z\}$,
$c_2=u\left(wm_1y^2+m_2yz+vm_3z^2\right)$,
moreover, the values of m_1,m_2,m_3 are

$$\begin{aligned}
m_1&=w\left[(u+v)u^2+(2v^2+vw+w^2)u+(v+w)(v-w)^2\right]+u(v+u)(u-v)^2,\\
m_2&=-2vw\left[(v^2+wv+w^2)u+(v+w)(v-w)^2\right],\\
m_3&=u\left[(v+w)v^2+(2w^2+wu+u^2)v+(w+u)(w-u)^2\right]+v(v+w)(v-w)^2.
\end{aligned}$$

First we show that $c_2>0$. Since $wm_1>0, vm_3>0$ and

$$\begin{aligned}
\Delta_2(y)&\equiv(m_2z)^2-4(wm_1)(vm_3z^2)\\
&=-4uvw\sum u\sum u^3\left[\sum u(u-v)(u-w)+3uvw\right]z^2<0,
\end{aligned}$$

where we have used Schur's inequality (2.8), so that $c_2>0$. Taking into account $a_2>0, c_2>0$ and

$$\Delta_2(x)\equiv b_2^2-4a_2c_2=-4uvw(vz-wy)^2\sum u\sum u^3\left[\sum u(u-v)(u-w)+3uvw\right]\leq 0,$$

hence $a_2x^2+b_2x+c_2\geq 0$ is true for all real numbers x, inequality $F_2\geq 0$ is proved. Then, the same argument in the proof of Theorem 2 shows that,

equality in (2.13) holds if and only if $x : y : z = u : v : w$. The proof of the Theorem 3 is complete.
Before giving the next result which is similar to Theorem 3, we first prove two inequalities.

Lemma 2.1. Let u, v, w be positive real numbers. Then

$$\sum u(u^2 - v^2)(u^2 - w^2) \geq 0, \tag{2.15}$$

with equality if and only if $u = v = w$.
*P*roof. For any positive real numbers u, v, w and real number k the following inequality holds:

$$\sum u^k(u - v)(u - w) \geq 0, \tag{2.16}$$

this is famous Schur's inequality (see [6]). Putting $k = \frac{1}{2}$ and replacing $u \to u^2, v \to v^2, w \to w^2$, the claimed inequality follows.

Lemma 2.2. Let u, v, w be positive real numbers. Then

$$\sum (u - v)(u - w)(v + w)u^2 \geq 0. \tag{2.17}$$

with equality if and only if $u = v = w$.
*P*roof We will use the method of difference substitution (see [7], [8]) to prove desired inequality. Without loss of generality suppose that $u \geq v \geq w$, putting $v = w + p, u = v + q (p \geq 0, q \geq 0)$, then plugging $v = w + p, u = w + p + q$ into the right-hand side of (2.17). One may easily check the identity:

$$\sum (u - v)(u - w)(v + w)u^2 = 2(p^2 + pq + q^2)w^3 + (2p + q)(p^2 + pq + 4q^2)w^2 +$$

$$+2q^2(2p + q)^2 w + pq^2(p + q)(2p + q), \tag{2.18}$$

since $p \geq 0, q \geq 0, w > 0$, we get the the inequality (2.17).
Now, we prove the following Theorem:

Theorem 4. Let x, y, z be real numbers and let u, v, w be positive numbers. Then

$$\frac{x^2}{u} + \frac{y^2}{v} + \frac{z^2}{w} \geq$$

$$\geq \frac{(x+y+z)^2}{u+v+w} + \frac{4\left[(vz-wy)^2+(wx-uz)^2+(uy-vx)^2\right]}{(u+v+w)^3}, \tag{2.19}$$

with equality if and only if $x:y:z = u:v:w$.
*P*roof. Letting

$$F_3 \equiv \sum \frac{x^2}{u} - \frac{\left(\sum x\right)^2}{\sum u} - \frac{4\sum (vz-wy)^2}{\left(\sum u\right)^3}.$$

After some computations we get the following identity

$$F_3 = \frac{a_3x^2+b_3x+c_3}{uvw(u+v+w)^3}, \tag{2.20}$$

where

$$\begin{aligned}
a_3 &= vw[(3v+3w+4u)vw+v(u-v)^2+w(w-u)^2],\\
b_3 &= -2uvw[(y+z)u^2-2(v-w)(y-z)u+(y+z)(v+w)^2],\\
c_3 &= u\left\{w[(3w+3u+4v)wu+w(v-w)^2+u(u-v)^2]y^2\right.\\
&\quad -2vwyz[u^2+2(v+w)u+(v-w)^2]\\
&\quad \left.+v[(3u+3v+4w)uv+u(w-u)^2+v(v-w)^2]z^2\right\}.
\end{aligned}$$

To prove $F_3 \geq 0$, we first prove that $c_3 > 0$, it suffices to show that

$$w\left[(3w+3u+4v)wu+w(v-w)^2+u(u-v)^2\right]y^2-$$

$$-2vwyz\left[u^2+2(v+w)u+(v-w)^2\right]+$$

$$+v\left[(3u+3v+4w)uv+u(w-u)^2+v(v-w)^2\right]z^2>0. \tag{2.21}$$

Note that

$$\begin{aligned}
\Delta_3(y) \equiv{}& \left\{-2vwz[u^2+2(v+w)u+(v-w)^2]\right\}^2\\
&-4vw\left[(3w+3u+4v)wu+w(v-w)^2+u(u-v)^2\right][(3u+3v+4w)uv\\
&+u(w-u)^2+v(v-w)^2\left.\right]\\
={}& -4uvwMz^2,
\end{aligned}$$

where

$$M=\sum u^5+\sum(v+w)u^4-2\sum(v+w)v^2w^2+20uvw\sum u^2+6uvw\sum vw.$$

Again, it is easy to verify the the following identity:

$$\sum u^5+\sum(v+w)u^4-2\sum(v+w)v^2w^2+3uvw\sum vw=$$

$$=\sum u(u^2-v^2)(u^2-w^2)+\sum(u-v)(u-w)(v+w)u^2+2uvw\sum u^2. \quad (2.22)$$

By Lemma 2.1 and Lemma 2.2 we see that $M>0$, hence $\Delta_3(y)<0$, thus inequality (2.21) holds for all real numbers y,z and positive numbers u,v,w. Since $a_3>0, c_3>0$ and

$$\Delta_3(x)\equiv b_3^2-4a_3c_3=-4uvwM(vz-wy)^2\leq 0.$$

Thus $a_3x^2+b_3x+c_3\geq 0$ holds for arbitrary real numbers x. So, the inequality of Theorem 4 is proved. As in the proof of the Theorem 2, we conclude that equality (2.19) holds if and only if $x:y:z=u:v:w$. Our proof is complete.

Remark 2.2 *Applying the method of undetermined coefficients, we can easily prove that the constant coefficient 4 in the right hand side of the inequality of Theorem 4 is the best possible. In addition, the analogous constant coefficients of others five Theorems in this paper are all the best possible.*

TWO REVERSE INEQUALITIES

In this section, we will establish two inverse inequalities of Bergström's Inequality (1.1).

Theorem 5. Let x,y,z be real numbers and let u,v,w be positive numbers. Then

$$\frac{x^2}{u}+\frac{y^2}{v}+\frac{z^2}{w}\leq\frac{(x+y+z)^2}{u+v+w}+\frac{(vz-wy)^2}{vw(v+w)}+\frac{(wx-uz)^2}{wu(w+u)}+\frac{(uy-vx)^2}{uv(u+v)}, \quad (3.1)$$

with equality if and only if $x:y:z=u:v:w$.

*P*roof. Setting

$$F_4 = \frac{(\sum x)^2}{\sum u} + \sum \frac{(vz - wy)^2}{vw(v+w)} - \sum \frac{x^2}{u},$$

then we get the following identity:

$$F_4 = \frac{a_4 x^2 + b_4 x + c_4}{uvw(v+w)(w+u)(u+v)}, \tag{3.2}$$

where

$$\begin{aligned}
&a_4 = (v+w)(v+w+2u)v^2w^2,\\
&b_4 = -2uvw(v+w)(yw^2 + zv^2 + uyw + uzv),\\
&c_4 = u^2\left[(w+u)(w+u+2v)w^2y^2 - 2vwyz(u+v)(w+u)+\right.\\
&\left.+(u+v)(u+v+2w)v^2z^2\right].
\end{aligned}$$

First, we will prove that $c_1 > 0$. Since

$$\begin{aligned}
&\Delta_4(y) \equiv [-2vwz(u+v)(w+u)]^2 - 4\left[(w+u)(w+u+2v)w^2\right] \cdot\\
&\cdot \left[(u+v)(u+v+2w)v^2z^2\right] =\\
&\qquad = -8(v+w)(w+u)(u+v)(u+v+w)v^2w^2z^2 < 0,
\end{aligned}$$

it follows that

$$(u+v)(u+v+2w)v^2z^2 - 2vwy(u+v)(w+u)z+$$

$$+(w+u)(w+u+2v)y^2w^2 > 0. \tag{3.3}$$

Hence $c_4 > 0$. Note that again $a_4 > 0$ and

$$\Delta_4(x) \equiv b_4^2 - 4a_4c_4 = -(u+v+w)(v+w)(w+u)(u+v)(uvw)^2(yw-vz)^2 \le 0,$$

so we have $a_4x^2 + b_4x + c_4 \ge 0$. Therefore $F_4 \ge 0$ and (3.1) are proved. Equality in (3.1) holds when $x : y : z = u : v : w$ and the proof is complete.

Remark 3.1 *F*rom identity (2.2), the inequality of Theorem 5 is equivalent to

$$\frac{x^2}{u}+\frac{y^2}{v}+\frac{z^2}{w}+\frac{(x+y+z)^2}{u+v+w}\geq\frac{(y+z)^2}{v+w}+\frac{(z+x)^2}{w+u}+\frac{(x+y)^2}{u+v} \tag{3.14}$$

Remark 3.2 *C*ombining Theorem 1 and Theorem 5, we get the following double inequalities:

$$\frac{(vz-wy)^2}{2yz(y+z)}+\frac{(wx-uz)^2}{2zx(z+x)}+\frac{(uy-vx)^2}{2xy(x+y)}\leq\frac{x^2}{u}+\frac{y^2}{v}+\frac{z^2}{w}-\frac{(x+y+z)^2}{u+v+w}\leq$$

$$\leq\frac{(vz-wy)^2}{yz(y+z)}+\frac{(wx-uz)^2}{zx(z+x)}+\frac{(uy-vx)^2}{xy(x+y)}. \tag{3.5}$$

In the sequel we give another reverse inequality:

Theorem 6. Let x, y, z be real numbers and let u, v, w be positive numbers, then

$$\frac{x^2}{u}+\frac{y^2}{v}+\frac{z^2}{w}\leq\frac{(x+y+z)^2}{u+v+w}+\frac{(vz-wy)^2+(wx-uz)^2+(uy-vx)^2}{2uvw}, \tag{3.6}$$

with equality if and only if $x:y:z=u:v:w$.
*P*roof. Letting

$$F_5=\frac{\left(\sum x\right)^2}{\sum u}-\frac{\sum(vz-wy)^2}{2uvw}-\sum\frac{x^2}{u},$$

then we have the following identity:

$$F_5=\frac{a_5x^2+b_5x+c_5}{2uvw(u+v+w)}, \tag{3.7}$$

where

$$\begin{aligned}
&a_5=u(v^2+w^2)+(v+w)(v-w)^2,\\
&b_5=-2u[u(zw+vy)+(v-w)(vy-zw)],\\
&c_5=\left[v(w^2+u^2)+(w+u)(w-u)^2\right]y^2-2vwz(v+w-u)y\\
&\quad+\left[w(u^2+v^2)+(u+v)(u-v)^2\right]z^2.
\end{aligned}$$

First we prove that $c_5 > 0$. Since

$$\begin{aligned}\Delta_5(y) &\equiv [-2vwz(v+w-u)]^2 - 4\left[v(w^2+u^2)+(w+u)(w-u)^2\right]\left[w(u^2+v^2)\right.\\ &\quad\left.+(u+v)(u-v)^2\right]z^2\\ &= -4u^2z^2\left[\sum u^3 - \sum(v+w)u^2 + 4uvw\right]\sum u\\ &= -4u^2z^2\left[\sum u(u-v)(u-w) + uvw\right]\sum u < 0,\end{aligned}$$

where we have applied Schur's inequality (2.8). Therefore, we know that $\Delta_5(y) < 0$ holds for arbitrary real numbers y, z. Hence $c_5 > 0$ is true. On the other hand, since $a_5 > 0, c_5 > 0$ and

$$\Delta_5(x) \equiv b_5^2 - 4a_5c_5 = -4(vz-wy)^2\left[\sum u(u-v)(u-w) + uvw\right]\sum u \leq 0,$$

thus $a_5x^2 + b_5x + c_5 \geq 0$ holds for arbitrary real number x. Inequality $F_5 \geq 0$ and (3.6) are proved. Clearly, the equality in (3.6) holds if and only if $x : y : z = u : v : w$. Hence Theorem 6 has been proved completely.
Let ABC be an acute triangle. In (3.6) we take $u = \tan A, v = \tan B, w = \tan C$, multiplying by $\tan A \tan B \tan C$ in both sides, then using the well known identity:

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C, \tag{3.8}$$

we get

Corollary 3.1. Let ABC be an acute triangle and $x, y, z > 0$, then

$$x^2 \tan B \tan C + y^2 \tan C \tan A + z^2 \tan A \tan B \leq (x+y+z)^2 +$$

$$+\frac{1}{2}[(y\tan C - z\tan B)^2 + (z\tan A - x\tan C)^2 + (x\tan B - y\tan A)^2], \tag{3.9}$$

with equality if and only if $x : y : z = \tan A : \tan B : \tan C$.

4. TWO CONJECTURES

Finally, we propose two conjectures. The first is about the triangle:

Conjecture 4.1. Let x, y, z be positive real numbers and let a, b, c be the sides of $\triangle ABC$, then

$$\frac{(b+c)^2}{y+z}+\frac{(c+a)^2}{z+x}+\frac{(a+b)^2}{x+y}\geq\frac{2(a+b+c)^2}{x+y+z}+\frac{(bz-cy)^2}{4(y+z)^3}+\frac{(cx-az)^2}{4(z+x)^3}+$$

$$+\frac{(ay-bx)^2}{4(x+y)^3} \tag{4.1}$$

with equality if and only if $x:y:z=a:b:c$.

The second is a generalization of Lemma 2.2.

Conjecture 4.2. Let k be arbitrary real numbers and let x,y,z,p,q be real numbers such that $x>0,y>0,z>0,p\geq 0,p\geq q+1$. Then

$$\sum(x^k-y^k)(x^k-z^k)(y^q+z^q)x^p\geq 0, \tag{4.2}$$

$$\sum(x^k-y^k)(x^k-z^k)(y+z)^q x^p\geq 0. \tag{4.3}$$

If we take $k=1,q=1,p=2$ in the conjecture, then both (4.2) and (4.3) become the inequality (2.18) of Lemma 2.2. If we put $q=0$, then we obtain actually Schur's inequality (2.16) from (4.2) or (4.3).

REFERENCES

[1] Bergström, H., *T*riangle inequality for matrices, Den Elfte Skandinaviske Matematikerkongress, Trodheim, 1949, Johan Grundt Tanums Forlag, Oslo, 1952, 264-267.

[2] Bellman, R., *N*otes on Matrix Theory-IV(An Inequality Due to Bergström), Amer.Math.Monthly, Vol.62(1955)172-173.

[3] Fan, Ky, *G*eneralization of Bergström's inequality, Amer.Math.Monthly, Vol.66,No.2(1959),153-154.

[4] Beckencbach, E.F., and Bellman, R., *I*nequalities, Springer, Berlin,Göttingen and Heidelberg, 1961.

[5] Mărghidanu, D., Díaz-Barrero, J.L., and Rădulescu, S., *N*ew refinements of some classical inequalities, Mathematical Inequalities Applications, Vol.12, 3(2009), 513-518.

[6] Hardy, G.H., Littlewood, J.E., and Pólya, G., *I*nequalities, Cambridge, 1934.

[7] Yang, L., *D*ifference substitution and automated inequality proving, J.Guangzhou Univ. (Natural Sciences Edition), 5(2)(2006), 1-7. (in Chinese)

[8] Yu-Dong Wu, Zhi-hua Zhang, and Yu-Rui Zhang, *P*roving inequalities in acute triangle with difference substitution, Inequal.Pure Appl. Math., 8(3)(2007), Art.81.

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New refinements for some classical inequalities

Mihály Bencze and Yu-Dong Wu[23]

ABSTRACT. In this paper we present new refinements for some classical inequalities.

INTRODUCTION

First we study the following inequality:

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\geq\frac{x+y+z}{\sqrt[3]{xyz}}$$

where $x, y, z > 0$. Using a new proof we give an extension for this. This inequality was studied by S. Arslanagic too.

MAIN RESULTS

Theorem 1. If $x, y, z > 0$, then

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\geq\frac{x+y+z}{\sqrt[3]{xyz}}\geq 3$$

Proof. Using the AM-GM inequality, we obtain:

$$\begin{cases}\frac{1}{3}\left(\frac{x}{y}+\frac{x}{y}+\frac{y}{z}\right)\geq\sqrt[3]{\frac{x^2}{yz}}=\frac{x}{\sqrt[3]{xyz}}\\ \frac{1}{3}\left(\frac{y}{z}+\frac{y}{z}+\frac{z}{x}\right)\geq\frac{y}{\sqrt[3]{xyz}}\\ \frac{1}{3}\left(\frac{z}{x}+\frac{z}{x}+\frac{x}{y}\right)\geq\frac{z}{\sqrt[3]{xyz}}\end{cases}$$

After addition we get the desired inequality.

Corollary 1.1. In all triangle ABC holds

1). $\sum\frac{a}{b}\geq\frac{2s}{\sqrt[3]{4sRr}}\geq 3$

2). $\sum\frac{-a+b+c}{a-b+c}\geq\frac{s}{\sqrt[3]{sr^2}}\geq 3$

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3). $\sum \frac{r_a}{r_b} \geq \frac{4R+r}{\sqrt[3]{s^2r}} \geq 3$
4). $\sum ctg\frac{A}{2}tg\frac{B}{2} \geq \sqrt[3]{\frac{s^2}{r^2}} \geq 3$
5). $\sum tg\frac{A}{2}ctg\frac{B}{2} \geq \frac{4R+r}{\sqrt[3]{s^2r}} \geq 3$
6). $\sum \left(\frac{\sin\frac{A}{2}}{\sin\frac{B}{2}}\right)^2 \geq (2R-r)\sqrt[3]{\frac{2}{Rr^2}} \geq 3$
7). $\sum \left(\frac{\cos\frac{A}{2}}{\cos\frac{B}{2}}\right)^2 \geq (4R+r)\sqrt[3]{\frac{2}{Rs^2}} \geq 3$

Proof. In Theorem 1 we take $(x,y,z) \in$
$\left\{(a,b,c);(s-a,s-b,s-c);(r_a,r_b,r_c);\left(ctg\frac{A}{2},ctg\frac{B}{2},ctg\frac{C}{2}\right);\right.$
$\left.\left(tg\frac{A}{2},tg\frac{B}{2},tg\frac{C}{2}\right);\left(\sin^2\frac{A}{2},\sin^2\frac{B}{2},\sin^2\frac{C}{2}\right);\left(\cos^2\frac{A}{2},\cos^2\frac{B}{2},\cos^2\frac{C}{2}\right)\right\}$
These are new refinements for a lot of classical geometrical inequalities.

Theorem 2. If

$$F(a,b,c) =$$

$$= \left\{\frac{a\sqrt[3]{b}+\sqrt[3]{ab^2c}+\sqrt[3]{a^2c^2}}{3\sqrt[3]{a}};\frac{b\sqrt[3]{c}+\sqrt[3]{bc^2a}+\sqrt[3]{b^2a^2}}{3\sqrt[3]{b}};\frac{c\sqrt[3]{a}+\sqrt[3]{ca^2b}+\sqrt[3]{c^2b^2}}{3\sqrt[3]{c}}\right\},$$

where $a,b,c>0$, then

$$\frac{a+b+c}{3} \geq F(a,b,c) \geq \sqrt[3]{abc},$$

which offer a new refinement for AM-GM inequality.
Proof. In Theorem 1 we consider $\frac{x}{y}=\frac{a}{\sqrt[3]{abc}}$, $\frac{y}{z}=\frac{b}{\sqrt[3]{abc}}$, $\frac{z}{x}=\frac{c}{\sqrt[3]{abc}}$ or
$x=\frac{abz}{\sqrt[3]{a^2b^2c^2}}$, $y=\frac{bz}{\sqrt[3]{abc}}$, therefore

$$\frac{a+b+c}{3} \geq \frac{a\sqrt[3]{b}+\sqrt[3]{ab^2c}+\sqrt[3]{a^2c^2}}{3\sqrt[3]{a}} \geq \sqrt[3]{abc}$$

and his permutations.

Corollary 2.1. In all triangle ABC holds
1). $\frac{2s}{3} \geq F(a,b,c) \geq \sqrt[3]{4sRr}$
2). $\frac{s}{3} \geq F(s-a,s-b,s-c) \geq \sqrt[3]{sr^2}$
3). $\frac{s^2+r^2+4Rr}{6R} \geq F(h_a,h_b,h_c,h_c) \geq \sqrt[3]{\frac{2s^2r^2}{R}}$
4). $\frac{4R+r}{3} \geq F(r_a,r_b,r_c) \geq \sqrt[3]{s^2r}$
5). $\frac{s}{3r} \geq F\left(ctg\frac{A}{2},ctg\frac{B}{2},ctg\frac{C}{2}\right) \geq \sqrt[3]{\frac{s}{r}}$
6). $\frac{4R+r}{3s} \geq F\left(tg\frac{A}{2},tg\frac{B}{2},tg\frac{C}{2}\right) \geq \sqrt[3]{\frac{r}{s}}$

7). $\frac{2R-r}{6R} \geq F\left(\sin^2\frac{A}{2}, \sin^2\frac{B}{2}, \sin^2\frac{C}{2}\right) \geq \sqrt[3]{\frac{r^2}{16R^2}}$
8). $\frac{4R+r}{6R} \geq F\left(\cos^2\frac{A}{2}, \cos^2\frac{B}{2}, \cos^2\frac{C}{2}\right) \geq \sqrt[3]{\frac{s^2}{16R^2}}$
which are new refinements for a lot of classical geometrical inequalities.

Theorem 3. If $x, y, z, t > 0$, then

$$\frac{x+y}{z} + \frac{y+z}{t} + \frac{z+t}{x} + \frac{t+x}{y} \geq \frac{2(x+y+z+t)}{\sqrt[4]{xyzt}} \geq 8$$

Proof. Using the AM-GM inequality we obtain.

$$\begin{cases} \frac{x}{y} + \frac{x}{y} + \frac{x}{z} + \frac{y}{t} \geq \frac{4x}{\sqrt[4]{xyzt}} \\ \frac{y}{z} + \frac{y}{z} + \frac{y}{t} + \frac{z}{x} \geq \frac{4y}{\sqrt[4]{xyzt}} \\ \frac{z}{t} + \frac{z}{t} + \frac{z}{x} + \frac{t}{y} \geq \frac{4z}{\sqrt[4]{xyzt}} \\ \frac{t}{x} + \frac{t}{x} + \frac{t}{y} + \frac{x}{z} \geq \frac{4t}{\sqrt[4]{xyzt}} \end{cases}$$

After addition we get

$$2\sum \frac{x}{y} + \sum \frac{x}{z} \geq \frac{4\sum x}{\sqrt[4]{xyzt}} \text{ or } \sum \frac{x+y}{z} \geq \frac{2\sum x}{\sqrt[4]{xyzt}} \geq 8$$

Corollary 3.1. In all tetrahedron $ABCD$ holds
1). $\sum \frac{(h_a+h_b)h_c}{h_a h_b} \geq \frac{2}{r}\sqrt[4]{h_a h_b h_c h_c} \geq 8$
2). $\sum \frac{(r_a+r_b)r_c}{r_a r_b} \geq \frac{4}{r}\sqrt[4]{r_a r_b r_c r_c} \geq 8$

Proof. In Theorem 3 we take $(x, y, z, t) \in \{(h_a, h_b, h_c, h_c); (r_a, r_b, r_c, r_c)\}$ and we consider the relations

$$\sum \frac{1}{h_a} = \frac{1}{r}, \ \sum \frac{1}{r_a} = \frac{2}{r}$$

Theorem 4. If

$$F(a,b,c,d) = \left\{ \frac{ab\sqrt[4]{c} + b\sqrt[4]{abc^2d} + \sqrt[4]{a^2b^2c^3d^2} + \sqrt[4]{a^3b^3d^3}}{2\sqrt[4]{ab^2\sqrt{abcd}}}; \right.$$

$$\frac{bc\sqrt[4]{d} + c\sqrt[4]{bcd^2a} + \sqrt[4]{b^2c^2d^3a^2} + \sqrt[4]{b^3c^3a^3}}{2\sqrt[4]{bc^2\sqrt{abcd}}};$$

$$\frac{cd\sqrt[4]{a} + d\sqrt[4]{cda^2b} + \sqrt[4]{c^2d^2a^3b^2} + \sqrt[4]{c^3d^3a^3}}{2\sqrt[4]{cd^2\sqrt{abcd}}};$$

$$\left. \frac{da\sqrt[4]{b} + a\sqrt[4]{dab^2c} + \sqrt[4]{d^2a^2b^3c^2} + \sqrt[4]{d^3a^3b^3}}{2\sqrt[4]{da^2\sqrt{abcd}}} \right\}$$

when $a, b, c, d > 0$, then

$$\frac{a+b+c+d}{4}+\frac{(a+c)(b+d)}{\sqrt[4]{abcd}} \geq F(a,b,c,d) \geq 2\sqrt[4]{abcd}$$

Proof. In Theorem 3 we consider $\frac{x}{y}=\frac{a}{\sqrt[4]{abcd}}$, $\frac{y}{z}=\frac{b}{\sqrt[4]{abcd}}$; $\frac{z}{t}=\frac{c}{\sqrt[4]{abcd}}$; $\frac{t}{x}=\frac{d}{\sqrt[4]{abcd}}$ and $x=\frac{abct}{\sqrt[4]{a^3b^3c^3d^3}}$, $y=\frac{bct}{\sqrt[4]{a^2b^2c^2d^2}}$, $z=\frac{ct}{\sqrt[4]{abcd}}$, then $\sum\frac{x}{y}=\frac{\sum a}{\sqrt[4]{abcd}}$, $\sum\frac{x}{z}=\frac{(a+c)(b+d)}{\sqrt[4]{a^2b^2c^2d^2}}$, and

$$\frac{2\sum x}{\sqrt[4]{xyzt}}=\frac{2\left(ab\sqrt[4]{c}+b\sqrt[4]{abc^2d}+\sqrt[4]{a^2b^2c^3d^2}+\sqrt[4]{a^3b^3d^3}\right)}{\sqrt[4]{ab^2\sqrt{abcd}}\cdot\sqrt[4]{abcd}}$$

therefore

$$\frac{\sum a}{\sqrt[4]{abcd}}+\frac{(a+c)(b+d)}{\sqrt[4]{a^2b^2c^2d^2}} \geq \frac{2\left(ab\sqrt[4]{c}+b\sqrt[4]{abc^2d}+\sqrt[4]{a^2b^2c^3d^2}+\sqrt[4]{a^3b^3d^3}\right)}{\sqrt[4]{ab^2\sqrt{abcd}}\cdot\sqrt[4]{abcd}} \geq 8$$

or

$$\frac{a+b+c+d}{4}+\frac{(a+c)(b+d)}{\sqrt[4]{abcd}} \geq F(a,b,c,d) \geq 2\sqrt[4]{abcd}$$

which offer a new refinement for AM-GM inequality.

Corollary 4.1. In all tetrahedron $ABCD$ holds

1). $\frac{1}{4r}+\frac{(h_a+h_c)(h_b+h_d)}{\sqrt[4]{h_a^3h_b^3h_c^3h_d^3}} \geq F\left(\frac{1}{h_a},\frac{1}{h_b},\frac{1}{h_c},\frac{1}{h_d}\right) \geq \frac{2}{\sqrt[4]{h_ah_bh_ch_d}}$

2). $\frac{1}{2r}+\frac{(r_a+r_c)(r_b+r_d)}{\sqrt[4]{r_a^3r_b^3r_c^3r_d^3}} \geq F\left(\frac{1}{r_a},\frac{1}{r_b},\frac{1}{r_c},\frac{1}{r_d}\right) \geq \frac{2}{\sqrt[4]{r_ar_br_cr_d}}$

Theorem 5. If $x_k > 0$ $(k=1,2,...,n)$, then

$$2\sum\frac{x_1}{x_2}+\sum\frac{x_1}{x_3}+...+\sum\frac{x_1}{x_{n-2}}+2\sum\frac{x_1}{x_{n-1}} \geq \frac{n\sum\limits_{k=1}^{n}x_k}{\sqrt[n]{\prod\limits_{k=1}^{n}x_k}} \geq n$$

Proof. Using the AM-GM inequality we obtain:

$$\begin{cases} \frac{x_1}{x_2}+\frac{x_1}{x_2}+\frac{x_1}{x_3}+...+\frac{x_1}{x_{n-1}}+\frac{x_2}{x_n} \geq \frac{nx_1}{\sqrt[n]{\prod\limits_{k=1}^{n}x_k}} \\ -------------------- \\ \frac{x_n}{x_1}+\frac{x_n}{x_1}+\frac{x_n}{x_2}+...+\frac{x_n}{x_{n-2}}+\frac{x_1}{x_{n-1}} \geq \frac{nx_n}{\sqrt[n]{\prod\limits_{k=1}^{n}x_k}} \end{cases}$$

After addition we get the desired inequality, because $\sum \frac{x_1}{x_{n-1}} = \sum \frac{x_2}{x_n}$.

Theorem 6. If $a_k > 0$ $(k = 1, 2, ..., n)$ and $P = \sqrt[n]{\prod_{k=1}^{n} a_k}$, then

$$\frac{2\sum\limits_{k=1}^{n} a_k}{P} + \sum \frac{a_1a_2}{P^2} + ... + \sum \frac{a_1a_2...a_{n-3}}{P^{n-3}} + 2\sum \frac{P^2}{a_{n-1}a_n} \geq$$

$$\geq \frac{n\left(a_1a_2...a_{n-1} + a_2...a_{n-1}P + ... + a_{n-1}P^{n-2} + P^{n-1}\right)}{\sqrt{P^{n-1}}\sqrt[n]{a_1a_2^2a_3^3...a_{n-1}^{n-1}}} \geq n$$

Proof. In Theorem 5 we consider $\frac{x_1}{x_2} = \frac{a_1}{P}, \frac{x_2}{x_3} = \frac{a_2}{P}, ..., \frac{x_{n-1}}{x_n} = \frac{a_{n-1}}{P}, \frac{x_n}{x_1} = \frac{a_n}{P}$ and $x_1 = \frac{a_1a_2...a_{n-1}x_n}{P^{n-1}}$,
$x_2 = \frac{a_2a_3...a_{n-1}x_n}{P}, ..., x_{n-2} = \frac{a_{n-2}a_{n-1}x_n}{P^2}, x_{n-1} = \frac{a_{n-1}x_n}{P}$.

REFERENCE

[1] Octogon Mathematical Magazine (1993-2009)

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On the equation $ax^2 + by^2 = z^2$, where $a + b = c^2$

József Sándor and Ildikó Bakcsi[24]

ABSTRACT. Let a, b, c begiven positive integers such that $a + b = c^2$. We offer a simple method of solving the diophantine equation $ax^2 + by^2 = z^2$, based on the Euler-Bell-Kalmár lemma.

Let $a + b = c^2$. Then remark tat the equation $ax^2 + by^2 = z^2$ may be written equivalently as

$$ax^2 + by^2 = z^2 - c^2y^2; \text{ i.e} \tag{1}$$

$$(ax - ay)(x + y) = (z - cy)(z + cy)$$

According to the Euler-Bell-Kalmár lemma (see [1]), the general solutions to (1) may be written as follows:

$$ax - ay = mn, x + y = kp, z - cy = mp, z + cy = nk \tag{2}$$

where m, n, k, p are arbitrary positive integers, with $(n, p) = 1$. Solving equations (2) we get:

$$x = \frac{mn + akp}{2a}, y = \frac{akp - mn}{2a}, z = \frac{mp + nk}{2}, y = \frac{nk - mp}{2c} \tag{2}$$

Therefore we must have

$$y = \frac{akp - mn}{2a} = \frac{nk - mp}{2c},$$

implying $c(akp - mn) = a(nk - mp)$; or also $p(cak + am) = n(ak + cm)$. As $(n, p) = 1$; we obtain that

$$cak + am = tn, ak + cm = tp; t \text{ arbitrary} \tag{3}$$

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Here a, c are given. Solving this system of unknowns k and m, we get:

$$k = \frac{t\,(cn - ap)}{ab}, m = \frac{t\,(ct - n)}{b} \tag{4}$$

Replacing these values of k, m from (4) into (2); after some transformations, we get:

$$x = \frac{t\left(n^2 - nct - cpn + ap^2\right)}{2ab}; y = \frac{t\left(n^2 - nct + cpn - ap^2\right)}{2ab};$$

$$z = \frac{t\left(cn^2 - 2nap + capt\right)}{2ab} \tag{5}$$

Here $(n, p) = 1$ and t should be selected in such a manner that x, y, z of (5) to be integers. For example when t is a multiple of $2ab$, etc.

REFERENCE

[1] Sándor, J., *Geometric theorems, diophantine equations, and arithmetic functions,* New Mexico, 2002

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About Dumitru Acu‘s inequality

Mihály Bencze and Yu-Dong Wu[25]

ABSTRACT. In [1] is presented the following inequality

$$\frac{1+x^2+...+x^{2n}}{x+x^3+...+x^{2n-1}} \geq \frac{n+1}{n}$$

for all $x > 0$. In this paper we give new proofs, refinement and some applications.

MAIN RESULTS

Theorem 1. If $x > 0$, then

$$\frac{1+x+x^2+...+x^n}{1+x+x^2+...+x^{n-1}} \geq \frac{(n+1)(x+1)}{2n}$$

Proof. After elementary computations we get

$$(n-1)(1+x^n) \geq 2\left(x+x^2+...+x^{n-1}\right)$$

or

$$(x-1)\left(x^{n-1}-1\right)+\left(x^2-1\right)\left(x^{n-2}-1\right)+...+\left(x^{n-1}-1\right)(x-1) \geq 0$$

and after descomposition we obtain:

$$(x-1)^2\left(x^{n-2}+x^{n-3}+...+x+1\right)+$$

$$+(x-1)^2(x+1)\left(x^{n-3}+x^{n-4}+...+x+1\right)+... \geq 0$$

Corollary 1.1. If $x > 0$, then

$$\frac{1+x^2+x^4+...+x^{2n}}{x+x^3+...+x^{2n-1}} \geq \frac{n+1}{n}$$

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(see [1]).
Proof. Because $\frac{x^2+1}{2} \geq x$, then from Theorem 1 with substitution $x \to x^2$, we get:

$$\frac{1+x^2+...+x^{2n}}{1+x^2+...+x^{2n-1}} \geq \frac{(n-1)\left(x^2+1\right)}{2n} \geq \frac{(n+1)\,x}{2} \text{ or}$$

$$\frac{1+x^2+...+x^{2n}}{x+x^3+...+x^{2n-1}} \geq \frac{n+1}{n},$$

therefore Theorem 1 is a refinement of inequality from [1].
If $f(x) = \frac{1+x+...+x^n}{1+x+...+x^{n-1}}$, then $f'(x) = \frac{x^{n-1}g(x)}{(x^n-1)^2}$, where $g(x) = x^{n+1} - (n+1)\,x + n = (x-1)^2\left(\left(x^{n-1}+x^{n-2}+...+x+1\right) + \left(x^{n-2}+x^{n-3}+...+x+1\right)+...+(x+1)+1\right) \geq 0$ therefore $x=1$ is a minium point for the function f and $f(x) \geq f(1) = \frac{n+1}{n}$ which is a new solution for [1], but this method no offer a stronger result like Theorem 1.
Corollary 1.2. If $a, b > 0$ and

$$F_n(a,b) = \begin{cases} \frac{n\left(a^{n+1}-b^{n+1}\right)}{(n+1)(a^n-b^n)} & \text{if } \ a \neq b \\ a & \text{if } \ a = b \end{cases}$$

then $F_n(a,b) \geq \frac{a+b}{2}$.
Proof. In Theorem 1 we take $x = \frac{a}{b}$.
Open Question 1. If $a, b > 0$, then determine all $k \in N$ for which

$$(F_n(a,b))^{2^k} \geq \frac{a^{2^k}+b^{2^k}}{2}$$

for all $n \in N$.
Open Question 2. Determine all $x > 0$ for which

$$n\left(x^{n+1}-1\right)^2 (x^n-1) \leq (n+1)(x^n-1)\left(x^{2n+2}-1\right)$$

for all $n \in N$.
Open Question 3. Determine all $a, b > 0$ such that

$$F_n\left(a^2, b^2\right) \geq F_n^2(a,b)$$

for all $n \in N$.
Open Question 4. Determine all $a_k, b_k > 0$ $(k = 1, 2, ..., m)$ for which

$$\prod_{k=1}^{m} F_n\left(a_k, b_k\right) \geq F_n^m\left(\sqrt[m]{\prod_{k=1}^{m} a_k};\ \sqrt[m]{\prod_{k=1}^{m} b_k}\right)$$

for all $n, m \in N^*$.

Corollary 1.3. If $0 < a < b$, then exist $c, d \in (a, b)$ such that

$$\frac{c^{kn+k-1}}{d^{kn-1}} \geq \frac{a^k + b^k}{2}$$

for all $k, n \in N$.

Proof. In Theorem 1 we take $x = \left(\frac{a}{b}\right)^k$ and we obtain

$$\frac{a^{kn+k} - b^{kn+k}}{a^{kn} - b^{kn}} \geq \frac{(n+1)\left(a^k + b^k\right)}{2n}$$

but from Lagrange theorem exist $c, d \in (a, b)$ such that $a^{kn+k} - b^{kn+k} = (a-b)(kn+k)\, c^{kn+k-1}$ and $a^{kn} - b^{kn} = (a-b)(kn)\, d^{kn-1}$.

Corollary 1.4. If $x > 0$, then

$$(n-1)\left(\frac{x^{kn+k}}{(kn+k+1)^p} - 1\right) \geq (n+1)\left(\frac{x^{kn}}{(kn+1)^p} - \frac{x^k}{(k+1)^p}\right)$$

for all $n, k, p \in N$.

Proof. The inequality from Theorem 1 can be written in the following form:

$$(n-1)\left(x^{n+1} - 1\right) \geq (n+1)\left(x^n - x\right)$$

if $x = t^k$, then

$$\int_0^x (n-1)\left(t^{kn+k} - 1\right) dt \geq \int_0^x (n+1)\left(t^{kn} - t\right) dt \text{ or}$$

$$(n-1)\left(\frac{x^{kn+k}}{kn+k+1} - 1\right) \geq (n+1)\left(\frac{x^{kn}}{kn+1} - \frac{x^k}{k+1}\right)$$

and

$$\int_0^x (n-1)\left(\frac{t^{kn+k}}{kn+k+1} - 1\right) dt \geq \int_0^x (n+1)\left(\frac{t^{kn}}{kn+1} - \frac{t^k}{k+1}\right) dt \text{ or}$$

$$(n-1)\left(\frac{x^{kn+k}}{(kn+k+1)^2}-1\right)\geq(n+1)\left(\frac{x^{kn}}{(kn+1)^2}-\frac{x^k}{(k+1)^2}\right)$$

and this iteration can be continued.

Corollary 1.5. If $t\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, then

$$\frac{(\sin t)^{2n+2}-(\cos t)^{2n+2}}{(\sin t)^{2n}-(\cos t)^{2n}}\geq\frac{n+1}{2n}$$

for all $n\in N^*$.
Proof. In Theorem 1 we take $x=tg^2t$.
Corollary 1.6. If $t\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, then

$$\frac{1-(\sin t)^{2n+2}}{1-(\sin t)^{2n}}+\frac{1-(\cot s)^{2n+2}}{1-(\cot s)^{2n}}\geq\frac{3(n+1)}{2n}$$

for all $n\in N^*$.
Proof. In Theorem 1 we take the substitutions $x=\sin^2 t$ and $x=\cos^2 t$ and we obtain

$$\frac{1-(\sin t)^{2n+2}}{1-(\sin t)^{2n}}+\frac{1-(\cot s)^{2n+2}}{1-(\cot s)^{2n}}\geq\frac{(n+1)\left(1+\sin^2 t\right)}{2n}+$$

$$+\frac{(n+1)\left(1+\cos^2 t\right)}{2n}=\frac{3(n+1)}{2n}$$

Open Question 5. If $x>0$ then determine all $a_k,b_k>0$ $(k=0,1,...,n)$ such that

$$\left(\sum_{k=0}^{n}b_k\right)\left(\sum_{k=0}^{n}a_kx^k\right)\geq\left(\sum_{k=0}^{n}a_k\right)\left(\sum_{k=0}^{n}b_kx^k\right)$$

for all $n\in N^*$.
If $a_k=b_k=1$ $(k=0,1,...,n-1)$, $a_n=1,b_n=0$, then we obtain Corollary 1.1.
Open Question 6. If $x>0$ then determine all $a_k,b_k>0$ $(k=0,1,...,n)$ such that

$$\left(\sum_{k=1}^{n}ka_kx^{k-1}\right)\left(\sum_{k=0}^{n}b_kx^k\right)\geq\left(\sum_{k=1}^{n}kb_kx^{k-1}\right)\left(\sum_{k=0}^{n}a_kx^k\right)$$

for all $n\in N^*$.

Corollary 1.7. If $0 < a < b$, then exist $c, d \in (a, b)$ such that

$$(n-1)\,c^{n+1} + \frac{(n+1)\,(a+b)}{2} \geq (n+1)\,d^n + n - 1$$

for all $n \in N^*$.
Proof. We have $b^{n+2} - a^{n+2} = (b-a)\,(n+2)\,c^{n+1}$,
$b^{n+1} - a^{n+1} = (b-a)\,(n+1)\,d^n$ and

$$\int_a^b 2n\left(x^{n+1} - 1\right) dx \geq \int_a^b (n+1)\left(x^{n+1} + x^n - x - 1\right) dx$$

Open Question 7. If $x > 0$ then determine all $a_k > 0$ $(k = 0, 1, ..., n+m)$ and all $m \in N$, $b_p > 0$ $(p = 0, 1, ..., n)$ such that

$$\frac{\left(\sum\limits_{p=0}^{n} b_p\right)\left(\sum\limits_{k=0}^{n+m} a_k x^k\right)}{\left(\sum\limits_{k=0}^{n+m} a_k\right)\left(\sum\limits_{p=0}^{n} b_p x^p\right)} \geq \frac{x^m + x^{m-1} + ... + x + 1}{m+1}$$

for all $n \in N$, which offer a genertalization of Theorem 1.
Theorem 2. If $x > 0$, then

$$\frac{1 + x + x^2 + ... + x^n}{1 + x + x^2 + ... + x^{n-1}} \geq \frac{(n+1)\,x}{n-1}$$

for all $n \in N$, $n \geq 2$.
Proof. From Theorem 1 we get

$$\frac{x^{n+1} - 1}{x^n - 1} \geq \frac{(n+1)\,(x+1)}{2n} \text{ or}$$

$$(n-1)\left(x^{n+1} - 1\right) \geq (n+1)\,(x^n - 1)\,x \text{ or}$$

$$\frac{x^{n+1} - 1}{x^n - 1} \geq \frac{(n+1)\,x}{n-1}$$

and finally

$$\frac{1 + x + x^2 + ... + x^n}{1 + x + x^2 + ... + x^{n-1}} \geq \frac{(n+1)\,x}{n-1}$$

CONNECTIONS WITH IDENTRIC MEAN

Corollary 1.8. If $I(a,b)=\frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}$, where $0<a<b$ is the identric mean, then

$$\exp\left(\frac{n}{n+1}\int\limits_a^b\frac{1+(\ln x)^2+...+(\ln x)^{2n}}{1+(\ln x)^2+...+(\ln x)^{2n-2}}dx\right)\geq (I(a,b))^{b-a}$$

Proof. In Corollary 1.1 we take $x\to\ln x$ so we obtain

$$\frac{n}{n+1}\int\limits_a^b\frac{1+(\ln x)^2+...+(\ln x)^{2n}}{1+(\ln x)^2+...+(\ln x)^{2n-2}}dx\geq\int\limits_a^b\ln x dx=(b-a)\ln I(a,b)$$

Corollary 1.9. If $0<a<b$, then

$$\exp\left(\frac{2n}{n+1}\int\limits_a^b\frac{1+\ln x+...+(\ln x)^n}{1+\ln x+...+(\ln x)^{n-1}}dx-(b-a)\right)\geq (I(a,b))^{b-a}$$

Proof. In Theorem 1 we take $x\to\ln x$ and so we obtain

$$\frac{2n}{n+1}\int\limits_a^b\frac{1+\ln x+...+(\ln x)^n}{1+\ln x+...+(\ln x)^{n-1}}dx\geq\int\limits_a^b(1+\ln x)dx=b-a+(b-a)\ln I(a,b)$$

Corollary 1.10. If $0<a<b$, then

$$I(a,b)^{2(b-a)}\geq$$

$$\geq\exp\left(b\ln^2 b-a\ln^2 a+b-a-\frac{2n}{n+1}\int\limits_a^b\frac{1+(\ln x)^2+...+(\ln x)^{2n}}{1+(\ln x)^2+...+(\ln x)^{2n-2}}dx\right)$$

Proof. In Theorem 1 we take $x\to\ln^2 x$, therefore

$$\frac{2n}{n+1}\int\limits_a^b\frac{1+(\ln x)^2+...+(\ln x)^{2n}}{1+(\ln x)^2+...+(\ln x)^{2n-2}}dx\geq\int\limits_a^b\left(1+\ln^2 x\right)dx=$$

$$=b-a+\left(x\ln^2 x-2(x\ln x-x)\right)|_a^b=b-a+b\ln^2 b-a\ln^2 a-2(b-a)\ln I(a,b)$$

Corollary 1.11. If $0 < a < b$, then

$$I^3(a,b) \le \exp\left(1 + \frac{b\ln^2 b - a\ln^2 a}{b-a}\right)$$

Proof. We start from

$$\int_a^b \frac{\ln^2 x + 1}{2} dx \ge \int_a^b \ln x dx$$

Corollary 1.12. If $0 < a < b$, then

$$\exp\left(n - 1 + \frac{(-1)^n n!}{b-a}\left(b\sum_{r=0}^{n}\frac{(-\ln b)^r}{r!} - a\sum_{r=0}^{n}\frac{(-\ln a)^r}{r!}\right)\right) \ge I^n(a,b)$$

Proof. We have

$$\int_a^b \frac{n - 1 + (\ln x)^n}{n} dx \ge \int_a^b \ln x dx, \text{ and}$$

$$\int (\ln x)^n dx = (-1)^n n! x \sum_{r=0}^{n} \frac{(-\ln x)^r}{r!}$$

Corollary 1.13. If $0 < a < b$, then

$$\exp\left(\frac{b^2}{2}\left(\ln^2 b - \ln b\right) - \frac{a^2}{2}\left(\ln^2 a - \ln a\right) + A(a,b) + \frac{1}{L(a,b)}\right) \ge I^2(a,b)$$

Proof. We have

$$\int_a^b \left(x\ln^2 x + \frac{1}{x}\right) dx \ge 2\int_a^b \ln x dx, \text{ and}$$

$$A(a,b) = \frac{a+b}{2},\ L(a,b) = \frac{b-a}{\ln b - \ln a}$$

Corollary 1.14. If $0 < a < b$ and $m \in N$, $m \ne n-1$, then

$$\exp\left((-1)^n \frac{n!}{m+1}\left(b^{m+1}\sum_{r=0}^{n}\frac{(-\ln b)^r}{r!(m+1)^{n-r}} - a^{m+1}\sum_{r=0}^{n}\frac{(-\ln a)^r}{r!(m+1)^{n-r}}\right) + \right.$$

$$+\frac{n-1}{n-m-1}\left(b^{1-\frac{m}{n-1}}-a^{1-\frac{m}{n-1}}\right)\Bigg)\geq\left(I\left(a,b\right)\right)^{n(b-a)}$$

Proof. We have

$$\int\limits_a^b\left(x^m\left(\ln x\right)^n+\frac{n-1}{x^{\frac{m}{n-1}}}\right)dx\geq n\int\limits_a^b\ln xdx$$

REFERENCES

[1] Acu, D., *O inegalitate interesanta,* (in Romanian), Mathematical Educational (Educatia Matematica), Vol. 3, Nr. 1-2, 2007, pp. 103-106
[2] Panaitopol,L., Bandila, V., Lascu, M., *Inegalitati,* (in Romanian), Editura Gil, Zalau, 1995, Problem 1.47, pp. 5.
[3] Octogon Mathematical Magazine (1993-2009)

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Euler and music. A forgotten arithmetic function by Euler

József Sándor[26]

Dedicated to the 100^{th} Anniversary of the famous musician David Lerner (1909-)

ABSTRACT. We study certain properties of an arithmetic function by Euler, having application in the theory of music.

MAIN RESULTS

1. Since the time of Ancient Greece, mathematicians and non-mathematicians have tried to find connections between mathematic and music. Especially are well-known the findings of Pythagora and his followers on the relations of natural numbers, the lengths of a vibrating string, and the pitches produced by this string.

The Pythagoreans were interested also in the number mysticism and studied these relations by experimenting with a monochored.

They discovered that a string whose lenght is subdivided in a ratio represented by a fraction of two positive integers produces a note that is in harmony with the note produced by the full string: if the ratio is 1:2 then the result is an octave, with 2:3 one gets a perfect fifth, with 3:4 a perfect fourth, etc.

Of particular importance was the discovery of the so-called Pythagorean comma. In all pitch systems that are based on perfect octaves and perfect fifths there is a discrepancy between the interval of seven octaves and the

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interval of twelve fifths, although both have to be considered as equal in musical terms.
In musical practice the Pythagorean comma causes serious problems. So in the past numerous approaches were developed to find tunings for instruments that reduce these problems to a minimum. The tuning that today is known best and used most often in European music is the equal temperament or well temperament tuning. Tjis tuning become popular during the baroque are and most notably by "The well-tmperated Clavier", Bach‘s grand collection of preludes and fugues that impressively demonstrated the possibility of letting all keys sound equally well. Of course one could say also equally bad, since in the equal temperament none of the intervals but the octaves are perfect any more, i.e. the ratio mentoned above are no longer valid.

In the equal temperament every octave is subdivided into twelve half-steps allof which have the same frequency ratio of $2^{1/12}$, where in the terminology above the 2 is to be reads as 2:1, i.e., the frequency ratio of an octave. All frequencies of the pitches of the equal tempered twelve-tone scale can be expressed by the geometric sequence

$$f_i = f_0 \cdot 2^{i/12},$$

where f_0 is a fixed frequency, e.g., the standard pitch $a'\,(440Hz)$ and i is the half-step distance of the target note from the note with the frequency f_0. Then, f_i is the frwquency of the target note.
In odern times, Leonard Euler (1707-1783) was one of the first who tried to use mathematical methods in order to deal with the consonance/dissonance problem. In his work, too, ratios of natural numbers, reflecting frequency ratio of intervals, play an important role. In his paper "Tentamen novae theoriae musicae" (see [1]) of 1739, Euler defines the following arithmetic function ("Gradus-suavitalis function"). Let n be a positive integer and suppose its prime factorization is

$$n = p_1^{a_1} p_2^{a_2} ... p_r^{a_r} \ \ (p_i \text{ distinct primes, } a_i \geq 1)$$

Put

$$E\,(n) = 1 + \sum_{k=1}^{r} a_k\,(p_k - 1) \tag{1}$$

Let

$$E\,(1) = 1, \text{ by definitin}$$

In what follows, we will study this forgotten arithmetical function by Euler, but first note that for musical application Euler defined the function E also for the reduced fraction $\frac{x}{y}$ by

$$E\left(\frac{x}{y}\right) = E\,(x \cdot y)$$

Inserting fractions that represent ratios of musical intervals into his formula, we obtain the following values:

octave : $E\left(\frac{1}{2}\right) = 2$
fifth : $E\left(\frac{2}{3}\right) = 4$
fourth : $\left(\frac{3}{4}\right) = 5$
major third : $E\left(\frac{4}{5}\right) = 7$
minor third : $E\left(\frac{5}{6}\right) = 8$
major second : $E\left(\frac{9}{10}\right) = 10$
minor second : $E\left(\frac{15}{16}\right) = 11$
tritone : $E\left(\frac{32}{45}\right) = 14$

According to Euler, these numbers are a measure for the pleasentness of an interval; the smaler the value the more pleasing the interval. Indeed, this is more or less in a accordance with our European listening habit, with one exception: the perfect fourth is heard as a dissonance in some contrapuntal and functional harmonic contexts (see [3]).

Remark. *Euler used the notation* $\Gamma\,(n)$ *for his function, in place of* $E\,(n)$*. We have adopted this notation, as there is another important function introduced also by Euler in mathematics, the famous "Gamma function."*

2. In what follows we will study the arithmetical function $E\,(n)$ of positive integers, defined by relation (1).

If the canonical factorization of n is $n = p_1^{a_1}...p_r^{a_r}$, then there are some well-known arithmetical functions, which are connected to the function $E\,(n)$.

Let $p\,(n)\,, P\,(n)$ denote respectivelly the least and the greates prime factors of n.

Let $\omega\,(n)\,, \Omega\,(n)$ denote the number of distinct, respectivelly total number, of prime factors of n. Then clearly, $\omega\,(n) = r$, $\Omega\,(n) = a_1 + ... + a_r$.

The following arithmetical function $B\,(n)$ has been intensively studied, too (see e.g. [4], [2]):

$$B(n)=\sum_{k=1}^{r}a_kp_k$$

Proposition 1. One has for $n>1$

$$E(n)=1+B(n)-\Omega(n) \tag{2}$$

$$E(n)\geq 1+\Omega(n) \tag{3}$$

Proof. Relation (2) is a consequence of (1) and the above introduced arithmetic functions. As

$$B(n)\geq\sum_{k=1}^{r}a_k\cdot 2=2\Omega(n),$$

inequality (3) follows by (2).

Proposition 2. For $n\geq 2$ one has the double inequality

$$1+\Omega(n)(p(n)-1)\leq E(n)\leq 1+\Omega(n)(P(n)-1) \tag{4}$$

Proof. Remark that

$$B(n)\leq\max\{p_1,...,p_r\}\sum_{k=1}^{r}a_k=P(n)\Omega(n),$$

and similarly

$$B(n)\geq\min\{p_1,...,p_r\}\sum_{k=1}^{r}a_k=P(n)\Omega(n)$$

From identity (2) we can deduce the double inequality (4).

Remark. *(4) may be written also as*

$$p(n)\leq 1+\frac{E(n)-1}{\Omega(n)}\leq P(n) \tag{5}$$

Remarking that $E(p)=1+(p-1)=p$ *for each prime* p*, one could ask for the fixed points of the function* E.

Proposition 3. The fix points of the function E are only the prime numbers. In other words, one has

$$E(n)=n \text{ if } n= \text{ prime}$$

Proof. We need the following two lemmas.

Lemma 1. $p^a \geq pa$ for all $p \geq 2$, $a \geq 1$; with equality only for $p = 2$, $a = 1$.

Proof. The inequality $p^{a-1} \geq a$ is true, as $p^{a-1} \geq 2^{a-1} \geq a$, which follows at once by mathematical induction.

Lemma 2. Let $x_i > 1 \left(i = \overline{1, r}\right)$. Then one has

$$r + x_1 x_2 ... x_r \geq 1 + x_1 + ... + x_r \tag{6}$$

with equality only for $r = 1$.

Proof. For $r = 1$ there is equality; while for $r = 2$ the inequalityis strict, as $2 + x_1 x_2 > 1 + x_1 + x_2$ by $(x_1 - 1)(x_2 - 1) > 0$, valid as $x_1 - 1 > 0$, $x_2 - 1 > 0$.

Assume now that (6) is true for $r \geq 2$ fixed, with a strict inequality. Then for $x_{r+1} > 1$ one has

$$r + 1 + x_1 x_2 ... x_r x_{r+1} > r + 1 + x_{r+1} (1 - r + x_1 + ... + x_r) =$$

$$= r - r x_{r+1} + (1 + x_{r+1} + x_1 + ... + x_r) + (x_{r+1} - 1)(x_1 + ... + x_r) >$$

$$> 1 + x_1 + ... + x_r + x_{r+1}$$

as

$$r - r x_{r+1} + (x_{r+1} - 1)(x_1 + ... + x_r) = (x_{r+1} - 1)(x_1 + ... + x_r - r) > 0$$

as $x_1 + ... + x_r > r$ and $x_{r+1} > 1$. By induction, we get that (6) is true for all r.

Proof of Proposition 3. One has

$$E(n) = 1 + a_1 (p_1 - 1) + ... + a_r (p_r - 1) \leq a_1 p_1 + ... + a_r p_r - r + 1 \leq p_1^{a_1} ... p_r^{a_r} = n,$$

with equality only for $r = 1, a_1 = 1$, i.e. when n is a prime. We have used Lemma 1 and Lemma 2.

Proposition 4. One has for $n \geq 2$,

$$E(n!) \leq 1 + n\pi(n), \tag{7}$$

where $\pi(n)$ denotes the number of all primes $\leq n$.

Proof. Let $n! = \prod_{p|n!} p^{a_p}$ be the prime factorization of $n!$.
By Legendre‘s theorem one has

$$a_p = \sum_{j=1}^{\infty}\left[\frac{n}{p^j}\right] \leq \sum_{j=1}^{\infty}\frac{n}{p^j} = \frac{n}{p}\left(1+\frac{1}{p}+\frac{1}{p^2}+...\right) = \frac{n}{p}\cdot\frac{1}{1-\frac{1}{p}} = \frac{n}{p-1}$$

Thus

$$E\left(n!\right) = 1 + \sum a_p\left(p-1\right)$$

where in the sum we have $\omega\left(n!\right)$ terms. Remark that $\omega\left(n!\right) = \pi\left(n\right)$, as in $n! = 1\cdot 2\cdot ...\cdot n$ the number of distinct prime divisors is exactly the number of primes $\leq n$. As $a_p \leq \frac{n}{p-1}$, relation (7) follows.
Finally, we will obtain the overage order of the function $E\left(n\right)$:
Proposition 5. One has

$$\sum_{n\leq x} E\left(n\right) = \frac{\pi^2}{12}\cdot\frac{x^2}{\log x} + O\left(\frac{x^2}{\log^2 x}\right) \tag{8}$$

Proof. By the famous result of Hardy and Ramanujan (see e.g. [1]) one has

$$\sum_{n\leq x} \Omega\left(n\right) = x\log\lg x + K\cdot x + O\left(\frac{x}{\log x}\right) \tag{9}$$

where K is a constant.
On the other hand, by a result of Alladi and Erdős (see [2], [4]) one has

$$\sum_{n\leq x} B\left(n\right) = \frac{\pi^2}{12}\cdot\frac{x^2}{\lg x} + O\left(\frac{x^2}{\log^2 x}\right) \tag{10}$$

Now, by Proposition 1, relation (2) the expression (8) follows by remarking that

$$x\lg\lg x + K\cdot x = O\left(\frac{x^2}{\log^2 x}\right)$$

and

$$O\left(\frac{x}{\lg x}\right) = O\left(\frac{x^2}{\log^2 x}\right)$$

REFERENCES

[1] Euler, L., *Opera omnia. Series tertia: Opera physica.,* Vol. 1, Commentationes physicae ad physicam generalem et ad theoriam soni pertinentes. Ediderunt E. Bernoulli, R. Bernoulli, F. Rudio, A. Speiser, Leipzig, B.G. Teubner, 1926.
[2] Alladis, K., and Erdős, P., *On an additive arithmetic function,* Pacific J. Math. 71(1977), pp. 275-294.
[3]. Mazzola, G., *Geometrie der Tone. Elemente der mathematischen Musiktheorie,* Basel: Birkhauser (1990).
[4]. Sándor, J., Mitrinovic, D.S., and Crstici, B., *Handbook of number theory I,* Springer Verlag, 2006

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About a partition inequality

Mihály Bencze[27]

ABSTRACT. In this paper we present a new type inequality cathegory generated from a partition.

MAIN RESULTS

Theorem 1. If $x_k > 0$ $(k = 1, 2, ..., n)$ and $f_k : R^m \to (0, +\infty)$ $(k = 1, 2, ..., n)$, such that

$$\left(\sum_{k=1}^{n} x_k\right)^m = \sum_{k=1}^{m} f_k\left(x_1, x_2, ..., x_n\right)$$

then

$$\left(\sum_{k=1}^{n} x_k\right)^{\frac{m}{\alpha}} \geq n^{\frac{1}{\alpha}-1} \sum_{k=1}^{n} \left(f_k\left(x_1, x_2, ..., x_n\right)\right)^{\frac{1}{\alpha}}$$

for all $\alpha \in (-\infty, 0] \cup [1, +\infty)$ and if $\alpha \in (0, 1)$, then holds the reverse inequality.

Proof. Using the Jensen's inequality we have:

$$\left(\sum_{k=1}^{n} x_k\right)^m = \sum_{k=1}^{m} f_k\left(x_1, x_2, ..., x_n\right) = \sum_{k=1}^{m} \left(\left(f_k\left(x_1, x_2, ..., x_n\right)\right)^{\frac{1}{\alpha}}\right)^{\alpha} \geq$$
$$\geq n^{1-\alpha} \left(\sum_{k=1}^{n} \left(f_k\left(x_1, x_2, ..., x_n\right)\right)^{\frac{1}{\alpha}}\right)^{\alpha}$$

or

$$\left(\sum_{k=1}^{n} x_k\right)^{\frac{m}{\alpha}} \geq n^{\frac{1}{\alpha}-1} \sum_{k=1}^{n} \left(f_k\left(x_1, x_2, ..., x_n\right)\right)^{\frac{1}{\alpha}}$$

for all $\alpha\,(-\infty, 0] \cup [1, +\infty)$. If $\alpha \in [0, 1]$ holds the reverse inequality.

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Corollary 1. If $x_k > 0$ $(k = 1, 2, ..., 2n + 1)$, then

$$\sqrt{2n+1}\sum_{k=1}^{2n+1} x_k \geq \sum_{cyclic} \sqrt{x_1^2 + 2x_{i_1}x_{j_1} + ... + 2x_{i_n}x_{j_n}}$$

where $1 \leq i_t, j_t \leq 2n + 1$, $i_t \neq j_t$ $(t = 1, 2, ..., n)$, $i_1 \neq i_2 \neq ... \neq i_n$, $j_1 \neq j_2 \neq ... \neq j_n$ and the cyclic sums explicited form is the following

$$\sqrt{x_1^2 + 2x_{i_1}x_{j_1} + ... + 2x_{i_n}x_{j_n}} + \sqrt{x_2^2 + 2x_{i_1+1} + x_{j_1+1} + ... + 2x_{i_m+1} + x_{j_m+1}} + ...$$

Proof. In Theorem 1 we take $m = \alpha = 2$,
$f_1(x_1, x_2, ..., x_{2n+1}) = x_1^2 + 2x_{i_1}x_{j_1} + ... + 2x_{i_n}x_{j_n}$ etc.

Corollary 1.1. If $x, y, z > 0$, then

$$\sqrt{3}\,(x + y + z) \geq \sqrt{x^2 + 2yz} + \sqrt{y^2 + 2zx} + \sqrt{z^2 + 2xy}$$

Proof. In Corollary 1 we take $n = 1$,
$x_1 = x, x_2 = y, x_3 = z, f_1(x, y, z) = x^2 + 2yz$ etc.

Corollary 1.2. If $x, y, z, t, u > 0$, then

$$\sqrt{5}\,(x + y + z + t + u) \geq \sum \sqrt{x^2 + 2\,(y + u)\,z}$$

Proof. In Corollary 1 we take $n = 2$,
$x_1 = x, x_2 = y, x_3 = z, x_4 = t, x_5 = u, f_1(x, y, z, t, u) = x^2 + 2\,(y + u)\,z$ etc.

Corollary 2. If $x_k > 0$ $(k = 1, 2, ..., 2n)$, then

$$\sqrt{2n}\sum_{k=1}^{2n} x_k \geq \sum_{cyclic} \sqrt{x_1^2 + 2\left(x_{i_1}x_{j_1} + ... + x_{i_{n-1}x_{j_{n-1}}}\right) + x_{i_n}x_{j_n}}$$

Proof. In Theorem 1 we take $m = \alpha = 2$,
$f_1(x_{11}, ..., x_{2n}) = x_1^2 + 2\left(x_{i_1}x_{j_1} + ... + x_{i_{n-1}}x_{j_{n-1}}\right) + x_{i_n}x_{j_n}$ etc.

Corollary 2.1. If $x, y > 0$, then

$$\sqrt{2}(x + y) \geq \sqrt{x^2 + xy} + \sqrt{y^2 + yx}$$

Proof. In Corollary 2 we take $n = 1$ and $f_1(x, y) = x^2 + xy$.

Corollary 2.2. If $x, y, z, t > 0$, then

$$2\,(x + y + z + t) \geq \sqrt{x^2 + y\,(2z + t)} + \sqrt{y^2 + z\,(2t + x)} + \sqrt{z^2 + t\,(2x + y)} +$$

$$+\sqrt{t^2+x\left(2y+z\right)}$$

Proof. In Corollary 2 we take $n=2$ and $f_1\left(x,y,z,t\right)=x^2+y\left(2z+t\right)$ etc.
Corollary 3. If $x,y,z>0$, then

$$\sqrt[3]{9}\left(x+y+z\right)\geq\sum\sqrt[3]{x^3+3\left(x+z\right)y^2+2xyz}$$

Proof. In Theorem 1 we take
$n=3,m=3,\alpha=3,f_1\left(x,y,z\right)=x^3+3\left(x+z\right)y^2+2xyz$ etc.

REFERENCE

[1] Octogon Mathematical Magazine (1993-2009)

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A divisibility property of $\sigma_k(n)$

József Sándor[28]

ABSTRACT. Let $\sigma_k(n)$ be the sum of k^{th} powers of divisors of n. We will prove that if p is a prime of the form $4m+3$ $(m \geq 0)$, with $2m+1 = prime$, then $\sigma_{2m+1}(pk-1)$ is divisible by p for any $k \geq 1$.

MAIN RESULTS

Let $\sigma_a(n) = \sum\limits_{d|n} d^a$ be the sum of a^{th} powers of divisors of the positive integer n. The main result of this note is contained in the following.

Theorem. Let p be an odd prioe of the form $p = 4m+3$, where $m \geq 0$ is an integer. Assume that $2m+1$ is prime. Then for any integer $k \geq 1$,

$$\sigma_{2m+1}(pk-1) \text{ is divisible by } p$$

Proof. Let $q = 2m+1$. Then $p-1 = 4m+2 = 2q$, so $p = 2q+1$. By Fermat‘s divisibility theorem we have that

$$x^{2q} \equiv 1 \pmod{p} \text{ for any } 0 < x < p \tag{1}$$

Indeed, as $(x,p) = 1$ and $2q = p-1$, (1) holds true.

Now, let d and d' two complementar divisors of $pk-1$, i.e. $pk-1 = d \cdot d'$, where $d = pA + r$, $d' = pB + r'$ $(0 < r,\ r' < p)$.

Since $d^q + d'^q \equiv r^q + r'^q \pmod{p}$; and as clearly $rr' \equiv -1 \pmod{p}$; it is sufficient to prove that

$$r^q + r'^q \equiv 0 \pmod{p} \tag{2}$$

Since $rr' \equiv -1 \pmod{p}$ and q is odd, so to have (2) satisfied, we have to be true the following:

$$r^{2q} \equiv 1 \pmod{p} \tag{3}$$

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As $0 < r < p$, (3) holds true by relation (1).
We have finally to notice that $pk-1$ cannot be a perfect square, as if $pk-1=t^2$, then p would be a divisor of t^2+1. This is impossible, since $p \equiv 3 \pmod 4$ (see e.g. [1]).
As $\sigma_q(n) = \sum\limits_{d|n, d<\sqrt{n}} (d^q + d'^q)$, by (2) the proof of the theorem is finished.

Corollaries.
1). 7 divides $\sigma_3(7k-1)$ $(m=1)$
2). 11 divides $\sigma_5(11k-1)$ $(m=2)$

Remark. *Though* $2m+1=1$ *for* $m=0$ *is not a prime, the proof above works, so we get that*

$$\sigma(3k-1) \text{ is divisible by } 3$$

For an application of this result, see e.g. [2].

REFERENCES

[1] Niven, I., and Zuckerman, H.S., *An introduction to the theory of numbers*, Hungarian translations 1978 by Muszaki Kiado, Budapest.
[2] Sandor, J., *On the composition of some arithmetic functions,* Studia Univ. Babes-Bolyai, Math. 34(1984),pp. 7-14.

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New refinements for AM-HM type inequality

Mihály Bencze and D.M. Bătineţu-Giurgiu[29]

ABSTRACT. In this paper we present a new generalization, and a new refinement for AM-HM inequality based on ideas of [1] and [2.]

MAIN RESULTS

Let be $a_k, b_k, x_k > 0$ $(k = 1, 2, ..., n)$ and $A_n = \frac{1}{n}\sum\limits_{k=1}^{n} x_k$, $H_n = \frac{n}{\sum\limits_{k=1}^{n}\frac{1}{x_k}}$, $C_n = \sum\limits_{k=1}^{n} a_k x_k$, $D_n = \sum\limits_{k=1}^{n}\frac{b_k}{x_k}$. It is known that
$A_n \geq H_n \Leftrightarrow \left(\sum\limits_{k=1}^{n} x_k\right)\left(\sum\limits_{k=1}^{n}\frac{1}{x_k}\right) \geq n^2$.

Theorem 1. Holds the following inequalities

$$C_n D_n \geq \sum_{k=1}^{n} a_k b_k + n\sqrt[n]{\prod_{k=1}^{n}\frac{b_k\left(C_n - a_k x_k\right)}{x_k}} \geq$$

$$\geq \sum_{k=1}^{n} a_k b_k + n\left(n-1\right)\sqrt[n]{\prod_{k=1}^{n} a_k b_k}$$

Proof 1. We have:

$$C_n D_n = \sum_{k=1}^{n} a_k b_k + \sum_{k=1}^{n}\frac{b_k\left(C_n - a_k x_k\right)}{x_k} \geq \sum_{k=1}^{n} a_k b_k + n\sqrt[n]{\prod_{k=1}^{n}\frac{b_k\left(C_n - a_k x_k\right)}{x_k}} \geq$$

$$\geq \sum_{k=1}^{n} a_k b_k + n\sqrt[n]{\prod_{k=1}^{n}\frac{b_k}{x_k}\left(\left(n-1\right)\sqrt[n-1]{\prod_{k=1}^{n}\frac{a_1 x_1 ... a_n x_n}{a_k x_k}}\right)} =$$

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$$= \sum_{k=1}^{n} a_k b_k + n \sqrt[n]{(n-1)^n \prod_{k=1}^{n} \frac{b_k}{x_k} \sqrt[n-1]{\frac{\prod\limits_{k=1}^{n} (a_k x_k)^n}{\prod\limits_{k=1}^{n} a_k x_k}}} =$$

$$= \sum_{k=1}^{n} a_k b_k + n\,(n-1) \sqrt[n]{\prod_{k=1}^{n} a_k b_k}$$

Proof 2. We have:

$$C_n D_n = \sum_{k=1}^{n} a_k b_k + \sum_{k=1}^{n} \frac{b_k\,(C_n - a_k x_k)}{x_k} \geq \sum_{k=1}^{n} a_k b_k + (n-1) \sum_{k=1}^{n} \frac{b_k}{x_k} \sqrt[n-1]{\frac{\prod\limits_{k=1}^{n} a_k x_k}{a_k x_k}} \geq$$

$$\geq \sum_{k=1}^{n} a_k b_k + n\,(n-1) \sqrt[n]{\prod_{k=1}^{n} \frac{b_k}{x_k} \sqrt[n-1]{\frac{\prod\limits_{k=1}^{n} a_k x_k}{a_k x_k}}} = \sum_{k=1}^{n} a_k b_k + n\,(n-1) \sqrt[n]{\prod_{k=1}^{n} a_k b_k}$$

Corollary 1.1. If $x_k > 0\ (k = 1, 2, ..., n)$, then

$$\left(\sum_{k=1}^{n} x_k\right) \left(\sum_{k=1}^{n} \frac{1}{x_k}\right) \geq n + n \sqrt[n]{\prod_{cyclic} \frac{x_2 + x_3 + ... + x_n}{x_1}} \geq n^2$$

Proof. In Theorem 1 we take $a_k = b_k = 1\ (k = 1, 2, ..., n)$.

Corollary 1.2. If $x_k, a_{k,} b_k > 0\ (k = 1, 2, ..., n)$ and

$$\prod_{k=1}^{n} a_k = \prod_{k=1}^{n} b_k = 1, \text{ then}$$

$$\left(\sum_{k=1}^{n} a_k x_k\right) \left(\sum_{k=1}^{n} \frac{b_k}{x_k}\right) \geq \sum_{k=1}^{n} a_k b_k + n \sqrt[n]{\prod_{cyclic} \frac{b_1\,(a_2 x_2 + ... + a_n x_n)}{x_1}} \geq n^2$$

Proof. It‘s immediately from the Theorem 1.

Corollary 1.3. If $x_k > 0$ $(k = 1, 2, ..., n)$ and $\alpha \in R$, then

$$\left(\sum_{k=1}^{n} k^{\alpha} x_k\right)\left(\sum_{k=1}^{n} \frac{k^{\alpha}}{x_k}\right) \geq \sum_{k=1}^{n} k^{2\alpha} + n\,(n-1)\,\sqrt[n]{(n!)^{2\alpha}}$$

Proof. We take $a_k = b_k = k^{\alpha}$ $(k = 1, 2, ..., n)$. If $\alpha = \frac{1}{2}$, $\alpha = 1$, $\alpha = \frac{3}{2}$, $\alpha = 2$, then we obtain the followings:

1). $\left(\sum\limits_{k=1}^{n} \sqrt{k} x_k\right)\left(\sum\limits_{k=1}^{n} \frac{\sqrt{k}}{x_k}\right) \geq \frac{n(n+1)}{2} + n\,(n-1)\,\sqrt[n]{n!}$

2). $\left(\sum\limits_{k=1}^{n} k x_k\right)\left(\sum\limits_{k=1}^{n} \frac{k}{x_k}\right) \geq \frac{n(n+1)(2n+1)}{6} + n\,(n-1)\,\sqrt[n]{(n!)^2}$

3). $\left(\sum\limits_{k=1}^{n} k\sqrt{k} x_k\right)\left(\sum\limits_{k=1}^{n} \frac{k\sqrt{k}}{x_k}\right) \geq \frac{n^2(n+1)^2}{4} + n\,(n-1)\,\sqrt[n]{(n!)^3}$

4). $\left(\sum\limits_{k=1}^{n} k^2 x_k\right)\left(\sum\limits_{k=1}^{n} \frac{k^2}{x_k}\right) \geq \frac{n(n+1)(2n+1)\left(3n^2+3n-1\right)}{30} + n\,(n-1)\,\sqrt[n]{(n!)^4}$

Theorem 2. We have the following inequalities:

$$C_n D_n \geq \sum_{k=1}^{n} a_k b_k + n^{1-\alpha}\left(\sum_{k=1}^{n}\left(\frac{b_k\,(C_n - a_k x_k)}{x_k}\right)^{\frac{1}{\alpha}}\right)^{\alpha} \geq$$

$$\geq \sum_{k=1}^{n} a_k b_k + n\,(n-1)\,\sqrt[n]{\prod_{k=1}^{n} a_k b_k}$$

for all $\alpha \in (-\infty, 0] \cup [1, +\infty)$ and

$$C_n D_n \leq \sum_{k=1}^{n} a_k b_k + n^{1-\alpha}\left(\sum_{k=1}^{n}\left(\frac{b_k\,(C_n - a_k x_k)}{x_k}\right)^{\frac{1}{\alpha}}\right)^{\alpha}$$

for all $\alpha \in (0, 1)$.

Proof. We have

$$C_n D_n = \sum_{k=1}^{n} a_k b_k + \sum_{k=1}^{n}\left(\left(\frac{b_k\,(C_n - a_k x_k)}{x_k}\right)^{\frac{1}{\alpha}}\right)^{\alpha} \geq$$

$$\geq \sum_{k=1}^{n} a_k b_k + n^{1-\alpha}\left(\sum_{k=1}^{n}\left(\frac{b_k\,(C_n - a_k x_k)}{x_k}\right)^{\frac{1}{\alpha}}\right)^{\alpha} \geq$$

$$\geq \sum_{k=1}^{n} a_k b_k + n(n-1)\sqrt[n]{\prod_{k=1}^{n} a_k b_k}$$

Theorem 3. We have the following inequalities:

$$C_n D_n \geq \sum_{k=1}^{n} a_k b_k + (n-1)\, n^{1-\alpha}\left(\sum_{k=1}^{n}\left(\frac{b_k}{x_k}\sqrt[n-1]{\frac{\prod\limits_{k=1}^{n} a_k x_k}{a_k x_k}}\right)^{\frac{1}{\alpha}}\right)^{\alpha} \geq$$

$$\geq \sum_{k=1}^{n} a_k b_k + n(n-1)\sqrt[n]{\prod_{k=1}^{n} a_k b_k}$$

for all $\alpha \in (-\infty, 0] \cup [1, +\infty)$.

Proof. We have

$$C_n D_n = \sum_{k=1}^{n} a_k b_k + \sum_{k=1}^{n} \frac{b_k (C_n - a_k x_k)}{x_k} \geq$$

$$\geq \sum_{k=1}^{n} a_k b_k + (n-1)\sum_{k=1}^{n}\left(\left(\frac{b_k}{x_k}\sqrt[n-1]{\frac{\prod\limits_{k=1}^{n} a_k x_k}{a_k x_k}}\right)^{\frac{1}{\alpha}}\right)^{\alpha} \geq$$

$$\geq \sum_{k=1}^{n} a_k b_k + (n-1)\, n^{1-\alpha}\left(\sum_{k=1}^{n}\left(\frac{b_k}{x_k}\sqrt[n-1]{\frac{\prod\limits_{k=1}^{n} a_k x_k}{a_k x_k}}\right)^{\frac{1}{\alpha}}\right)^{\alpha} \geq$$

$$\geq \sum_{k=1}^{n} a_k b_k + n(n-1)\sqrt[n]{\prod_{k=1}^{n} a_k b_k}$$

REFERENCES

[1] Bencze, M., *New Cauchy-type inequalities*, Octogon Mathematical Magazine, Vol. 16, No. 1, April 2008, pp. 137-140.
[2] Bencze, M., *New refinements for Cauchy-Schwarz inequality*, Octogon Mathematical Magazine, Vol. 13, No. 1, April 2005, pp. 139-149.

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Characteristics of triangular numbers

K.P.Pandeyend[30]

INTRODUCTION

In this paper we have studied about the characteristics of the triangular numbers which are of course positive integers obtained by taking the sum of consecutive positive integers starting from unity.

DISCUSSION

Theorem 1. The sum of any two consecutive triangular numbers is necessarily a perfect square.
Proof. We have

$$T_n + T_{n+1} = \frac{n(n+1)}{2} + \frac{(n+1)(n+2)}{2} = \frac{(n+1)(n+n+2)}{2} =$$

$$= (n+1)(n+1) = (n+1)^2$$

which is a perfect square for any positive integer n, hence the result.
Theorem 2. Any triangular number can never be expressed as the sum of two consecutive squares.
Proof. Supposing contrary, let for any positive integer n, $\left\{n^2 + (n+1)^2\right\}$ is a triangular number, then definitely

$$\begin{aligned} & 8\left\{n^2+(n+1)^2\right\}+1 && \text{must be a perfect square,} \\ & \quad \Rightarrow 8\left\{n^2+(n+1)^2\right\}+1 = m^2 && \text{(say)} \\ & \Rightarrow 8\left(n^2+n^2+2n+1\right)+1 = m^2 && \\ & \Rightarrow 16n^2+16n+9 = m^2 && \\ & \Rightarrow (4n+2)^2+5 = m^2 && \\ & \Rightarrow (4n+2)^2 = m^2-5 && \\ & \qquad\qquad = 9-5 = 4 && \text{for } m=3, \\ & \Rightarrow 4n+2 = 2 \Rightarrow n = 0 && \end{aligned}$$

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which is a contradiction of the fact that $n > 0$, hence the result.

Theorem 3. The product of any two consecutive triangular numbers can never be a perfect square.

Proof. For any positive integer n, we have

$$T_n T_{n+1} = \frac{n(n+1)}{2}\frac{(n+1)(n+2)}{2} = \frac{(n+1)^2(n^2+2n)}{4} =$$

$$= \frac{(n+1)^2\left\{(n+1)^2-1\right\}}{4}$$

due to the term $\left\{(n+1)^2-1\right\}$, the R.H.S. can never be a perfect square for any positive integer n, hence the result.

Theorem 4. The product of any two consecutive triangular numbers is just the half of another triangular number.

Proof. Let n be any positive integer, then we have

$$T_n T_{n+1} = \frac{n(n+1)}{2}\frac{(n+1)(n+2)}{2} = \frac{(n+1)^2(n^2+2n)}{4} =$$

$$= \frac{\left\{(n^2+2n)+1\right\}(n^2+2n)}{4} = \frac{1}{2}\left\{\frac{\left\{(n^2+2n)+1\right\}(n^2+2n)}{2}\right\} =$$

$$= \frac{1}{2}\frac{m(m+1)}{2}, \text{ where } n^2+2n = m \text{ (say) } \frac{1}{2}T_m,$$

hence the result.

Theorem 5. Product of any two alternate triangular numbers is just the double of another triangular number.

Proof. Let n be any positive integer, then

$$T_n T_{n+2} = \frac{n(n+1)}{2}\frac{(n+2)(n+3)}{2} = \frac{n(n+3)(n+1)(n+2)}{4} =$$

$$= \frac{(n^2+3n)(n^2+3n+2)}{4} = \left(\frac{n^2+3n}{2}\right)\left(\frac{n^2+3n}{2}+1\right) =$$

$$= 2\left\{\frac{\left(\frac{n^2+3n}{2}\right)\left\{\left(\frac{n^2+3n}{2}\right)+1\right\}}{2}\right\} = 2T_m, \text{ where } \left(\frac{n^2+3n}{2}\right) = m.$$

Hence the result.

Theorem 6. For $n \geq 2$, we have

$$T_{(n+1)^2} - T_{n^2} = T_{n+1}^2 - T_{n-1}^2$$

Proof. We have

$$T_{(n+1)^2} - T_{n^2} = \frac{(n+1)^2 \left\{(n+1)^2 + 1\right\}}{2} - \frac{n^2 \left(n^2+1\right)}{2} =$$

$$= \frac{\left(n^2+2n+1\right)\left(n^2+2n+2\right) - n^2\left(n^2+1\right)}{2} = \frac{2\left(2n^3+3n^2+3n+1\right)}{2} =$$

$$= 2n^3 + 3n_2 + 3n + 1$$

And,

$$T_{n+1}^2 - T_{n-1}^2 = \left\{\frac{(n+1)(n+2)}{2}\right\}^2 - \left\{\frac{(n-1)n}{2}\right\}^2 = \frac{(n+1)^2(n+2)^2}{4} -$$

$$-\frac{\left(n^2-n\right)^2}{4} = \frac{4\left(2n^3+3n^2+3n+1\right)}{4} = 2n^3+3n^2+3n+1$$

Thus

$$T_{(n+1)^2} - T_{n^2} = T_{n+1}^2 - T_{n-1}^2.$$

Hence the result.

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A double-inequality for $\sigma_k(n)$

József Sándor[31]

ABSTRACT. Let $\sigma_k(n)$ denote the sum of k^{th} powers of divisors of n. We prove that for $k > 1$ and $n > 1$ one has the double inequality
$\sigma^k(n)/d^{k-1}(n) < \sigma_k(n) < n^k \cdot \zeta(k)$, where $d(n)$ is the number of divisors of n; and $\zeta(k)$ is the Riemann zeta function value at k. Certain corollaries, which improve known bounds are pointed out, too.

MAIN RESULTS

Let $\sigma_k(n) = \sum\limits_{d|n} d^k$ denote the sum of k^{th} powers of divisors of n.
Here $k \geq 1$ and $n \geq 1$ are positive integers; but we can assume that k is a real number. The main aim of this note is to prove the following result:
Theorem. Let $n > 1$ and $k > 1$. Then

$$\frac{(\sigma(n'))^k}{(d(n'))^{k-1}} < \sigma_k(n) < n^k \cdot \zeta(k) \tag{1}$$

where $\zeta(k) = \sum\limits_{d=1}^{\infty} \frac{1}{d^k}$ is the value of the Riemann zeta function ζ at k.
Particularly, one has

$$\frac{(\sigma(n'))^2}{d(n)} < \sigma_2(n) < \frac{\pi^2}{6} \cdot n^2 \tag{2}$$

Proof. Since the function $x \to x^k$ is strictly convex for $k > 1$ we get that

$$\left(\frac{x_1 + ... + x_r}{r}\right)^k \leq \frac{x_1^k + ... + x_r^k}{r} \ (x_i > 0, r \geq 1),$$

we get that

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$$(x_1 + ... + x_r)^k \leq r^{k-1} \left(x_1^k + ... + x_r^k\right), \tag{3}$$

with equality only for $r = 1$ or $x_1 = ... = x_r$. When $x_i = d_i$ $\left(i = \overline{1, d(n)}\right)$ are the distinct divisors of n, then for $n > 1$ we get from (3)

$$(\sigma(n))^k < (d(n))^{k-1} \cdot \sigma_k(n),$$

which implies the left side of inequality (1).
On the other hand, remark that

$$\sigma_k(n) = \sum_{d|n} d^k = \sum_{d|n} \left(\frac{n}{d}\right)^k = n^k \sum_{d|n} \frac{1}{d^k} \leq n^k \sum_{d|n} \frac{1}{d^k} < n^k \sum_{d=1}^{\infty} \frac{1}{d^k} = n^k \cdot \zeta(k)$$

Remark. *The proof shows that (1) holds true for any real number* $k > 1$.
Since $\zeta(2) = \frac{\pi^2}{6}$, for $k = 2$ we get relation (2).
Corollary. For any $n > 1$ and $k > 1$ one has

$$\frac{\sigma(n)}{n} < (d(n))^{1-\frac{1}{k}} (\zeta(k))^{\frac{1}{k}} \tag{4}$$

Particularly, as $d(n) < 4n^{1/3}$ (see [3]), for $k = 2$ we get fro (4) that

$$\frac{\sigma(n)}{n} < \sqrt{d(n) \cdot \frac{\pi^2}{6}} < \pi\sqrt{\frac{2}{3}} \cdot n^{1/6} \tag{5}$$

Since $\pi\sqrt{\frac{2}{3}} < 2,6$; and $\frac{6}{\pi^2} \approx 0,609 < 0,7$; inequality (5) refines the V. Annapurna (see [2]) result

$$\frac{\sigma(n)}{n} < \frac{6}{\pi^2} \cdot n^{1/2} \text{ for } n \geq 9 \tag{6}$$

and of course the C.C. Lindner (see [2]) result

$$\frac{\sigma(n)}{n} < n^{1/2} \text{ for } n \geq 3 \tag{7}$$

Remark. *In paper [1] we have proved among others that if* $\omega(n) \geq 2$, *then*

$$\frac{\sigma_k(n)}{d(n)} \leq \frac{n^k}{2} \tag{8}$$

for any $k \geq 1$, and for any n,

$$\frac{\sigma_k(n)}{d(n)} \le \frac{n^k+1}{2} \tag{9}$$

a result of M. Bencze.
By applying the left side of (1), combined with (8), we get that

$$(\sigma(n))^k < \frac{n^k}{2}(d(n))^k \text{ for } \omega(n) \ge 2, \text{ i.e.}$$

$$\frac{\sigma(n)}{n} < \frac{1}{2^k}d(n) < \frac{1}{2}d(n) \tag{10}$$

for $k > 1$ and $\omega(n) \ge 2$, where $\omega(n)$ denotes the number of distinct prime factors of n.

REFERENCES

[1] Sándor, J. and Cristici, B., *An application of the Jensen-Hadamarad inequality*, Nieuw Arch. Wiskunde (4), 8(1990), pp. 63-66.
[2] Sándor, J. et al, *Handbook of number theory I,* Springer Verlag, 2006.
[3] Sándor, J. and Kovács, L., *An inequality for the number of divisors of* n, submitted to Octogon Mathematical Magazine.

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Improvement of one of Sándor‘s inequalities

Nicuşor Minculete[32]

ABSTRACT. The objective of this paper is to present an improvement of Sándor‘s inequality $\frac{\sqrt{\sigma_k(n)\cdot\sigma_1(n)}}{\sigma_{\frac{k-1}{2}}(n)} \le n^{\frac{-(k-l)}{4}} \cdot \frac{n^{\frac{k+l}{2}}+1}{2}$, for any $n, k, l \in N^*$, where $\sigma_k\left(n\right)$ is the sum of kth powers of divisors of n, so $\sigma_k\left(n\right) = \sum\limits_{d|n} d^k$.

INTRODUCTION

Let n be a positive integer, $n \ge 1$. We note with $\sigma_k\left(n\right)$ the sum of kth powers of divisors of n, so, $\sigma_k\left(n\right) = \sum\limits_{d|n} d^k$, whence we obtain the following equalities: $\sigma_1\left(n\right) = \sigma\left(n\right)$ and $\sigma_0\left(n\right) = \tau\left(n\right) -$ the number of divisors of n.

In [1], J. Sándor shows that

$$\frac{\sqrt{\sigma_k\left(n\right)\cdot\sigma_l\left(n\right)}}{\sigma_{\frac{k-l}{2}}\left(n\right)} \le n^{\frac{-(k-l)}{4}} \cdot \frac{n^{\frac{k+l}{2}}+1}{2}, \text{ for any } n, k, l \in N^* \tag{1}$$

In [2] an inequality which is due to J.B. Diaz and F.T. Metcalf is proved, namely:

Lemma 1.1 Let n be a positive integer, $n \ge 2$. For every $a_1, a_2, ..., a_n \in R$ and for every $b_1, b_2, ..., b_n \in R^*$ with $m \le \frac{a_i}{b_i} \le M$ and $m, M \in R$, we have the following inequality:

$$\sum_{i=1}^{n} a_i^2 + mM\sum_{i=1}^{n} b_i^2 \le \left(m+M\right)\sum_{i=1}^{n} a_i b_i. \tag{2}$$

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1. MAIN RESULT

Theorem 1.2. For every $n, k, l \in N$ with $n \geq 2$ and $\frac{k-l}{2} \in N$ the following relation

$$\frac{\sqrt{\sigma_k(n) \cdot \sigma_l(n)}}{\sigma_{\frac{k-l}{2}}(n)} \leq \frac{n^{\frac{l-k}{4}} \sigma_k(n) + n^{\frac{k-l}{4}} \sigma_l(n)}{2\sigma_{\frac{k-l}{2}}(n)} \leq n^{\frac{-(k-l)}{4}} \cdot \frac{n^{\frac{k+l}{2}}+1}{2}, \text{ is true.} \quad (3)$$

Proof. In the Lemma 1.1, making the substitution $a_i = \sqrt{d_i^k}$ and $b_i = \frac{1}{\sqrt{d_i^l}}$, where d_i is the divisor of n, for any $i = \overline{1, \tau(n)}$. Since $1 \leq \frac{a_i}{b_i} = \sqrt{d_i^{k+l}} \leq n^{\frac{k+l}{2}}$ and $a_i b_i = d_i^{\frac{k-l}{2}}$, we take $m = 1$ and $M = n^{\frac{k+l}{2}}$. Therefore, inequality (2) becomes

$$\sum_{i=1}^{\tau(n)} d_i^k + n^{\frac{k+l}{2}} \sum_{i=1}^{\tau(n)} \frac{1}{d_i^l} \leq \left(1 + n^{\frac{k+l}{2}}\right) \sum_{i=1}^{\tau(n)} d_i^{\frac{k-l}{2}}$$

which is equivalent to

$$\sigma_k(n) + n^{\frac{k+l}{2}} \frac{\sigma_l(n)}{n^l} \leq \left(1 + n^{\frac{k+l}{2}}\right) \sigma_{\frac{k-l}{2}}(n)$$

so that

$$\sigma_k(n) + n^{\frac{k-l}{2}} \sigma_l(n) \leq \left(1 + n^{\frac{k+l}{2}}\right) \sigma_{\frac{k-l}{2}}(n), \quad (4)$$

for every $n, k, l \in N$ with $n \geq 2$.
The arithmetical mean is greater than the geometrical mean or they are equal, so for every $n, k, l \in N$ with $n \geq 2$, we have

$$\sqrt{n^{\frac{k-l}{2}} \sigma_k(n) \sigma_l(n)} \leq \frac{\sigma_k(n) + n^{\frac{k-l}{2}} \sigma_l(n)}{2}. \quad (5)$$

Consequently, from the relations (4) and (5), we deduce the inequality

$$\frac{\sqrt{\sigma_k(n) \sigma_l(n)}}{\sigma_{\frac{k-l}{2}}(n)} \leq \frac{n^{\frac{l-k}{4}} \sigma_k(n) + n^{\frac{k-l}{4}} \sigma_l(n)}{2\sigma_{\frac{k-l}{2}}(n)} \leq n^{\frac{-(k-l)}{4}} \cdot \frac{n^{\frac{k+l}{2}}+l}{2}.$$

Remark *For $k \to k+2$ and $l \to k$ we obtain the relation*

$$\frac{\sqrt{\sigma_{k+2}(n) \cdot \sigma_k(n)}}{\sigma(n)} \leq \frac{\frac{1}{\sqrt{n}} \sigma_{k+2}(n) + \sqrt{n} \sigma_k(n)}{2\sigma(n)} \leq \frac{1}{\sqrt{n}} \cdot \frac{n^{k+1}+1}{2}, \quad (6)$$

for every $n, k \in N$ *with* $n \geq 2$.
For $k = l$, we deduce another inequality which is due to Sándor, namely,

$$\frac{\sigma_k(n)}{\tau(n)} \leq \frac{n^k + 1}{2}, \tag{7}$$

for every $n, k \in N$ with $n \geq 2$.

REFERENCES

[1] Sándor, J., *On Jordan's Arithmetical Function*, Gazeta Matematică nr. 2-3/1993.
[2] Drimbe, M.O., *Inegalităţi. Idei şi metode*, Editura GIL, Zalău, 2003.

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About one algebraic inequality

Šefket Arslanagić[33]

ABSTRACT. In this paper we present the error by the proof of one algebraic inequality.

INTRODUCTION

In [2], p. 39, problem AQ.10.; in [3], p.15, problem 80. and in [4], p.5, problem A1991-6. we have the next problem:
Prove the inequality

$$\frac{x^2y}{z}+\frac{y^2z}{x}+\frac{z^2x}{y}\geq x^2+y^2+z^2 \tag{1}$$

for all positive numbers x, y, z.

Remark. *In [2] and [3] in the place x, y, z we have a, b, c.*

MAIN RESULTS

In [2], p. 69 and [3], p. 38, we have only this phrase as the solution:

With the Cauchy-Buniakowsky-Schwarzs inequality, we get:

$$\left(\frac{x^2y}{z}+\frac{y^2z}{x}+\frac{z^2x}{y}\right)\left(\frac{x^2z}{y}+\frac{y^2x}{z}+\frac{z^2y}{x}\right)\geq\left(x^2+y^2+z^2\right)^2 \tag{2}$$

But , I see not in what way to receive the proof of the given inequality (1) from the inequality (2)?!
In [4], p. 41 we have this solution:

Without lost of generality we may assume that $x \geq y \geq z > 0$. We have

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$$\frac{x^2y}{z}+\frac{y^2z}{x}+\frac{z^2x}{y}\geq x^2+y^2+z^2\Leftrightarrow x^3y^2+y^3z^2+z^3x^2\geq x^3yz+y^3zx+z^3xy$$

$$\Leftrightarrow x^3y\left(y-z\right)+y^2z^2\left(y-z\right)+z^3\left(y^2-2yx+x^2\right)-xyz\left(y^2-z^2\right)\geq 0$$

$$\Leftrightarrow\left(y-z\right)\left(x-z\right)\left[x^2y+yz\left(x-y\right)\right]+z^3\left(x-y\right)^2\geq 0$$

Then the last inequality holds.
Unfortunately, this proof is not complete. Why? The inequality (1) is cyclic and homogeneous, but this inequality is not symmetric! We can not take only that is $x\geq y\geq z>0$.
It is not heawily give one contraexample, i.e. show that this inequality is not exact, for example so $x=16$, $y=1$, $z=2$ $(x\geq z\geq y>0)$; now we have:

$$\frac{256}{2}+\frac{1}{8}+64\geq 256+1+4\Leftrightarrow 192\frac{1}{8}\geq 261(?!)$$

In general, for $x=n^4, y=1, z=n$; $(n\in i;\ n\geq 2)$ we have of (1):

$$n^7+\frac{1}{n^3}+n^6\geq n^8+1+n^2\Leftrightarrow n^{10}+n^9+1\geq n^{11}+n^5+n^3\Leftrightarrow$$

$$\Leftrightarrow n^{10}\left(n-1\right)+n^5\left(1-n^4\right)+n^3-1\leq 0\Leftrightarrow$$

$$\Leftrightarrow n^{10}\left(n-1\right)-n^5\left(n^4-1\right)+\left(n-1\right)\left(n^2+n+1\right)\leq 0\Leftrightarrow$$

$$\Leftrightarrow\left(n-1\right)\left[n^{10}-n^5\left(n+1\right)\left(n^2+1\right)+n^2+n+1\right]\leq 0\Leftrightarrow$$

$$\Leftrightarrow\left(n-1\right)\left(n^{10}+n^2+n+1-n^8-n^7-n^6-n^5\right)\leq 0\Leftrightarrow$$

$$\Leftrightarrow\left(n-1\right)\left(n^8+n^7+n^6+n^5\right)\left(\frac{n^{10}+n^2+n+1}{n^8+n^7+n^6+n^5}-1\right)\leq 0\Leftrightarrow$$

$$\Leftrightarrow n^5\left(n-1\right)\left(n^3+n^2+n+1\right)\left[n\left(n-1\right)-1+\frac{n^6+n^2+n+1}{n^8+n^7+n^6+n^5}\right]\leq 0$$

what is not exact because $n\geq 2$.

The inequality (1) not holds too for $0 < x \le y \le z$, because for $x = 1$, $y = 2$, $z = 16$, we get of (1):

$$\frac{1}{8} + 64 + 128 \ge 1 + 4 + 256 \Leftrightarrow 192\frac{1}{8} \ge 261,$$

what is not true.
Therefore, the inequality (1) not holds for all $x, y, z > 0$. This inequality holds for $x \ge y \ge z > 0$.

REFERENCES

[1] Arslanagić, Š., *Matematika za nadarene*, Bosanska rijec, Sarajevo, 2005.
[2] Maftei, I.V., Popescu, P.G., Piticari, M., Lupu, C., Tataram, M.A., *Inegalitati alese in matematica Inegalitati clasice,* Editura Niculescu, Bucuresti, 2005.
[3] Mortici, C., *600 de probleme*, Editura Gil, Zalau, 2001.
[4] *The Vietnamese Mathematical Olympiad (1990-2006), Selected Problems*, Education Publishing House, Hanoi-Vietnam, 2007.

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On certain inequalities for the $\sigma-$function

József Sándor[34]

ABSTRACT. We prove that $\frac{\sigma(n)}{n} < \frac{P(n)}{p(n)-1}$, where $P(n)$ and $p(n)$ denote the greatest, respectively, least-prime factors of n

MAIN RESULTS

Let $\sigma(n)$ denote the sum of positive divisors of n. The main aim of this note is to prove the following inequality

Theorem. Let $n \geq 2$ be a positive integer. Then

$$\frac{\sigma(n)}{n} < \frac{P(n)}{p(n)-1}, \tag{1}$$

where $p(n)$ denotes the least prime factor of n, and $P(n)$ the greatest prime factor.

Corollary. For all $n \geq 2$ one has

$$\sigma(n!) < n \cdot n!, \tag{2}$$

$$\varphi(n!) \geq (n-1)! \tag{3}$$

where φ denotes Euler‘s totient function.

Proof. We shall use the following well-known inequality (see e.g. [1]):

$$\frac{\sigma(n)\varphi(n)}{n^2} < 1.$$

Therefore, one has

$$\frac{\sigma(n)}{n} < \frac{n}{\varphi(n)} \tag{4}$$

On the other hand, it is well-known that,

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$$\frac{\varphi(n)}{n} = \prod_{p|n}\left(1-\frac{1}{p}\right).$$

Let $p_1 < p_2 < ... < p_r$ denote all distinct prime factors of n. Then

$$\frac{n}{\varphi(n)} = \frac{p_1}{p_1-1}...\frac{p_r}{p_r-1} \leq \frac{p_2-1}{p_1-1}\cdot\frac{p_3-1}{p_2-1}...\frac{p_r}{p_r-1},$$

where we have used the fact that $p_1 \leq p_2-1, ..., p_{r-1} \leq p_r-1$.
Therefore, we have obtained that

$$\frac{n}{\varphi(n)} \leq \frac{p_r}{p_1-1} = \frac{P(n)}{p(n)-1} \tag{5}$$

By inequalities (4) and (5), relation (1) follows.
Letting $n = m!$ in (5), and remarking that $p(m!) = 2$, $P(m!) \leq m$, from (5) we get (3) for n replaced with m.
From (1) applied to $n = m!$, we get similarly relation (2).
Remark 1. *As* $n \cdot n! < (n+1)\,n! = (n+1)!$, we get the inequality

$$\sigma(n!) < (n+1)! \tag{6}$$

i.e. $\sigma(1\cdot 2\cdot ...\cdot n) < 2\cdot 3\cdot ...\cdot(n+1)$; i.e. the inequality

$$\sigma(a_1a_2...a_n) < (a_1+1)(a_2+1)...(a_n+1) \tag{7}$$

is valid for the particular case $a_i = i\ \left(i=\overline{1,n}\right)$.
As by (3), one has $\varphi(1\cdot 2\cdot ...\cdot(n+1)) \geq 1\cdot 2\cdot 3\cdot ...\cdot n$, we get that

$$\varphi(a_1a_2...a_{n+1}) \geq (a_1-1)(a_2-1)...(a_{n+1}-1) \tag{8}$$

is valid for $a_i = 1\ \left(i=\overline{1,n}\right)$, $n \geq 1$.
As $\sigma(k) \geq k+1$ and $\varphi(k) \leq k-1$, by (7) and (8) we can write also for these particular cases:

$$\sigma(a_1a_2...a_n) < (a_1+1)(a_2+1)..(a_n+1)\,\sigma(a_1)\,\sigma(a_2)...\sigma(a_n) \tag{9}$$

respectively.

$$\varphi(a_1a_2...a_n) \geq (a_1-1)(a_2-1)..(a_{n+1}-1) \geq \varphi(a_1)\,\varphi(a_2)...\varphi(a_n) \tag{10}$$

Remark 2. *If* $n \geq 2$ *is even, we get from (1) that*

$$\sigma(n) < nP(n) \tag{11}$$

while if n *is odd, sivisible by 3, one has*

$$\sigma(n) < \frac{nP(n)}{2} \tag{12}$$

REFERENCE

[1] Sándor, J. et.al., *Handbook of number theory I,* Springer Verlag, 2006.

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A note on the inequality $\left(x_1+x_2+...+x_n\right)^2 \leq n\left(x_1^2+...+x_n^2\right)$

József Sándor[35]

ABSTRACT. A lemma proved in [1] follows from the inequality of the title.

Recently, Zlatko Udovicic [1] proved certain inequalities for the sequence of arithmetical means. In his proof, he used a basic inequality, as follows:

Let $x_k \in R\ (1 \leq k \leq n)$, and suppose that $x_1^2+...+x_n^2 \leq r^2$. Then

$$\left(x_1+...+x_n\right)^2 \leq n \cdot r^2 \tag{1}$$

Instead the quite special and complicated proof of (1) shown in [1], we want tooffer inequalities of the title, which trivially yield (1).

Let $f:(0,\infty) \to R,\ f(x)=x^k,\ x \in (0,\infty)$. As $f''(x)=k(k-1)x^{k-2}$, we get that $f''(x) \geq 0$, if $k \in (-\infty,0] \cup [1,\infty)$ and $f''(x) \leq 0$, if $k \in [0,1]$.

By Jensen‘s inequality for convex functions, we can write the inequality

$$f\left(\frac{x_1+...x_n}{n}\right) \leq \frac{f\left(x_1\right)+...+f\left(x_n\right)}{n}\ \left(x_i>0,\ \overline{i=1,n}\right) \tag{2}$$

so we get the inequality

$$\left(x_1+...+x_n\right)^k \leq n^{k-1}\left(x_1^k+...+x_n^k\right), \tag{3}$$

when $k \in (-\infty,0] \cup [1,\infty)$.

Clearly, the inequality in (3) is rewersed, when $k \in [0,1]$.

There are well-known facts.Put e.g. $k=2$ in (3). Then we get the inequality

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$$(x_1 + ... + x_n)^2 \leq n\left(x_1^2 + ... + x_n^2\right) \quad (4)$$

Clearly, if soe of x_i (or all) are < 0, then (4) holds true, by letting $x_i = -y_i$ ($y_i > 0$) and using (4) for y_i as well as the modulus inequality.

Inequality (4) implies at one relation (1).

REFERENCE

[1] Udovicic, Z., *Three inequalities with the sequence of arithmetical means,* Octogon Mathematical Magazine, Vol. 16, No. 2, October 2008, pp. 1027-1030.

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A note on inequalities for the logarithmic function

József Sándor[36]

ABSTRACT. We show that the logarithmic inequalities from [1] and [2] are equivalent with known inequalities for means

Let $a, b > 0$ and $A = A(a,b) = \frac{a+b}{2}$, $G = G(a,b) = \sqrt{ab}$,
$L = L(a,b) = \frac{b-a}{\ln b - \ln a}\ (a \neq b)\,,\ I = I(a,b) = \frac{1}{e}\left(b^b/a^a\right)^{1/(b-a)}\ (a \neq b)\,,$
$L(a,a) = I(a,a) = a$
be the well-known arithmetic, geometric,logarithmic, respectivelly identric means of arguments a and b.

In papers [1], [2] certain logarithmic inequalities are offered.

However these inequalities are well-known, since they are in fact equivalent with certain inequalities for the above considered means.

The left side of Theorem 1 of [1] states that

$$\frac{3\left(x^2-1\right)}{x^2+4x+1} < \ln x,\ \ x > 1 \tag{1}$$

Put $x := \sqrt{\frac{a}{b}}$ in (1) (as in Corollary 1.14), where $a > 1$.
Then becomes

$$L < \frac{2G+A}{3} \tag{2}$$

where $L = L(a,b)$, etc. This is a known inequality of Pólya and Szegő (see the References from [4], or [7]); and rediscovered by B.C. Carlson.

But inequality (2) implies also (1)! Put $a = x^2 b$ in inequality (2). Then

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reducing with b, after some easy computations, we get (1).
The right side of Theorem 1 is

$$\ln x < \frac{\left(x^3-1\right)(x+1)}{33x\left(x^2+1\right)},\ x>1 \tag{3}$$

Letting $x=\sqrt{\frac{a}{b}}$, where $a>b$; we get that (3) is equivalent with

$$L > \frac{3AG}{2A+G} \tag{4}$$

(and not $L > \frac{3AG}{2(2A+G)}$, as is stated in Corollary 1.14 of [1]).

As $\frac{3AG}{2A+G} > G$, inequality (4) is stronger than $L>G$, but weaker than the inequality

$$L > \sqrt[3]{G^2A} \tag{5}$$

due to Leach and Scholander ([4]). This is exactly Theorem 2 of [1], attributed to W. Janous.
To show that $\sqrt[3]{G^2A} > \frac{3AG}{2A+G}$, one has to verify the inequality
$8A^3-15A^2G+6AG^2+G^3>0$, or dividing with G^3, and letting

$$\frac{A}{G}=t : 8t^3-15t^2+6t+1>0$$

This is true, as

$$(t-1)^2(8t+1)>0$$

We do not enter into all inequalities presented in [1], [2] but note that the identity

$$\frac{A}{L}-1=\ln\frac{I}{G} \tag{6}$$

in page 983 of [2] is due to H.J. Seiffert ([5]). The proof which appears here has been discovered by the author in 1993 [5]

REFERENCES

[1] Bencze, M., *New inequalities for the function* $\ln x$ *(1),* Octogon Mathematical Magazine, Vol. 16, No. 2, October 2008, pp. 965-980.
[2] Bencze, M., *New inequality for the function* $\ln x$ *and its applications* (2)*,* Octogon Mathematical Magazine, Vol. 16, No. 2, October 2008, pp. 981-983.

[3] Carlson, B.C., *The logarithmic mean,* Amer. Math. Monthly 79(1972), pp. 615-618.
[4] Sándor, J., *On the identric and logarithmic means,* Aequationes Math. 40(1990), pp. 261-270.
[5] Sándor, J., *On certain identities for means,* Studia Univ. Babes-Bolyai, Math., 18(1993), No.4., pp. 7-14.
[6] Sándor, J., *Some simple integral inequalities,* Octogon Mathematical Magazine, Vol. 16, No. 2, October 2008, pp. 925-933.

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On the inequality $(f(x))^k < f\left(x^k\right)$

József Sándor[37]

ABSTRACT. Let $f : [0, a] \rightarrow 1$, where $a > 1$ and $f(a) \leq 1, f(0) = 0$. We prove that if f is a two-times differentiable, strictly increasing and strictly concave function, such that $0 < f(x) < x$ for all $x \in (0, a]$, then the inequality of the title holds true for any x in $(0, \sqrt[k]{a}]$ for any $k > 1$.

MAIN RESULTS

Let $a > 1$ and f a real-valued function defined on $(0, a]$. Suppose that $0 < f(x) < x$. If $1 \leq x \leq \sqrt[k]{a}$, then as $x \leq x^k$, and f is strictly increasing, we can write $f(x) \leq f\left(x^k\right)$. Thus $(f(x))^k < f(x) \leq f\left(x^k\right)$, as $(f(x))^{k-1} < 1$, by $k > 1$, $f(x) > 0$ and $f(x) < 1$, since $f(x) < f(a) \leq 1$ for $x < a$. Clearly $a \neq a^k$, so there is strict inequality.

Assume now that $x \in (0, 1)$. Then remark first that $(f(t))^{k-1} < t^{k-1}$ for $t \in (0, x)$. This follows by $0 < f(t) < t$ and $k > 1$. On the other hand, as f is strictly concave, we have $f''(t) < 0$ on $(0, x)$, so $f'(t)$ is strictly decreasing, implying $f'(t) < f'\left(t^k\right)$ since $0 < t^k < t < 1$. Therefore we can write

$$k\,(f(t))^{k-1} f'(t) < k \cdot t^{k-1} \cdot f'\left(t^k\right) \qquad (*)$$

for any $t \in (0, x)$; on base of the above two inequalities.

Integrating the inequality (*) on $(0, x)$; and remarking that $\frac{d}{dt}(f(t))^k = k\,(f(t))^{k-1} f'(t)$ and $\frac{d}{dt} f\left(r^k\right) = kt^{k-1} f'\left(t^k\right)$, and using $f(0) = 0$.

This proves the theorem.

Remark. *Without assuming* $f(0) = 0$, *we get the relation*

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$$(f(x))^k < f\left(x^k\right) + (f(0))^k \tag{1}$$

Application. Put $f(x) = \sin x$, $a = \frac{\pi}{2} > 1$. As $f(0) = 0$, $f(a) = 1$ and $f'(x) > 0$, $f''(x) < 0$ and $0 < \sin x < x$, we get by (1)

$$(\sin x)^k < \sin x^k \text{ for any } x \in \left(0, \sqrt[k]{\frac{\pi}{2}}\right] \tag{2}$$

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A note on Bang‘s and Zsigmond‘s theorems

József Sándor[38]

ABSTRACT. In a recent note [2], an application of the so-called Birkhoff-Vandier theorem was given: We offer a history of this theorem, due to Bang and Zsygmondy.

Recently, in note [2], the following theorem of Zsygmondy from 1892 (see [4]) has been applied:

Theorem. If a, b and n are integers with $a > b > 0$, $gdc\,(a, b) = 1$ and $n > 2$, then there is a prime divisor p of $a^n - b^n$ such that p is not a divisor of $a^k - b^k$ for any integer with $1 \leq k < n$, except for the case $a = 2$, $b = 1$, $n = 6$.

The case $b = 1$ is due to Bang [1], who discovered it in 1886.

Both Bang‘s theorem and Zsigmond‘s theorem have been rediscovered many times in the XXth century. A partial list of references is given in [5], p. 361. It should be noted that Zsygmond‘s theorem has itself been generalized to algebraic number fields. A list of references on generalizations of Zsigmond‘s theorem can be found in [3], from which earlier references may be obtained.

REFERENCES

[1]. Bang. A.S., *Taltheoretiske Undersogelser*, Tidsskrifft Math., 5IV (1886), 70-80 and 130-137.
[2] Le. M., and Bencze,M., *An application of the Birkhoff-Vandiver theorem*, Octogon Mathematical Magazine, Vol. 16, No. 2, October 2008, pp. 1357-1360.
[3] Stewart, C.L., *On divisors of terms of linear recurrence sequences*, J. Reine Angen. Math., 333, (1982),pp. 12-31.

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[4] Zsigmondy, K., Zur Theorie der Potenzreste, Monatsch Math. Phys. 3(1982), pp. 265-284.
[5] Dandapat, G.G., Hunsucker, J.L., and Poerance, C., Some new results on odd perfect numbers, Pacific J. Math. 57(1975), pp. 359-364.

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József Wildt International Mathematical Competition

The XIXth Edition, 2009 [39]

The solutions of the problems W.1-W.30 must be mailed before 30.October 2009, to Mihály Bencze, Str. Hărmanului 6, 505600 Săcele-Négyfalu, Jud. Brasov, Romania, E-mail: benczemihaly@yahoo.com

W.1. Let a, b, c be positive real numbers such that $a+b+c=1$. Prove that

$$\sqrt[3]{\left(\frac{1+a}{b+c}\right)^{\frac{1-a}{bc}}\left(\frac{1+b}{c+a}\right)^{\frac{1-b}{ca}}\left(\frac{1+c}{a+b}\right)^{\frac{1-c}{ab}}} \geq 64.$$

José Luis Diaz-Barrero, Barcelona, Spain

W.2. Find the area of the set $A=\{(x,y)\,|1\leq x\leq e, 0\leq y\leq f(x)\}$, where

$$f(x)=\begin{vmatrix} 1 & 1 & 1 & 1\\ \ln x & 2\ln x & 3\ln x & 4\ln x\\ (\ln x)^2 & 4(\ln x)^2 & 9(\ln x)^2 & 16(\ln x)^2\\ (\ln x)^3 & 8(\ln x)^3 & 27(\ln x)^3 & 64(\ln x)^3 \end{vmatrix}$$

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José Luis Diaz-Barrero

W.3. Let Φ and Ψ denote the Euler totient and Dedekind‘s totient, respectively. Determine all n such that $\Phi(n)$ divides $n+\Psi(n)$.

József Sándor and Lehel Kovács

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Key words and phrases. Contest.

W.4. Let Φ denote the Euler totient function.Prove that for infinitely many k we has $\Phi\left(2^k+1\right)<2^{k-1}$ and that for infinitely many m one has $\Phi\left(2^m+1\right)>2^{m-1}$.

József Sándor

W.5. Let p_1, p_2 be two odd prime numbers and α, n integers with $\alpha>1$ and $n>1$. Prove that if the equation $\left(\frac{p_2-1}{2}\right)^{p_1}+\left(\frac{p_2+1}{2}\right)^{p_1}=\alpha^n$ does not accept integer solutions in the case $p_1=p_2$, then the equation does not also have integer solutions for the case $p_1\neq p_2$.

Michael Th. Rassias, Athens, Greece

W.6. Prove that

$$p\left(n\right)=2+\left(p\left(1\right)+...+p\left(\left[\frac{n}{2}\right]+\chi_1\left(n\right)\right)+\left(p'_2\left(n\right)+...+p'_{\left[\frac{n}{2}\right]-1}\left(n\right)\right)\right)$$

for every $n\in N$ with $n>2$, where $\chi_1\left(n\right)$ denotes the principal character Dirichlet modulo 2, i.e. $\chi_1\left(n\right)=\begin{cases}1, & \text{if} \quad (n,k)=1\\ 0, & \text{if} \quad (n,k)=0\end{cases}$ with $p'(n)$ we denote the number of partitions of n in exactly m sumands.

Michael Th. Rassias

W.7. If $0<a<b$, then

$$\int\limits_a^b \frac{\left(x^2-\left(\frac{a+b}{2}\right)^2\right)\ln\frac{x}{a}\ln\frac{x}{b}dx}{\left(x^2+a^2\right)\left(x^2+b^2\right)}>0.$$

György Szöllősy, Máramarossziget, Romania

W.8. If $n,p,q\in N$, $p<q$ then

$$\binom{(p+q)\,n}{n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{(p+q-1)\,n}{pn-k}=$$

$$=\binom{(p+q)\,n}{pn}\sum_{k=0}^{\left[\frac{n}{2}\right]}(-1)^k\binom{pn}{k}\binom{(q-p)\,n}{n-2k}$$

György Szöllősy

W.9. Let the series

$$s(n,x)=\sum_{n\geq 0}\frac{(1-x)(1-2x)...(1-nx)}{n!}$$

Find a real set on which this series is convergent, and then compute its sum. Find also $\lim\limits_{(n,x)\to(\infty,0)} s(n,x)$.

Laurenţiu Modan, Bucharest, Romania

W.10. Let consider the following function set
$\mathcal{F}=\{f|f:\{1,2,...,n\}\to\{1,2,...,n\}\}$

1). Find $|\mathcal{F}|$
2). For $n=2k$, prove that $|\mathcal{F}|<e(4k)^k$
3). Find n,if $|\mathcal{F}|=540$ and $n=2k$.

Laurenţiu Modan

W.11. Find all real numbers m such that

$$\frac{1-m}{2m}\in\left\{x\in R|m^2x^4+3mx^3+2x^2+x=1\right\}.$$

Cristinel Mortici, Tirgovişte, Romania

W.12. Find all functions $f:(0,+\infty)\cap Q\to(0,+\infty)\cap Q$ satisfying the following conditions:

1). $f(ax)\leq(f(x))^a$, for every $x\in(0,+\infty)\cap Q$ and $a\in(0,1)\cap Q$
2). $f(x+y)\leq f(x)f(y)$, for every $x,y\in(0,+\infty)\cap Q$.

Cristinel Mortici

W.13. If $a_k>0$ $(k=1,2,...,n)$, then prove the following inequality

$$\left(\sum_{k=1}^{n}a_k^5\right)^4\geq\frac{1}{n}\left(\frac{2}{n-1}\right)^5\left(\sum_{1\leq i<j\leq n}a_i^2a_j^2\right)^5$$

Róbert Szász, Marosvásárhely, Romania

W.14. If the function $f:[0,1]\to(0,+\infty)$ is increasing and continuous, then for every $a\geq 0$ the following inequality holds:

$$\int\limits_0^1 \frac{x^{a+1}}{f(x)}dx \leq \frac{a+1}{a+2}\int\limits_0^1 \frac{x^a dx}{f(x)}.$$

Róbert Szász

W.15. Let a triangle ABC and the real numbers $x,y,z>0$. Prove that

$$x^n\cos\frac{A}{2}+y^n\cos\frac{B}{2}+z^n\cos\frac{C}{2}\geq(yz)^{\frac{n}{2}}\sin A+(zx)^{\frac{n}{2}}\sin B+(xy)^{\frac{n}{2}}\sin C.$$

Nicuşor Minculete, Sfîntu-Gheorghe, Romania

W.16. Prove that

$$\sum_{k=1}^{n}\frac{1}{d(k)}>\sqrt{n+1}-1,$$

for every $n\geq 1$, where $d(n)$ is the number of divisors of n.

Nicuşor Minculete

W.17. If $a,b,c>0$ and $abc=1$, $\alpha=\max\{a,b,c\}$; $f,g:(0,+\infty)\to R$, where $f(x)=\frac{2(x+1)^2}{x}$, $g(x)=(x+1)\left(\frac{1}{\sqrt{x}}+1\right)^2$, then

$$(a+1)(b+1)(c+1)\geq\min\{\{f(x),g(x)\}\,|x\in\{a,b,c\}\setminus\{\alpha\}\}.$$

Ovidiu Pop and György Szöllősy

W.18. If $a,b,c>0$ and $abc=1$, then

$$\sum\frac{a+b+c^n}{a^{2n+3}+b^{2n+3}+ab}\leq a^{n+1}+b^{n+1}+c^{n+1}$$

for all $n\in N$.

Mihály Bencze

W.19. If $x_k > 0$ $(k = 1, 2, ..., n)$, then

$$\sum_{k=1}^{n} \left(\frac{x_k}{1 + x_1^2 + ... + x_k^2} \right)^2 \leq \frac{\sum\limits_{k=1}^{n} x_k^2}{1 + \sum\limits_{k=1}^{n} x_k^2}.$$

Mihály Bencze

W.20. If $x \in R \setminus \left\{ \frac{k\pi}{2} | k \in Z \right\}$, then

$$\left(\sum_{0 \leq j < k \leq n} \sin\left(2\left(j+k\right)x\right) \right)^2 + \left(\sum_{0 \leq j < k \leq n} \cos\left(2\left(j+k\right)x\right) \right)^2 =$$

$$= \frac{\sin^2 nx \sin^2 \left(n+1\right)x}{\sin^2 x \sin^2 2x}.$$

Mihály Bencze

W.21. If ζ denote the Riemann zeta function, and $s > 1$, then

$$\sum_{k=1}^{\infty} \frac{1}{k^s + 1} \geq \frac{\zeta\left(s\right)}{1 + \zeta\left(s\right)}.$$

Mihály Bencze

W.22. If $a_i > 0$ $(i = 1, 2, ..., n)$, then

$$\left(\frac{a_1}{a_2} \right)^k + \left(\frac{a_2}{a_3} \right)^k + ... + \left(\frac{a_n}{a_1} \right)^k \geq \frac{a_1}{a_2} + \frac{a_2}{a_3} + ... + \frac{a_n}{a_1}$$

for all $k \in N^*$.

Mihály Bencze

W.23. If $x_k \in R$ $(k = 1, 2, ..., n)$ and $m \in N$, then

1). $\sum\limits_{cyclic} \left(x_1^2 - x_1 x_2 + x_2^2 \right)^m \leq 3^m \sum\limits_{k=1}^{n} x_k^{2m}$

2). $\prod\limits_{cyclic} \left(x_1^2 - x_1 x_2 + x_2^2 \right)^m \leq \left(\frac{3^m}{n} \right)^m \left(\sum\limits_{k=1}^{n} x_k^{2m} \right)^n$

Mihály Bencze

W.24. If K, L, M denote the midpoints of sides AB, BC, CA, in triangle ABC, then for all P in the plane of triangle, we have

$$\frac{AB}{PK}+\frac{BC}{PL}+\frac{CA}{PM}\geq\frac{AB\cdot BC\cdot CA}{4PK\cdot PL\cdot P}.$$

Mihály Bencze

W.25. Let $ABCD$ be a quadrilateral in which $\widehat{A}=\widehat{C}=90^\circ$. Prove that

$$\frac{1}{BD}\left(AB+BC+CD+DA\right)+BD^2\left(\frac{1}{AB\cdot AD}+\frac{1}{CB\cdot CD}\right)\geq 2\left(2+\sqrt{2}\right).$$

Mihály Bencze

W.26. If $a_i>0$ $(i=1,2,...,n)$ and $\sum\limits_{i=1}^{n}a_i^k=1$, where $1\leq k\leq n+1$, then

$$\sum_{i=1}^{n}a_i+\frac{1}{\prod\limits_{i=1}^{n}a_i}\geq n^{1-\frac{1}{k}}+n^{\frac{n}{k}}.$$

Mihály Bencze

W.27. Let a, n be positive integers such that a^n is a perfect number. Prove that

$$a^{n/\mu}>\frac{\mu}{2},$$

where μ denotes the number of distinct prime divisors of a^n.

Michael Th. Rassias

W.28. Let θ and p $(p<1)$ be nonnegative real numbers.

Suppose that $f:X\to Y$ is a mapping with $f\left(0\right)=0$ and

$$\left\|2f\left(\frac{x+y}{2}\right)-f\left(x\right)-f\left(y\right)\right\|_Y\leq\theta\left(\|x\|_X^p\right)+\|y\|_X^p\right) \tag{1}$$

for all $x, y\in Z$ with $x\perp y$, where X is an orthogonality space and Y is a real Banach space.

Prove that there exists a unique orthogonally Jensen additive mapping $T:X\to Y$, namely a mapping T that satisfies the so-called orthogonally Jensen additive functional equation

$$2f\left(\frac{x+y}{2}\right)=f\left(x\right)+f\left(y\right)$$

for all $x,y\in X$ with $x\perp y$, satisfying the property

$$\left\|f\left(x\right)-T\left(x\right)\right\|_Y\leq\frac{2^p\theta}{2-2^p}\left\|x\right\|_X^p \tag{2}$$

for all $x\in X$.

Themistocles M. Rassias

W.29. In all triangle ABC holds

$$\sum\left(1-\sqrt{\sqrt{3}tg\frac{A}{2}}+\sqrt{3}tg\frac{A}{2}\right)\left(1-\sqrt{\sqrt{3}tg\frac{B}{2}}+\sqrt{3}tg\frac{B}{2}\right)\geq 3$$

Mihály Bencze

W.30. Prove that

$$\sum_{0\leq i<j\leq n}\left(i+j\right)\binom{n}{i}\binom{n}{j}=n\left(2^{2n-1}-\binom{2n}{n}\right)$$

Mihály Bencze

Book reviews

1. Selected chapters of geometry, analysis and number theory: Classical topics in new perspectives

Sándor, J., Babes-Bolyai University, Cluj and Miercurea Ciuc, Romania

This book focuses on some important classical parts of Geometry, Analysis and Number Theory.

The material is divided into ten chapters, including new advances on triangle or tetrahedral inequalitiesl special sequences and series of real numbers; various algebraic or analytic inequalities with applications; special functions (as Euler gamma and beta functions) and special means (as the logarithmic, identric or Seiffert‘s mean); arithmetic functions and arithmetic inequalities with connections to perfect numbers or related fields; and many more.
The majority of the presented topics are based on the original journal publications of the author.

This reference work eill be useful for undergraduate university or college students; as well as teachers, researchers and professors interested in these fields of mathematics.

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$ 114.00, sold by Amazon.com

Proposed problems

PP. 15249. [40] Denote M, N the midpoint of sides AB and AC in triangle ABC, and $AB + AC = \lambda \cdot BC$.

1). Determine all $\lambda > 0$ for which MN is tangent to the incircle.
2). Determine all $\lambda > 0$ for which MN lie on diameter of circumcircle.

Mihály Bencze

PP. 15250. If $a_k > 1$ $(k = 1, 2, ..., n)$ and $S = \sum\limits_{k=1}^{n} a_k$, then

1). $\prod\limits_{k=1}^{n} \log_{a_k} \frac{S-a_k}{n-1} \geq 1$ 2). $\sum\limits_{k=1}^{n} \log_{a_k} \frac{S-a_k}{n-1} \geq n$

Mihály Bencze

PP. 15251. In all triangle ABC holds:

1). $3s^2\left(s^2 + r^2 + 2Rr\right)^2 \geq 2Rr\left(5s^2 + r^2 + 4Rr\right)^2$
2). $12s^2R^2 \geq \left(s^2 + r^2 + 4Rr\right)^2$
3). $3s^2\left(s^2 + r^2 + 4Rr\right)\left(s^2 + r^2 + 2Rr\right)^2 \geq \left(\left(s^2 + r^2 + 4Rr\right)^2 + 8s^2Rr\right)^2$
4). $12\left(4R + r\right)s^2R^2 \geq r\left(\left(4R + r\right)^2 + s^2\right)^2$
5). $6\left(2R - r\right)\left(\left(2R - r\right)\left(s^2 + r^2 - 8Rr\right) - 2Rr^2\right)^2 \geq$
$4Rr^2\left(16R^2 - 24Rr + 5r^2 + s^2\right)^2$
6). $3\left(4R + r\right)\left(\left(4R + r\right)^3 + s^2\left(2R + r\right)^2\right) \geq 2s^2R\left(5\left(4R + r\right)^2 + s^2\right)^2$

Mihály Bencze

PP. 15252. Denote $F\left(k_1, k_2, ..., k_n, m\right)$ the last decimal of $\left(m + k_1\right)^m + \left(m + k_2\right)^m + ... + \left(m + k_n\right)^m$, when $m, k_1, k_2, ..., k_n \in N^*$. Prove that F is periodical, in raport of m.

Mihály Bencze

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[40] *Solution should be mailed to editor until* 30.12.2010. *No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new in sights on past problems.*

PP. 15253. In all triangle ABC holds $\sum \frac{ctg\frac{A}{2}}{\sqrt{s-r\cdot ctg\frac{A}{2}}} \geq \frac{1}{r}\sqrt{\frac{3s}{2}}$.

Mihály Bencze

PP. 15254. Determine all regular $n-$gon $A_1A_2...A_n$ in which $\frac{1}{A_iA_j^k}+\frac{1}{A_iA_p^k}=\frac{2k}{A_iA_r^k}$, when $k\in N^*$.

Mihály Bencze

PP. 15255. Determine all regular $n-$gon $A_1A_2...A_n$ in which the difference of the maximal and minimal diagonal is equal with sides of $n-$gon.

Mihály Bencze

PP. 15256. Let M be a random point on the circumcircle of regular $2n-$gon $A_1A_2...A_nA_{n+1}...A_{2n}$. Denote $B_1, B_2, ..., B_n$ the proiection of M to $A_1A_{n+1}, A_2A_{n+2}, ..., A_nA_{2n}$.

Prove that $\frac{\sigma[A_1A_2...A_{2n}]}{\sigma[B_1B_2...B_n]}=\frac{4}{\cos\frac{\pi}{n}}$.

Mihály Bencze

PP. 15257. If $z_k\in C$ $(k=1,2,...,n)$ and $x,y\in R$, then

$$(x+y)^2\sum_{k=1}^{n}|z_k|^2 - xy\left(|z_1+z_2|^2+|z_2+z_3|^2+...+|z_n+z_1|^2\right)=$$
$$=|yz_1-xz_2|^2+|yz_2-xz_3|^2+...+|yz_n-xz_1|^2.$$

Mihály Bencze

PP. 15258. Solve the following system: $\begin{cases} \left(x_1^3+1\right)^3=2\left(2x_2-1\right) \\ \left(x_2^3+1\right)^3=2\left(2x_3-1\right) \\ ----------- \\ \left(x_n^3+1\right)^3=2\left(2x_1-1\right) \end{cases}$.

Mihály Bencze

PP. 15259. In all triangle ABC holds $\sum\sqrt{ctg\frac{A}{2}} \geq \frac{3\sqrt{3}(4R+r)}{s}\sqrt{\frac{r}{s}}$.

Mihály Bencze

PP. 15260. Let be $a_k \in (0,1) \cup (1,+\infty)$ $(k = 1, 2, ..., n)$ and $f : R^{n-1} \to R^{n-1}$, where $f(x_1, x_2, ..., x_{n-1}) = a_1^{x_1} a_2^{x_2} ... a_{n-1}^{x_{n-1}} a_n^{1-x_1-x_2-...-x_n} + $ $+ a_2^{x_1} a_3^{x_2} ... a_n^{x_{n-1}} a_1^{1-x_1-...-x_n} + ... + a_n^{x_1} a_1^{x_2} ... a_{n-2}^{x_{n-1}} a_{n-1}^{1-x_1-...-x_n}$

1). Determine $E \subseteq R^{n-1}$ in which f is increasing

2). Determine $F \subseteq R^{n-1}$ in which f is decreasing.

Mihály Bencze

PP. 15261. Let $f : [0,+\infty) \to [0,+\infty)$ be a function, where $f(x) = \sum\limits_{k=1}^{n} a_k^x$, $a_k > 0$ $(k = 1, 2, ..., n)$

1). Prove that if $\prod\limits_{k=1}^{n} a_k \geq 1$, then f is increasing

2). Determine all $a_k > 0$ $(k = 1, 2, ..., n)$ for which f is decreasing

Mihály Bencze

PP. 15262. Let ABC be a triangle. Determine all $x > 0$ for which

$$\sum \frac{\ln\left(xtg\frac{A}{2}\left(\frac{4R+r}{s} - tg\frac{A}{2}\right)\right)}{\ln\left(x\frac{r}{s}ctg\frac{A}{2}\right)} \geq x.$$

Mihály Bencze

PP. 15263. If $x_k > 1$ $(k = 1, 2, ..., n)$ and $\sum\limits_{k=1}^{n} x_k = n(n-1)$, then $\sum\limits_{cyclic} \log_{x_1}(x_2 + x_3 + ... + x_n) \geq 2n$.

Mihály Bencze

PP. 15264. Solve the following system: $\left\{\begin{array}{l} x_1^{2x_1} = e^{1-x_2^2} \\ x_2^{2x_2} = e^{1-x_3^2} \\ ------ \\ x_n^{2x_n} = e^{1-x_1^2} \end{array}\right.$.

Mihály Bencze

PP. 15265. In all triangle ABC holds

$$\sum\left(\sqrt[3]{1-tg\frac{A}{2}}+\sqrt[3]{1+tg\frac{A}{2}}\right)\cos\frac{A}{2}\le 5+\frac{r}{2R}.$$

Mihály Bencze

PP. 15266. In all triangle ABC holds:

1). $\frac{9}{4(2R-r)}\le\sum\frac{1}{2R\left(1+\sin^2\frac{A}{2}\right)-r}<\frac{7}{3(2R-r)}$

2). $\frac{9}{4(4R+r)}\le\sum\frac{1}{2R\left(2+\cos^2\frac{A}{2}\right)+r}<\frac{7}{3(2R+r)}$

Mihály Bencze

PP. 15267. In all triangle ABC holds: $\frac{1}{2}\sum tg\frac{A}{2}ctg\frac{C}{2}\ge\sum\frac{\sqrt{ctg\frac{A}{2}}}{\sqrt{ctg\frac{B}{2}}+\sqrt{ctg\frac{C}{2}}}$.

Mihály Bencze

PP. 15268. In all triangle ABC holds:

$$\sum\left(\frac{tg^2\frac{A}{2}}{tg^3\frac{A}{2}+ctg\frac{A}{2}}+\frac{ctg^2\frac{A}{2}}{ctg^3\frac{A}{2}+tg\frac{A}{2}}\right)\ge 2\sum\frac{1}{tg^2\frac{A}{2}+ctg^2\frac{A}{2}}.$$

Mihály Bencze

PP. 15269. In all triangle ABC holds: $\prod\left(3+ctg\frac{A}{2}ctg\frac{B}{2}\right)\le\frac{4s^2R}{r^3}$.

Mihály Bencze

PP. 15270. If $a,b,c>1$, then solve the equation

$$\left(a^{-x}+a^{\frac{1}{x}}\right)\left(b^{-x}+b^{\frac{1}{x}}\right)\left(c^{-x}+c^{\frac{1}{x}}\right)=$$
$$=(abc)^{-x}+(abc)^{\frac{1}{x}}+\left(\frac{a}{bc}\right)^x+\left(\frac{bc}{a}\right)^{\frac{1}{x}}+\left(\frac{b}{ac}\right)^x+\left(\frac{ac}{b}\right)^{\frac{1}{x}}+\left(\frac{c}{ab}\right)^x+\left(\frac{ab}{c}\right)^{\frac{1}{x}}.$$

Mihály Bencze

PP. 15271. If $z_1, z_2, z_3 \in C$ are distinct such that

$\alpha \left|z_2 - z_3\right| = \left|z_1 - z_2\right| + \left|z_1 - z_3\right|$, then $\frac{\sqrt{\alpha}-1}{\sqrt{\alpha}+1} \leq \left|\frac{z_1-z_2}{z_1-z_3}\right| \leq \frac{\sqrt{\alpha}+1}{\sqrt{\alpha}-1}$.

Mihály Bencze

PP. 15272. Let $A_1A_2A_3A_4$ be a concyclic quadrilateral. If all triangles determiated by three vertexes of the given quadrilateral are isoscelles, then $A_1A_2A_3A_4$ is square or trapezium.

Mihály Bencze

PP. 15273. In all triangle ABC holds:

$$\sum \left(\sqrt{tg\frac{A}{2}ctg\frac{B}{2}} + \sqrt{tg\frac{B}{2}ctg\frac{A}{2}}\right) tg\frac{C}{2} \leq 2\sqrt{3}.$$

Mihály Bencze

PP. 15274. In all triangle ABC holds:

1). $\frac{9}{4(R+r)} \leq \sum \frac{1}{4R+r+r_a} < \frac{7}{3(4R+r)}$ 2). $\frac{9}{4s} \leq \sum \frac{1}{s+r\cdot ctg\frac{A}{2}} < \frac{7r}{3s}$

Mihály Bencze

PP. 15275. In all triangle ABC holds:

$$\frac{1}{b(a-b+c)} + \frac{2}{c(a+b-c)} + \frac{3}{a(-a+b+c)} \geq \frac{108}{(2b+c)(3a+b+2c)}.$$

Mihály Bencze

PP. 15276. If $z_k \in C$, $|z_k| \leq 1$ $(k = 1, 2, ..., n)$ and $p \in \{2, 3, ..., n-1\}$, then

$$\left|(1 - z_1z_2...z_p)(1 - z_2z_3...z_{p+1})...(1 - z_nz_1...z_{p-1})\left(1 - \prod_{k=1}^{n} z_k\right)\right| \geq$$
$$\geq \prod_{k=1}^{n} (1 - |z_k|)^{p+1}.$$

Mihály Bencze

PP. 15277. Let ABC be a triangle, and $M \in Int\,(ABC)$ such $AMB\measuredangle = BMC\measuredangle = CMA\measuredangle$. Prove that:

1). $2\sum MA^2 + \sum \frac{MA^3+MB^3}{MC} \geq \frac{16\sqrt{3}}{3}\sigma\,[ABC]$

2). $\left(\sum MA^3\right)\left(\sum \frac{1}{MA}\right) \geq 4\sqrt{3}\sigma\,[ABC]$

3). $\left(\sum MA + MB\right)^2 \geq \frac{16\sqrt{3}}{3}\sigma\,[ABC]$

4). $\sum \left(\frac{MA\cdot MB}{MA+MB}\right)^2 \leq \frac{\sqrt{3}}{3}\sigma\,[ABC]$.

Mihály Bencze

PP. 15278. Let be the triangles $A_kB_kC_k$ such that $M_k \in Int\,(A_kB_kC_k)$ and $A_kM_kB_k\measuredangle = B_kM_kC_k\measuredangle = C_kM_kA_k\measuredangle$ $(k = 1, 2, ..., n)$. Prove that:

$$\sqrt{\left(\sum_{k=1}^{n} M_kA_k^2\right)\left(\sum_{k=1}^{n} M_kB_k^2\right)} + \sqrt{\left(\sum_{k=1}^{n} M_kB_k^2\right)\left(\sum_{k=1}^{n} M_kC_k^2\right)} +$$

$$+\sqrt{\left(\sum_{k=1}^{n} M_kC_k^2\right)\left(\sum_{k=1}^{n} M_kA_k^2\right)} \geq \frac{4\sqrt{3}}{3}\sum_{k=1}^{n} \sigma\,[A_kB_kC_k].$$

Mihály Bencze

PP. 15279. Determine all $z_1, z_2, z_3 \in C$ for which $|z_1| = |z_2| = |z_3| = 1$ and

$\left|\frac{z_2-z_3}{z_1} + \frac{z_3-z_1}{z_2} + \frac{z_1-z_2}{z_3}\right| = 3\sqrt{3}.$

Mihály Bencze

PP. 15280. Prove that the triangle ABC is equilateral if and only if

$\sum \log_a \frac{8Rr\cos\frac{B-C}{2}}{(s^2+r^2+2Rr)\sin\frac{A}{2}} = 0.$

Mihály Bencze

PP. 15281. Determine all $z_k \in C$ $(k = 1, 2, ..., n)$ such that

$$\left(\left|1 - \prod_{k=1}^{n} z_k\right| - \sum_{cyclic} |z_1 - z_2|\right)^2 \leq$$

$$\leq \prod_{k=1}^{n} \left|1 - z_k^2\right| \leq \left(\left|1 - \prod_{k=1}^{n} z_k\right| + \sum_{cyclic} |z_1 - z_2|\right)^2.$$

Mihály Bencze

PP. 15282. If $z_1, z_2, z_3 \in C$ and $|z_k| \leq 1$ $(k = 1, 2, 3)$, then

$$|(1 - z_1 z_2)(1 - z_2 z_3)(1 - z_3 z_1)(1 - z_1 z_2 z_3)| \geq$$
$$\geq (1 - |z_1|)^3 (1 - |z_2|)^3 (1 - |z_3|)^3.$$

Mihály Bencze

PP. 15283. Let ABC and $A'B'C'$ be two triangles. Prove that $16 \sum h_a h'_a + x \sum (a - b)^2 + y \sum (a' - b')^2 \leq \left(\sum aa'\right)^2 \left(\sum \frac{1}{h_a h'_a}\right)$, where $x, y \geq 0$ are constant, which will be determinated.

Mihály Bencze

PP. 15284. Let ABC and $A'B'C'$ be two triangles. Prove that

$$\left(\sum \frac{\sin A'}{a}\right)\left(\sum \frac{\sin A}{a'}\right) \leq \frac{1}{4}\left(\frac{1}{R} + \frac{1}{r}\right)\left(\frac{1}{R'} + \frac{1}{r'}\right).$$

Mihály Bencze

PP. 15285. 1). If $A(x) = \begin{pmatrix} a_1 + b_1 x & a_2 + b_2 x \\ a_3 + b_3 x & a_4 + b_4 x \end{pmatrix}$, where $a_k, b_k \in C$ $(k = 1, 2, 3, 4)$, then determine all $a_k, b_k, a, b \in C$ $(k = 1, 2, 3, 4)$ such that $A^2(x) = A\left((x + a)^2 + b\right)$ for all $x \in C$.

2). If $a_k, b_k, a, b \in Z$ $(k = 1, 2, 3, 4)$, then solve in Z the equation $A^2(x) = A\left(y^2 + c^2\right)$, where $c \in Z$.

Mihály Bencze

PP. 15286. If $a, b, c \in (0, 1) \cup (1, +\infty)$, then

$$\sum \frac{(\log_a b)^2 + (\log_b c)^2}{(\log_a b)^4 + (\log_b c)^4} \leq \log_a b + \log_b c + \log_c a.$$

Mihály Bencze

PP. 15287. If $a_k \in (0,1) \cup (1,+\infty)$ $(k = 1,2,...,n)$, then

$\sum\limits_{cyclic} \left(\log_{a_1} a_2 + \log_{a_1 a_2} a_2\right) \geq \frac{3n}{2}$. When holds the equality?

Mihály Bencze

PP. 15288. If $x, y, z, t > 0$, then $\sum \frac{x^2}{y^2} \geq \sum \frac{1}{x+y+z} + \frac{1}{3} \sum \frac{3t-3y-1}{x} + 4$.

Mihály Bencze

PP. 15289. In all triangle ABC holds:

1). $\prod \left(a^2 + \frac{3}{4}\right)^2 \geq 2s\left(s^2 + r^2 + 2Rr\right)$

2). $\prod \left(h_a^2 + \frac{3}{4}\right)^2 \geq \frac{s^2 r\left(s^2+r^2+2Rr\right)}{R^2}$ 3). $\prod \left(r_a^2 + \frac{3}{4}\right)^2 \geq 4s^2 R$

Mihály Bencze

PP. 15290. In all triangle ABC holds:

1). $\sum \left(\frac{a}{b}\right)^2 \geq \frac{4\left(s^2-r^2-Rr\right)}{s^2+r^2+2Rr}$ 2). $\sum \left(\frac{s-a}{s-b}\right)^2 \geq \frac{s^2+r^2}{2Rr} - 4$

Mihály Bencze

PP. 15291. 1). If $z_k \in C$ $(k = 1,2,...,n)$ and $|z_k| = 1$ $(k = 1,2,...,n)$, then
$\sqrt{\sum\limits_{i=1}^{n} \left(\sum\limits_{j=1}^{n} \operatorname{Re}\left(\frac{z_i}{z_j}\right)\right)} \leq \sum\limits_{k=1}^{n} |z_k|$

2). Determine all $z_k \in C$ $(k = 1,2,...,n)$ such that
$\sum\limits_{i=1}^{n} \left(\sum\limits_{j=1}^{n} \operatorname{Re}\left(\frac{z_i}{z_j}\right)\right) = \left|\sum\limits_{k=1}^{n} z_k\right|^2$

Mihály Bencze

PP. 15292. Solve in Z the equations

1). $(1 - x + xy)(1 - y + xy) = 1$ 2). $\prod\limits_{k=1}^{n} (1 - x_k + x_1 x_2 ... x_n) = 1$

Mihály Bencze

PP. 15293. If $x \in \left(0, \frac{\pi}{2}\right)$ and $n \in N^*$, then

$\left(1 - \frac{1}{n}\right) \cos x + \frac{1}{n} \geq \cos\left(1 - \frac{1}{2n}\right) x$

2). We have $\sum\limits_{k=1}^{\infty} \left(\cos\left(1 - \frac{1}{2k^2}\right) x - \left(1 - \frac{1}{k^2}\right) \cos x\right) < \frac{\pi^2}{6}$

3). $\sum\limits_{k=1}^{n} \left(\cos\left(1 - \frac{1}{2k(k+1)}\right) x - \left(1 - \frac{1}{k(k+1)}\right) \cos x\right) < \frac{n}{n+1}$

4). Compute $\lim\limits_{n \to \infty} \left(\sum\limits_{k=1}^{n} \cos\left(1 - \frac{1}{2k(k+1)}\right) x - \left(1 - \frac{1}{k(k+1)}\right) \cos x\right)$

Mihály Bencze

PP. 15294. If $A_k B_k C_k$ $(k = 1, 2, ..., n)$ are triangles, then

$\sum w_{a_1} w_{a_2} ... w_{a_n} \leq \frac{3}{2^n} \left(\sum \left(a_1 a_2 ... a_n\right)\right)^{\frac{n}{2}}$.

Mihály Bencze

PP. 15295. Determine all $x, y, n \in N$ such that

$\left(\frac{3+\sqrt{17}}{2}\right)^n + \left(\frac{3-\sqrt{17}}{2}\right)^n = x^2 + y^2.$

Mihály Bencze

PP. 15296. If $a, b, c \in (0, 1) \cup (1, +\infty)$, then $\sum \frac{\log_c ca}{\log_b ba} \geq 3.$

Mihály Bencze

PP. 15297. If $\varepsilon = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}$, $n \in N$, $n \geq 2$ and

$s_k = 1^k + 2^k \varepsilon + 3^k \varepsilon^2 + ... + (n-1)^k \varepsilon^{n-2}$, then compute:

1). $|s_1| + |s_2| + ... + |s_k|$ 2). $|s_1 s_2| + |s_2 s_3| + ... + |s_k s_1|$

3). $\left|\frac{s_1}{s_2}\right| + \left|\frac{s_2}{s_3}\right| + ... + \left|\frac{s_k}{s_1}\right|$

Mihály Bencze

PP. 15298. Determine all $x, y, z \in \{0,1,2,3,4,5,6,7,8,9\}$ such that

$$\left(\underbrace{\overline{xx...x}}_{n-time}\right)^2 + \underbrace{\overline{yy...y}}_{n-time} = \underbrace{\overline{zz...z}}_{2n-time}.$$

Mihály Bencze

PP. 15299. If $f_k, g_k : [0,+\infty) \to [0,+\infty)$ $(k = 1,2,...,n)$ are continuous functions, such that $\left(\sum\limits_{k=1}^{n} a_k f_k(x)\right)\left(\sum\limits_{k=1}^{n} b_k g_k(x)\right)^{-1}$ is an increasing function, then determine all $a_k, b_k \in R$ $(k = 1,2,...,n)$ for which

$$\left(\sum_{k=1}^{n} a_k \int_0^x f_k(t)\,dt\right)\left(\sum_{k=1}^{n} b_k \int_0^x g_k(t)\,dt\right)^{-1}$$ is an increasing function too.

Mihály Bencze

PP. 15300. In all triangle ABC holds:

1). $\sum \frac{a^2}{r_a - r} = 2(4R + r)$ 2). $\sum \frac{a^2}{r_b + r_c} = 2(2R - r)$

Mihály Bencze

PP. 15301. If $a, b, c > 0$, then $abc \prod (4a + b + c) \le \left(\sum a\right)^3 \prod (a + b)$.

Mihály Bencze

PP. 15302. Determine all function $f : [0,+\infty) \to [0,+\infty)$ such that $f(x) + \sqrt[n]{f^n([x]) + f^n(\{x\})} = x$ for all $x \ge 0$, $n \in N^*$, when $[\cdot]$ and $\{\cdot\}$ denote the integer, respective the fractional part.

Mihály Bencze

PP. 15303. If $x_k > 0$ $(k = 1,2,...,n)$ and $\prod\limits_{k=1}^{n} x_k = 1$, then determine all $a, b > 0$ for which $\sum\limits_{cyclic} \frac{a + x_1 x_2 ... x_p}{b + x_{p+1} x_{p+2} ... x_n} \ge n$, when $p \in \{2,3,...,n-1\}$.

Mihály Bencze

PP. 15304. If $a, b, c, x, y, z > 0$, then $\left(\sqrt{\sum a}\right)\left(\sum \frac{x}{\sqrt{a}}\right) \ge \sqrt{3} \sum xy$.

Mihály Bencze

PP. 15305. If $a, b, c > 0$ and $abc = 1$, then

$$\prod\left(\frac{a^2}{a^3+b}+\frac{b^2}{b^3+a}\right)\prod\left(a^2-ab+b^2+1\right)\geq 8.$$

Mihály Bencze

PP. 15306. Compute $s_p = \lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{k=1}^{n}\left(\left\{\sqrt{2k\left(2k-1\right)}\right\}\right)^p$, where $\{\cdot\}$ denote the fractional part.

Mihály Bencze

PP. 15307. Determine all $n \in N$ for which $a_1, a_2, ..., a_n$ are in arithmetical progression if and only if $\sum\limits_{k=1}^{n} a_k^2 - a_1a_2 - a_2a_3 - ... - a_na_1 \in$ $\left\{n\left(a_1-a_2\right)\left(a_2-a_3\right); n\left(a_2-a_3\right)\left(a_3-a_4\right), ..., n\left(a_n-a_1\right)\left(a_1-a_2\right)\right\}.$

Mihály Bencze

PP. 15308. If $x \in R$, then

$$\frac{e^{\sin^2 x}+e^{\cos^2 x}}{e^{2\sin^2 x}+e^{2\cos^2 x}}+\frac{e^{-1}+e^{\sin^2 x}}{e^{-2}+e^{2\sin^2 x}}+\frac{e^{-1}+e^{\cos^2 x}}{e^{-2}+e^{2\cos^2 x}}\leq e^{\frac{1}{2}\sin^2 x}+e^{\frac{1}{2}\cos^2 x}+e^{-\frac{1}{2}}.$$

Mihály Bencze

PP. 15309. Determine all $k, p, n \in N$ for which

$1 < \left\{\sqrt{2k\left(2p-1\right)}\right\}+\left\{\sqrt{2p\left(2n-1\right)}\right\}+\left\{\sqrt{2n\left(2k-1\right)}\right\} < \frac{3}{2}$, when $\{\cdot\}$ denote the fractional part.

Mihály Bencze

PP. 15310. If $A = \left\{x \in R^* | ax^3 + bx^2 + cx + d = 0\right\}$,

$B = \left\{x \in R^* | bx^3 + cx^2 + dx + a = 0\right\}$,
$C = \left\{x \in R^* | cx^3 + dx^2 + ax + b = 0\right\}$,
$D = \left\{x \in R^* | dx^3 + ax^2 + bx + c = 0\right\}$ and $A \cap B \cap C \cap D \neq \emptyset$, then compute $A \cup B \cup C \cup D$.

Mihály Bencze

PP. 15311. In triangle ABC let be $D \in (BC)$, and denote G_1, G_2 the centroid of triangles ABD and ACD.

1). Determine all $D \in (BC)$ for which G_1, G_2, I are colloinear if and only if AB, BC, CA are in arithmetical progression.

2). What happens if AB, BC, CA are in geometrical progression?

Mihály Bencze

PP. 15312. If $x, y \in R$, then $\frac{e^{y-x}}{1+e^{x+2y}} + \frac{e^{x-y}}{1+e^{y+2x}} + \frac{e^{-2(x+y)}}{e^x+e^y} \geq \frac{3}{2}$.

Mihály Bencze

PP. 15313. If $x, y \in R$, then $\frac{2e^x}{e^{x+y}+1} + \frac{2e^{-x}}{1+e^y} + \frac{2e^{x+y}}{1+e^x} \leq e^{\frac{2x+y}{2}} + e^{\frac{y-x}{2}} + e^{-\frac{2x+y}{2}}$.

Mihály Bencze

PP. 15314. Determine all $n \in N^*$ for which $\left[(n-1)\,2^{-n}\sum\limits_{k=0}^{n}\frac{\binom{n}{k}}{2k+1}\right] = 1$, where $[\cdot]$ denote the integer part.

Mihály Bencze

PP. 15315. If $(x_n)_{n\geq 1}$ is a real numbers sequence for which $\lim\limits_{n\to\infty} x_n = 0$, then determine all continuous functions $f : R \to R$ for which

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}+x_n\right) = \int_0^1 f(x)\,dx.$$

Mihály Bencze

PP. 15316. In triangle ABC we take $D, F \in (BC)$ and denote G_1, G_2, G_3 the centroid of triangles ABD, ADE, AEC.

1). Determine all $D, E \in (BC)$ for which G_1, G_2, G_3 are collinear.

2). Determine all $D, E \in (BC)$ for which G_1, G_2, G_3, I are collinear.

Mihály Bencze

PP. 15317. If $a, b, c > 0$ and $a^4 + b^4 + c^4 = 1$, the determine all $\alpha \in R$ for which $a\left(ab\right)^{\alpha} + b\left(bc\right)^{\alpha} + c\left(ca\right)^{\alpha} \leq \frac{7}{4}$.

Mihály Bencze

PP. 15318. Solve the equation $4^{x+\frac{1}{x}} + 9^{x+\frac{1}{x}} + 25^{x+\frac{1}{x}} = 390900$.

Mihály Bencze

PP. 15319. In all triangle ABC holds:
1). $\sum \frac{\left(r_a+r_b\right)\left(r_a^2+r_b^2\right)}{r_c} \geq \left(\frac{(4R+r)^2-s^2}{s}\right)^2$
2). $\sum r_c^4\left(r_a+r_b\right)\left(r_a^2+r_b^2\right) \geq \frac{s^4\left(s^2-4Rr-r^2\right)^2}{4R+r}$

Mihály Bencze

PP. 15320. In all triangle ABC holds:
1). $\sum \frac{a}{s-a} \geq \frac{s^2-4Rr-r^2}{r(4R+r)}$
2). $\sum \frac{\left(\sin^2\frac{A}{2}+\sin^2\frac{B}{2}\right)\left(\sin^4\frac{A}{2}+\sin^4\frac{B}{2}\right)}{\sin^2\frac{C}{2}} \geq \frac{\left(16R^2+3r^2-8Rr-s^2\right)^2}{16R^2\left(s^2+r^2-8Rr\right)}$
3). $\sum \frac{\left(\cos^2\frac{A}{2}+\cos^2\frac{B}{2}\right)\left(\cos^4\frac{A}{2}+\cos^4\frac{B}{2}\right)}{\cos^2\frac{C}{2}} \geq \frac{\left(3(4R+r)^2-s^2\right)^2}{16R^2\left(s^2+(4R+r)^2\right)}$

Mihály Bencze

PP. 15321. In all triangle ABC holds
$4x^3sR^2r + x^2\left(\left(\frac{s^2+r^2+4Rr}{2}\right)^2 - 4s^2Rr\right) + xsr\left(s^2-4Rr-r^2\right) + s^2r^2 \leq$
$\leq \frac{1}{2}\left(1+\sqrt{4x^2+1}\right)^3 R^4$, for all $x > 0$.

Mihály Bencze

PP. 15322. Let ABC be a triangle. Denote A_1, B_1, C_1 the proiection of point $M \in Int\left(ABC\right)$ to the sides BC, CA, AB. Prove that
$MA_1 \cdot MB_1 \cdot MC_1 \leq \left(\frac{s^2-r^2-4Rr}{9R}\right)^3$.

Mihály Bencze

PP. 15323. In all triangle ABC holds
$4\sqrt{\lambda\left(\lambda-1\right)}sR + s^2 + r^2 + 4Rr \leq 12\lambda R^2$ for all $\lambda \geq 1$.

Mihály Bencze

PP. 15324. Let ABC be a triangle. Determine all $x, y > 0$ such that: $2(x+y)sR + s^2 + r^2 + 4Rr \le 6\left(1 + \frac{x+y+1}{\sqrt{2}}\right)$.

Mihály Bencze

PP. 15325. Solve in Z the equation $\left(x^k - 1\right)(y-1) = (y^n - 1)(x-1)$, where $n, k \in N$.

Mihály Bencze

PP. 15326. Determine all $k \in N$ such that for every positive integer n, there exists an integer m such that $k^m + m$ is divisible by n.

Mihály Bencze

PP. 15327. Determine all integers x, y, z such that

$$\left(x + y^2 + z^3\right)\left(y + z^2 + x^3\right)\left(z + x^2 + y^3\right) = (x+y+z)^{6.}$$

Mihály Bencze

PP. 15328. 1). If $f(n) = \frac{1}{n}\left(\sigma\left(\left[\frac{n}{1}\right]\right) + \sigma\left(\left[\frac{n}{2}\right]\right) + ... + \sigma\left(\left[\frac{n}{n}\right]\right)\right)$, where $[\cdot]$ denotethe integer part and σ the sum of divisors, then $f(n+1) > f(n)$ for infinitely many n, and $f(m+1) < f(m)$ for infinitely many m.

2). Study the general case for $f(n) = \frac{1}{n}\left(F\left(\left[\frac{n}{1}\right]\right) + F\left(\left[\frac{n}{2}\right]\right) + ...F\left(\left[\frac{n}{n}\right]\right)\right)$, where F is an arithmetical function.

Mihály Bencze

PP. 15329. 1). Prove that $\left(\sum\limits_{1 \le i < j \le n} \frac{ij}{i+j}\right)\left(\sum\limits_{1 \le i < j \le n} ij\right)^{-1} \le \frac{1}{n+1}$

2). Determine the maximum of

$$\left(\sum_{1 \le i_1 < ... < i_k \le n} \frac{i_1 ... i_k}{i_1 + ... + i_k}\right)\left(\sum_{1 \le i_1 < ... < i_k \le n} i_1 ... i_k\right)^{-1}.$$

Mihály Bencze

PP. 15330. 1). Prove that $\left(\sum\limits_{0\leq i<j\leq n}\frac{C_n^iC_n^j}{C_n^i+C_n^j}\right)\left(\sum\limits_{0\leq i<j\leq n}C_n^iC_n^j\right)^{-1}\leq\frac{n+1}{2^{n+1}}$

2). Determine the maximum of

$\left(\sum\limits_{1\leq i_1<...<i_k\leq n}\frac{C_n^{i_1}...C_n^{i_k}}{C_n^{i_1}+...+C_n^{i_k}}\right)\left(\sum\limits_{1\leq i_1<...<i_k\leq n}C_n^{i_1}...C_n^{i_k}\right)^{-1}$.

Mihály Bencze

PP. 15331. Let be $x_{n+2}=f\left(\left[x_n\right]\right)+g\left(\left\{x_{n-1}\right\}\right)$, when $x_0,x_1\in R$ are given, and $[\cdot]$, $\{\cdot\}$ denote the integer part, respective the fractional part. Determine all functions $f,g:R\to R$ for which the sequence $\left(x_n\right)_{n\geq1}$ is periodically.

Mihály Bencze

PP. 15332. 1). In all triangle ABC holds: $\sum\limits_{cyclic}\frac{\sqrt{m_a+m_b-m_c}}{\sqrt{m_a}+\sqrt{m_b}-\sqrt{m_c}}\leq3$

2). Determine all $\alpha>0$ for which $\sum\limits_{cyclic}\frac{\left(m_a+m_b-m_c\right)^\alpha}{m_a^\alpha+m_b^\alpha-m_c^\alpha}\leq3$.

Mihály Bencze

PP. 15333. If $x_0=1$ and $n^2x_n^2+\left(n^2-1\right)x_{n-1}^2\leq\left(2n^2-1\right)x_nx_{n-1}$ for all $n\geq1$, then $n-2+\frac{1}{n}\leq\sum\limits_{k=1}^n\frac{x_k}{x_{k-1}}\leq n$.

Mihály Bencze

PP. 15334. If $a,b,c>0$, then $\frac{1+ab}{1+a}+\frac{1+bc}{1+bc}+\frac{ab+256}{ab(1+c)}+\frac{a(bc+256)}{abc+256}\geq\frac{1028}{5}$.

Mihály Bencze

PP. 15335. If $a,b,c\in Z$ such that $a+b+c=1$, then solve in integers the equation $x^3=ax^2+bx+c$.

Mihály Bencze

PP. 15336. If $a_1,a_2,...,a_n,...>0$ is an arithmetical progression, $c>0$, and $b_1,b_2,...,b_n,...>0$ is a geometrical progression then

$\min\left\{\prod\limits_{k=1}^n\frac{c^{a_{k+8}}+c^{a_{k+4}}+c^{a_k}}{c^{a_{k+5}}+c^{a_{k+1}}};\prod\limits_{k=1}^n\frac{b_{k+8}+b_{k+4}+b_k}{b_{k+5}+b_{k+1}}\right\}\geq1.$

Mihály Bencze

PP. 15337. If $x > 0$, then $\prod\limits_{k=0}^{n} (x+k)^{k+1} \geq \frac{((n+1)!x)^{n+1}}{1!2!....n!}$.

Mihály Bencze

PP. 15338. If $\alpha > 0$ and $x, y > 0$, then $\left(\frac{x}{y}\right)^{\alpha} + \left(\frac{y}{x}\right)^{\alpha} - 2 \geq \alpha^2 \left(\frac{x}{y} + \frac{y}{x} - 2\right)$.

Mihály Bencze

PP. 15339. Determine all differentiable function $f : R \to (0, +\infty)$ such that $(\ln f(x))' = f(x) + \frac{1}{f(x)}$.

Mihály Bencze

PP. 15340. Determine all differentiable function $f : R \to (0, +\infty)$ such that $(\ln f(x))' + f(x) + \frac{1}{f(x)} = 0$.

Mihály Bencze

PP. 15341. If $f : R \backslash \left\{-\frac{d}{c}\right\} \to R \backslash \left\{\frac{a}{c}\right\}$, when $f(x) = \frac{ax+b}{cx+d}$, then determine all $a, b, c, d \in R$ for which

$$\begin{cases} f(a) + f(b) + f(c) + f(d) = 0 \\ (f \circ f)(a) + (f \circ f)(b) + (f \circ f)(c) + (f \circ f)(d) = 0 \\ f(a) - f(-c) = f(b) - f(-d) \\ (f \circ f)(a) - (f \circ f)(-c) = (f \circ f)(b) - (f \circ f)(-d) \end{cases}$$

Mihály Bencze

PP. 15342. If $M = \left\{n \in N^* | n! \text{ is divisible by } n^2+1\right\}$ then, compute $\sum\limits_{n \in M} \frac{1}{n}$.

Mihály Bencze

PP. 15343. Solve the following system: $\begin{cases} \left[x_1^2\right] + 1 = [x_2] + [x_3] \\ \left[x_2^2\right] + 1 = [x_3] + [x_4] \\ ---------- \\ \left[x_n^2\right] + 1 = [x_1] + [x_2] \end{cases}$, when $[\cdot]$ denote the integer part.

Mihály Bencze

PP. 15344. Solve the following system:

$$\begin{cases} \left[x_1^k\right]+k-1=[x_2]+[x_3]+...+[x_{k+1}] \\ \left[x_2^k\right]+k-1=[x_3]+[x_4]+...+[x_{k+2}] \\ ------------------ \\ \left[x_n^k\right]+k-1=[x_1]+[x_2]+...+[x_k] \end{cases}$$, when $[\cdot]$ denote the integer part.

Mihály Bencze

PP. 15345. In all triangle ABC holds
$\sum \frac{m_a-m_b+m_c}{(-m_a+m_b+m_c)^2(m_c^2+2m_am_b-m_am_c-m_bm_c)} \geq$
$\geq \frac{3}{2(-m_a+m_b+m_c)(m_a-m_b+m_c)(m_a+m_b-m_c)}$.

Mihály Bencze

PP. 15346. If $a_k>0$ $(k=1,2,...,n)$ and $\prod\limits_{k=1}^{n} a_k=1$, then

1). $\sum\limits_{cyclic} \frac{a_1+a_2+...+a_{n-1}}{a_1^2+a_2^2+...+a_{n-1}^2} \leq \sum\limits_{k=1}^{n} \sqrt[n-1]{a_k}$

2). $\sum\limits_{cyclic} \frac{(a_1+...+a_{n-1})^p}{a_1^k+...+a_{n-1}^k} \leq (n-1)^{p-1} \sum\limits_{i=1}^{n} a_i^{\frac{k-p}{n-1}}$, when $p \in \{0,1,...,k\}$.

Mihály Bencze

PP. 15347. If $a_k \in C$ $(k=1,2,...,n)$ and $p \in \{1,2,...,n\}$, then
$n\left|1+\prod\limits_{k=1}^{n} a_k\right| + \sum\limits_{cyclic} |a_1a_2...a_p+a_{p+1}...a_n| \geq$
$\geq \sum\limits_{cyclic} \sqrt{\left|a_1^2a_2^2...a_p^2-1\right| \cdot \left|a_{p+1}^2...a_n^2-1\right|}$.

Mihály Bencze

PP. 15348. If $x,y,z \in R$, then
$3\left|\cos\frac{x+y+z}{2}\right| + \left|\cos\frac{y+z-x}{2}\right| + \left|\cos\frac{z+x-y}{2}\right| + \left|\cos\frac{x+y-z}{2}\right| \geq$
$\geq \sqrt{|\sin x \sin(y+z)|} + \sqrt{|\sin y \sin(z+x)|} + \sqrt{|\sin z \sin(x+y)|}$.

Mihály Bencze

PP. 15349. If $a,b,c>0$, then $\sum \frac{a^6}{a^6+2b^6} \geq 1$.

Mihály Bencze

PP. 15350. If $x_1 \in (0, \alpha)$, $x_{n+1} = x_n - \alpha x_n^2$ for all $n \geq 1$, then determine all $\alpha > 0$ such that $\lim\limits_{n\to\infty} \left(\left(n + \frac{n^2}{\ln n} \right) x_n - \frac{n}{\alpha \ln n} \right) = 0$.

Mihály Bencze

PP. 15351. If $a, b, c \in N$ $(a \geq 1, b, c \geq 2)$ are given, then determine the function $f : N \to N$ for which $f(0) = 0$ and $f(n) = 1 + af\left(\left[\frac{n}{b} \right] + \left[\frac{n}{c} \right] \right)$ for all $n \geq 1$, where $[\cdot]$ denote the integer part.

Mihály Bencze

PP. 15352. If p, q are prime, then solve in Z the equations:
1). $x^3 - y^3 = pxy + q$ 2). $x^4 - y^4 = pxy(x+y) + q$

Mihály Bencze

PP. 15353. If $a, b, c > 0$, then $2a^2b^2c^2 \sum \frac{1}{b^2(a+c)} \leq \sum (ac)^{\frac{3}{2}}$.

Mihály Bencze

PP. 15354. Prove that $4n \sum\limits_{k=1}^{n} \frac{1}{(n+k)^2} \geq \left(\sum\limits_{k=1}^{n} \frac{1}{k(2k-1)} \right)^2 \geq \left(\frac{4n}{3n+1} \right)^2$.

Mihály Bencze

PP. 15355. If $a_k > 0$ $(k = 1, 2, ..., n)$, then $\sum\limits_{cyclic} \frac{a_1^2 + a_1a_2 + a_2^2}{a_2(a_1+a_2)} \geq \frac{3n}{2}$.

Mihály Bencze

PP. 15356. In all triangle ABC holds

$$\left(\sum \frac{1}{m_a} \right)^2 \leq 3 \sum \frac{1}{(m_a - m_b + m_c)(m_a + m_b - m_c)}.$$

Mihály Bencze

PP. 15357. Determine all $a_1, a_2, ..., a_k \in \left\{ \binom{n}{0}; \binom{n}{1}; ...; \binom{n}{n} \right\}$ for which $\frac{a_1}{a_1+a_2}, \frac{a_2}{a_2+a_3}, \frac{a_3}{a_3+a_4}, ..., \frac{a_{k-1}}{a_{k-1}+a_k}$ are in arithmetical progression.

Mihály Bencze

PP. 15358. If $a_i \in \left\{p^b q^c | p, q \text{ prime and } b, c \in N\right\}$ $(i = 1, 2, ..., n)$, then $\sum\limits_{i=1}^{n} \frac{1}{a_i} < \frac{pq}{(p-1)(q-1)}$.

Mihály Bencze

PP. 15359. Determine all $p \in N^*$ for which $\prod\limits_{k=1}^{n} \left(1 - \frac{1}{pk}\right) \le \frac{1}{\sqrt{(p+1)n+1}}$, for all $n \in N^*$.

Mihály Bencze

PP. 15360. If $a, b, c > 0$, then $\sum \frac{ac+b^2}{a(a^2c^2+b^4)} \le \frac{1}{abc} \sum \sqrt{\frac{a}{b}}$.

Mihály Bencze

PP. 15361. If $z_k \in C$ $(k = 1, 2, ..., n)$, then

$$n\left|\sum_{k=1}^{n} z_k\right|^2 + 8 \sum_{1 \le i < j \le n} \mathrm{Re}(z_i)\mathrm{Re}(z_j) \ge \frac{1}{n}\left|\sum_{k=1}^{n} \overline{z_k} - (n-1)\sum_{k=1}^{n} z_k\right|^2.$$

Mihály Bencze

PP. 15362. If $x \in R \backslash \left\{\frac{k\pi}{2} | k \in Z\right\}$, then

$$\sum_{0 \le j < k \le n} \cos\left(2\left(j+k\right)x\right) = \frac{\sin nx \sin(n+1)x \cos 2nx}{\sin x \sin 2x}.$$

Mihály Bencze

PP. 15363. Solve in R the following system:

$$\begin{cases} \frac{1}{(x_1+x_2)^3+2} + \frac{1}{(x_2+x_3)^3+2} + \frac{1}{(x_3+x_4)^3+2} = \frac{1}{6}\left(\frac{1}{x_5} + \frac{1}{x_6} + \frac{1}{x_7}\right) \\ \frac{1}{(x_2+x_3)^3+2} + \frac{1}{(x_3+x_4)^3+2} + \frac{1}{(x_4+x_5)^3+2} = \frac{1}{6}\left(\frac{1}{x_6} + \frac{1}{x_7} + \frac{1}{x_8}\right) \\ - \\ \frac{1}{(x_n+x_1)^3+2} + \frac{1}{(x_1+x_2)^3+2} + \frac{1}{(x_2+x_3)^3+2} = \frac{1}{6}\left(\frac{1}{x_4} + \frac{1}{x_5} + \frac{1}{x_6}\right) \end{cases}$$

Mihály Bencze

PP. 15364. If $x, y, z > 0$ and $x + y + z = 1$, then $2\left(1 + xy + yz + zx\right) \le$
$\le 3\left(x^2 + xz + z^2 + 3y\right)\left(y^2 + yx + x^2 + 3z\right)\left(z^2 + zy + y^2 + 3x\right)$.

Mihály Bencze and Shanhe Wu

PP. 15365. Denote D, E, F the contact points of incircle of triangle ABC with sides BC, CA, AB and $R_{DB}, R_{DC}, R_{EC}, R_{EA,}R_{FA}, R_{FB}$ the radii of circles inscribed in triangles ADB, ADC, BEC, BEA, CFA, CFB. Prove that:

1). $\sum \frac{\sqrt{R_{DB}\cdot R_{DC}}}{R_{DB}+R_{DC}} \le \frac{3}{2}\sin\frac{\widehat{ADB}+\widehat{BEC}+\widehat{CFA}}{3}$

2). $\prod \frac{\sqrt{R_{DB}\cdot R_{DC}}}{R_{DB}+R_{DC}} \le \frac{1}{8}\sin^3\frac{\widehat{ADB}+\widehat{BEC}+\widehat{CFA}}{3}$

Mihály Bencze

PP. 15366. If $x_0 = 1$ and $x_{n+1}^2 + x_n^2 = 2 + 4x_n x_{n+1}$ for all $n \ge 0$, then compute $\sum\limits_{k=0}^{\infty} \frac{1}{x_k^2}$.

Mihály Bencze and Zhao Changjian

PP. 15367. 1). If $a_k, b_k \in R$ $(k = 1, 2, ..., n)$, then

$\max\left\{\sum\limits_{k=1}^{n} a_k, \sum\limits_{k=1}^{n} b_k,\right\} \le \sum\limits_{k=1}^{n} \max\{a_k, b_k\}$

2). Determine all $\alpha \in R$ for which

$\max\left\{\sum\limits_{k=1}^{n} a_k^\alpha, \sum\limits_{k=1}^{n} b_k^\alpha\right\} \le \left(\sum\limits_{k=1}^{n} \max\{a_k, b_k\}\right)^\alpha$

for all $a_k, b_k > 0$ $(k = 1, 2, ..., n)$.

3). If $a_{ij} \in R$ $(i = 1, 2, ..., n; j = 1, 2, ..., m)$, then

$\max\left\{\sum\limits_{i=1}^{n} a_{i1}, \sum\limits_{i=1}^{n} a_{i2}, ..., \sum\limits_{i=1}^{n} a_{im}\right\} \le \sum\limits_{i=1}^{n} \max\{a_{i1}, a_{i2}, ..., a_{im}\}$.

Mihály Bencze

PP. 15368. Determine all $n \in N$ for which $\sqrt{n} < \frac{2^{2n}(n!)^2}{(2n)!}e^{-\frac{c}{2}} < 2\sqrt{n}$, when $c = 0,57...$ is the Euler's constant.

Mihály Bencze

PP. 15369. If $z_1, z_2, z_3 \in C$, then

$6\left(|z_1|^2 + |z_2|^2 + |z_3|^2\right) + 3|z_1 + z_2 + z_3|^2 \ge$

$\ge 12\mathrm{Re}(z_1z_2 + z_2z_3 + z_3z_1) + |\overline{z_1} + \overline{z_2} + \overline{z_3} - 2(z_1 + z_2 + z_3)|^2$.

Mihály Bencze

PP. 15370. In all triangle holds:
1). $\prod tg^2\left(\frac{A}{2}+\frac{\pi}{4}\right)=\frac{4R^2+4sR+s^2+r^2+4Rr+2sr}{4R^2-4sR+s^2+r^2+4Rr-2sr}$. 2). $s^2\geq 3r\left(4R+r\right)$

Mihály Bencze

PP. 15371. In all triangle ABC holds $\sum \sin\frac{A}{2}\sin A\cos\frac{B-C}{2}\geq\frac{2s}{R}$.

Mihály Bencze

PP. 15372. If $a,b,c>1$ and $x>0$ then
$\left(x+\sum\log_a b\right)^2\geq 3\log_b a+2x\sum\sqrt{\log_b a}+x^2$.

Mihály Bencze

PP. 15373. If $a,b\geq 1$, then solve the following equation
$a^{x+\frac{2}{x}}+b^{x+\frac{2}{x}}+3\left(ab\left(a+b\right)\right)^{\frac{2x}{3}+\frac{1}{3x}}=\left(a+b\right)^3$.

Mihály Bencze

PP. 15374. In all triangle ABC holds:
1). $s^2\geq 3r\left(4R+r\right)$
2). $\left(s^2+r^2+4Rr\right)^2\geq 24s^2Rr$
3). $4R+r\geq 3s$
4). $4\left(2R-r\right)^2\geq 3\left(s^2+r^2-8Rr\right)$
5). $4\left(4R+r\right)^2\geq 3\left(s^2+\left(4R+r\right)^2\right)$

Mihály Bencze

PP. 15375. In all triangle ABC holds: $\sum\frac{m_a^2}{a(b^2+c^2-a^2)}\geq\frac{9}{4s}$.

Mihály Bencze

PP. 15376. If $x\in R$, $n\in N$, then
$\int\limits_0^1\min\left\{\sum\limits_{k=0}^n x^k;\sum\limits_{k=0}^n(-x)^k\right\}e^{nx}dx\geq\frac{(n+1)(e^n-1)}{n(2n+1)}$.

Mihály Bencze

PP. 15377. In all triangle ABC holds $R\geq\frac{R\left(s^2+r^2+4Rr\right)}{2(s^2-r^2-4Rr)}\geq 2r$ (A refinement of Euler's inequality).

Mihály Bencze

PP. 15378. If $x_n = \sqrt{n + \sqrt{n - 1 + ... + \sqrt{2 + \sqrt{1}}}}$ and $y_n = \sqrt{1 + \sqrt{2 + ... + \sqrt{n - 1 + \sqrt{n}}}}$, then compute $\lim\limits_{n \to \infty} n^{\alpha} \left(x_n - y_n \right)$, when $\alpha \in R$.

Mihály Bencze

PP. 15379. If M is an interior point in triangle ABC, and R_a, R_b, R_c denote the circumradii of triangles MBC, MCA, MAB, then

1). $\sum \frac{MB \cdot MC}{R_b R_c} \leq 4 \left(1 - \frac{r}{2R} \right)$

2). $\prod \left(\frac{MB}{R_c} + \frac{MC}{R_b} \right) \leq \frac{16r}{R}$

3). $\prod \frac{MA}{R_a} \leq \frac{2r}{R}$

4). $\sum \frac{MB \cdot R_b + MC \cdot R_b}{R_b R_c \sin \frac{A}{2}} \leq 12$

5). $\sum \left(\left(\frac{MB}{R_c} \right)^{\beta} + \left(\frac{MC}{R_c} \right)^{\beta} \right) \leq 3 \cdot 2^{\beta}$ for all $\beta \geq 1$

6). $\sum \left(\frac{MB \cdot MC}{R_b R_c} \right)^{\frac{\alpha}{2}} \leq 3$ for all $\alpha \in [0, 1]$

Mihály Bencze

PP. 15380. 1). If $a, b, c \geq 2$ and $a, b, c \in N$ then $3 \left(a + b + c \right)^n \geq n \left(n^2 + n + 1 \right) \left(a + b + c \right) + 3$ for all $n \geq 2$

2). If $a_i \geq 2$, $a_i \in N$ $(i = 1, 2, ..., k)$, then $k \left(a_1 + a_2 + ... + a_k \right)^n \geq n \left(n^{k-1} + n^{k-2} + ... + n + 1 \right) \left(a_1 + a_2 + ... + a_k \right) + k$, for all $n \geq 2$

Mihály Bencze

PP. 15381. If $a, b, c \geq 2$ and $a, b, c \in N$ then determine all $n, m, k \in N$ and $n, m, k \geq 2$ for which $(a + b + c)^{n+m+k} \geq \left(an^3 + bm^2 + ck + 1 \right) \left(am^3 + bk^2 + cn + 1 \right) \left(ak^3 + bn^2 + cm + 1 \right)$.

Mihály Bencze

PP. 15382. If $x_0, y_0, z_0 \in \left(-\frac{1}{2}, +\infty \right)$ and $x_{n+1} \left(1 + y_n + z_n \right) = 1$, $y_{n+1} \left(1 + z_n + x_n \right) = 1$, $z_{n+1} \left(1 + x_n + y_n \right) = 1$ for all $n \geq 1$, then the sequences $\left(x_n \right)_{n \geq 0}$, $\left(y_n \right)_{n \geq 0}$, $\left(z_n \right)_{n \geq 0}$ are convergent, and compute its limit.

Mihály Bencze

PP. 15383. If $a_k \in N$, $a_k \geq 2$ $(k = 1, 2, ..., n)$, then solve in N the following system:

$$\begin{cases} a_1 x_1^n + a_2 x_1^{n-1} + ... + a_n x_1 + 1 = \left(\sum\limits_{k=1}^{n} a_k \right)^{x_2} \\ a_1 x_2^n + a_2 x_2^{n-1} + ... + a_n x_2 + 1 = \left(\sum\limits_{k=1}^{n} a_k \right)^{x_3} \\ ----------------------- \\ a_1 x_n^n + a_2 x_n^{n-1} + ... + a_n x_n + 1 = \left(\sum\limits_{k=1}^{n} a_k \right)^{x_1} \end{cases} .$$

Mihály Bencze

PP. 15384. Let be $a > 1$ and $\alpha_k, \beta_k \in R^*$ $(k = 1, 2, ..., n)$ such that $\sum\limits_{k=1}^{n} a^{\alpha_k x} \geq \sum\limits_{k=1}^{n} a^{\beta_k x}$ for all $x \in A \subset R$. If $A = (-\varepsilon, \varepsilon)$, when $\varepsilon > 0$ then $\sum\limits_{k=1}^{n} \alpha_k \geq \sum\limits_{k=1}^{n} \beta_k$.

Mihály Bencze

PP. 15385. If $x_0 \in [0, 1]$ and $x_{n+1}^2 + x_n^2 = x_n$ for all $n \geq 1$, then compute $\lim\limits_{n\to\infty} n \left(x_n - \frac{1}{2} \right)$.

Mihály Bencze

PP. 15386. If $x_n = \sum\limits_{k=0}^{n} \frac{1}{\binom{n}{k}}$ for all $n \geq 1$. Compute $\lim\limits_{n\to\infty} n \left(n a_n - 2 \left(n + 1 \right) \right)$.

Mihály Bencze

PP. 15387. If $x_1 > 0$ and $x_{n+1} = \ln \left(1 + x_n \right)$ for all $n \geq 1$, then compute:

1). $\lim\limits_{n\to\infty} n x_n$ 2). $\lim\limits_{n\to\infty} n \left(n x_n - 2 \right)$ 3). $\lim\limits_{n\to\infty} n \left(\frac{n(n x_n - 2)}{\ln n} - \frac{2}{3} \right)$

Mihály Bencze

PP. 15388. Let be $m_i = \left[\frac{in}{k} \right]$ $(i = 1, 2, ..., k - 1)$ where $[\cdot]$ denote the integer part. If $a_0 = 1$ and $a_n = a_{m_1} + a_{m_2} + ... + a_{m_{k-1}}$ for all $n \geq 1$, then determine the general term of the sequence $\left(a_n \right)_{n \geq 0}$.

Mihály Bencze

PP. 15389. If $x, y > 0$, then

$$2^{n+2}\left(x^{2n}+y^{2n}\right)\left(x^{3n}+y^{3n}\right) \geq (x+y)^n\left(x^n+y^n\right)^4.$$

Mihály Bencze

PP. 15390. 1). If $x_k > 0$ $(k = 1, 2, ..., n)$ and $x_1x_2 \leq 1$, $x_2x_3 \leq 1, ..., x_nx_1 \leq 1$, then $\sum\limits_{k=1}^{n} arctg x_k \leq \frac{n\pi}{4}$

2). If $x \in \left(0, \frac{\pi}{4}\right)$ $(k = 1, 2, ..., n)$, then $\sum\limits_{k=1}^{n} arctg\left(\frac{\sin x_k + \cos x_k}{1 - \sin x_k \cos x_k}\right) \leq \frac{n\pi}{2}$.

Mihály Bencze

PP. 15391. Solve the following system: $\sqrt[4]{2\left(49 - x_1\right)} + \sqrt{2\left(4 + x_2\right)} =$

$$= \sqrt[4]{2\left(49 - x_2\right)} + \sqrt{2\left(4 + x_3\right)} = ... = \sqrt[4]{2\left(49 - x_n\right)} + \sqrt{2\left(4 + x_1\right)} = 8.$$

Mihály Bencze

PP. 15392. For all $n \geq 1$, $n \in N$ the equation $\left(\frac{1}{n^k+1+x}\right)^{\frac{1}{k}} + \left(\frac{1}{n^k+2+x}\right)^{\frac{1}{k}} + ... + \left(\frac{1}{n^k+1+x}\right)^{\frac{1}{k}} = 1$ have an unique real solution, denoted x_n and $k \in N$, $k \geq 1$

1). Study the convergence of the sequence $\left(x_n\right)_{n\geq 1}$

2). Compute $\lim\limits_{n\to\infty} \frac{x_n}{n^\alpha}$, where $\alpha \in R$.

Mihály Bencze

PP. 15393. Let ABC be a triangle. Determine all $x, y \in R$ such that $\sum \frac{1}{\left(\sin\frac{A}{2}\right)^x\left(\cos\frac{A}{2}\right)^{x+y}\left(\cos\frac{B-C}{2}\right)^y} \geq \frac{64}{3}$.

Mihály Bencze

PP. 15394. If $a_k > 0$ $(k = 1, 2, ..., n)$, then

$$\left(\sum_{k=1}^{n}\frac{1}{a_k}\right)\left(\sum_{k=1}^{n}a_k^2\right) + \sum_{k=1}^{n}a_k\left(a_k - 1\right) \geq \left(\sum_{k=1}^{n}\frac{1}{(n-1)a_k+1}\right)\left(\sum_{k=1}^{n}a_k\right)^2.$$

Mihály Bencze

PP. 15395. In all triangle ABC holds:

1). $\left(tg\frac{A}{2}\right)^{tg\frac{B}{2}}\left(tg\frac{B}{2}\right)^{tg\frac{C}{2}}\left(tg\frac{C}{2}\right)^{tg\frac{A}{2}}+\left(tg\frac{A}{2}\right)^{tg\frac{C}{2}}\left(tg\frac{B}{2}\right)^{tg\frac{A}{2}}\left(tg\frac{C}{2}\right)^{tg\frac{B}{2}}\leq\frac{2s}{4R+r}$

2). $\left(ctg\frac{A}{2}\right)^{ctg\frac{B}{2}}\left(ctg\frac{B}{2}\right)^{ctg\frac{C}{2}}\left(ctg\frac{C}{2}\right)^{ctg\frac{A}{2}}+\left(ctg\frac{A}{2}\right)^{ctg\frac{C}{2}}\left(ctg\frac{B}{2}\right)^{ctg\frac{A}{2}}\left(ctg\frac{C}{2}\right)^{ctg\frac{B}{2}}\leq$
$\leq\frac{2(4R+r)}{s}$

Mihály Bencze

PP. 15396. In all triangle ABC holds $\sum\frac{1}{3+ctg\frac{A}{2}ctg\frac{B}{2}+\sqrt{3ctg\frac{A}{2}ctg\frac{B}{2}}}\leq\frac{1}{3}$.

Mihály Bencze

PP. 15397. Determine all $a,b,c\in R^*$ such that
$\left|ax^2+bx+c\right|+\left|bx^2+cx+a\right|+\left|cx^2+ax+b\right|\leq$
$\leq\left(x-\frac{1}{a}\right)^2+\left(x-\frac{1}{b}\right)^2+\left(x-\frac{1}{c}\right)^2$ for all $x\in R$.

Mihály Bencze

PP. 15398. In all triangle ABC holds
$3s^2+\sum\frac{\left(s^2-16Rr-4r^2+2srtg\frac{C}{2}+s^2tg^2\frac{A}{2}tg^2\frac{B}{2}\right)\cos\frac{A}{2}\cos\frac{B}{2}}{\cos\left(\frac{A-B}{2}\right)}\leq$
$\leq 4\left(s^2-8Rr-2r^2\right)\sum\frac{\cos\frac{A}{2}\cos\frac{B}{2}}{\cos\left(\frac{A-B}{2}\right)}$.

Mihály Bencze

PP. 15399. Let consider the following system:

$$\begin{cases}(x_1+2)\left(x_2^3+2\right)=3\left(2x_3^2+1\right)\\(x_2+2)\left(x_3^3+2\right)=3\left(2x_4^2+1\right)\\-------------\\(x_n+2)\left(x_1^3+2\right)=3\left(2x_2^2+1\right)\end{cases}$$

1). Solve in R^+ 2). Solve in R 3). Solve in C
4). Solve in N 5). Solve in Z 6). Solve in Q

Mihály Bencze

PP. 15400. If $a,b>0$, then $\int\limits_a^b\frac{dx}{x^3+2}\leq\frac{1}{12}\ln\frac{2a^2+1}{2b^2+1}+\frac{\sqrt{2}}{3}arctg\frac{\sqrt{2}(a-b)}{1+2ab}$.

Mihály Bencze

PP. 15401. If $x_k > 0$ $(k = 1, 2, ..., n)$, then determine all $a \in R$ such that $\sum\limits_{k=1}^{n} \frac{a+x_k}{ax_k^2+a-1} \geq (a+1) \sum\limits_{k=1}^{n} \frac{1}{a+x_k^3}$.

Mihály Bencze

PP. 15402. Determine all $x, y \in R$ such that $\frac{x^2+2y}{2x^2+1} + \frac{y^2+2x}{2y^2+1} \geq \frac{3x}{y^2+2} + \frac{3y}{x^3+2}$.

Mihály Bencze

PP. 15403. If $x_k > 0$ $(k = 1, 2, ..., n)$ and $\sum\limits_{k=1}^{n} x_k^3 = n$, then $\sum\limits_{k=1}^{n} \frac{x_k^2(x_k+2)}{2x_k^2+1} \geq \frac{1}{n} \left(\sum\limits_{k=1}^{n} x_k \right)^2$.

Mihály Bencze

PP. 15404. If $x_k > 0$ $(k = 1, 2, ..., n)$ and $\sum\limits_{k=1}^{n} \frac{1}{ch^2 x_k} = \alpha$, then $\sum\limits_{k=1}^{n} \frac{2+\sqrt[3]{2sh^2 x_k}}{1+2\sqrt[3]{4sh^4 x_k}} \geq \frac{3\alpha}{2}$.

Mihály Bencze

PP. 15405. If $x_k \in R$ $(k = 1, 2, ..., n)$ and $\sum\limits_{k=1}^{n} \cos^2 x_k = \alpha$, then $\sum\limits_{k=1}^{n} \frac{2+\sqrt[3]{2tg^2 x_k}}{1+2\sqrt[3]{4tg^4 x_k}} \geq \frac{3\alpha}{2}$.

Mihály Bencze

PP. 15406. If $x_k > 0$ $(k = 1, 2, ..., n)$ and $\sum\limits_{k=1}^{n} x_k^3 = \alpha$, then $\sum\limits_{k=1}^{n} \frac{x_k+2\sqrt{2}}{x_k^2+1} \geq \frac{12n^2}{\alpha+4n\sqrt{2}}$.

Mihály Bencze

PP. 15407. Determine all $a, b \in R$ such that $\sum\limits_{k=1}^{an+b} \frac{1}{n+k} \geq 1$ for all $n \in N^*$.

Mihály Bencze

PP. 15408. Let be $A \subset N$ and $f : A \to A$ an injective function and $f_n = \underbrace{f \circ f \circ ... \circ f}_{n-time}$. Determine the function f if exist $p_1, p_2, ..., p_k \in N^*$, $(p_1, p_2, ..., p_k) = 1$ such that $f_{p_1}(x) + f_{p_2}(x) + ... + f_{p_k}(x) = kx$ for all $x \in A$. What happend if $(p_1, p_2, ..., p_k) \neq 1$?

Mihály Bencze

PP. 15409. Determine all $a, b, c \in R$ such that $\sum\limits_{k=1}^{n} \left(\frac{a}{3k-1} + \frac{b}{3k} + \frac{c}{3k+1} \right) = \sum\limits_{p=1}^{2n+1} \frac{1}{n+p}$.

Mihály Bencze

PP. 15410. Determine all $a_1, a_2, ..., a_{2p+1} \in R$ such that

$$\sum_{k=1}^{n} \left(\frac{a_1}{(2p+1)k-p} + \frac{a_2}{(2p+1)k-p+1} + \frac{a_3}{(2p+1)k-p+2} + ... \right.$$
$$\left. + \frac{a_{2p-1}}{(2p+1)k+p-2} + \frac{a_{2p}}{(2p+1)k+p-1} + \frac{a_{2p+1}}{(2p+1)k+p} \right) = \sum_{t=1}^{2pn+p} \frac{1}{n+t}.$$

Mihály Bencze

PP. 15411. 1). Compute $\alpha = \lim\limits_{n\to\infty} \frac{n}{2^n} \sum\limits_{k=0}^{n} \frac{\binom{n}{k}}{2k+1}$

2). Compute $\lim\limits_{n\to\infty} n \left(\alpha - \frac{n}{2^n} \sum\limits_{k=0}^{n} \frac{\binom{n}{k}}{2k+1} \right)$

Mihály Bencze

PP. 15412. Solve in N the following equations:

1). $\sum\limits_{k=1}^{n} k^2 = m^3$ 2). $\sum\limits_{k=1}^{n} k^3 = m^4$ 3). $\sum\limits_{k=1}^{n} k^4 = m^5$

Mihály Bencze

PP. 15413. Determine all $a > 0$ for which if $a - \frac{1}{n} < x < a + \frac{1}{n}$, where $n \in N^*$, then $a - \frac{1}{n+k} < \sqrt{\underbrace{a + \sqrt{a + ... + \sqrt{a + x}}}_{k-time}} < a + \frac{1}{n+k}$ for all $k \in N^*$.

Mihály Bencze

PP. 15414. If $p \geq 3$ is a prime, then solve in Z the equation

$x^3 + y^3 = x^2y + xy^2 + p^n$, where $n \in N$.

Mihály Bencze

PP. 15415. If $x_k > 0$ $(k = 1, 2, ..., n)$ and $\sum\limits_{k=1}^{n} x_k^2 = 1$, then compute min and max of the expression $\frac{1}{\sqrt{4-x_1^2-x_2^3}} + \frac{1}{\sqrt{4-x_2^2-x_3^3}} + ... + \frac{1}{\sqrt{4-x_n^2-x_1^3}}$.

Mihály Bencze

PP. 15416. If $x_k \in \left(0, \frac{\pi}{2}\right)$ $(k = 1, 2, ..., n)$ and $\sum\limits_{k=1}^{n} x_k^2 = 1$, then compute min and max of the expression $\sum\limits_{k=1}^{n} \left(\left(\frac{\sin x_k}{x_k}\right)^2 + \frac{tgx_k}{x_k}\right)$.

Mihály Bencze

PP. 15417. Solve the following system: $\sqrt[3]{x_1 + 2} + \sqrt[3]{2x_2 + 2} + \sqrt[3]{3x_3 + 2} =$

$= \sqrt[3]{x_2 + 2} + \sqrt[3]{2x_3 + 2} + \sqrt[3]{3x_4 + 2} = ... = \sqrt[3]{x_n + 2} + \sqrt[3]{2x_1 + 2} + \sqrt[3]{3x_2 + 2} = 0.$

Mihály Bencze

PP. 15418. Prove that $\frac{\pi}{2} \cos \frac{x-y}{2} \left(\sin \frac{x+y}{2} + \cos \frac{x+y}{2}\right) \leq$
$\leq \int\limits_0^{\frac{\pi}{2}} \sqrt{(\cos x \sin t)^2 + (\sin x \cos t)^2} dt + \int\limits_0^{\frac{\pi}{2}} \sqrt{(\cos y \sin t)^2 + (\sin y \cos t)^2} dt \leq$
$\leq \frac{\pi\sqrt{2}}{2}$.

Mihály Bencze

PP. 15419. Determine all $n \in N^*$ and $x \in R$
$(68 \cos x + 55)^{\frac{1}{n}} + (68 \cos x - 55)^{\frac{1}{n}} = \sqrt{20}$.

Mihály Bencze

PP. 15420. Determine all $n \in N^*$ and $x \in R$ such that
$(45 + 58 \sin x)^{\frac{1}{n}} + (45 - 58 \cos x)^{\frac{1}{n}} = 6$.

Mihály Bencze

PP. 15421. Determine all $x, y \in R$ such that

$$\sqrt{\sqrt[4]{x^3}+\sqrt{\sqrt{y}-1}}-\sqrt{\sqrt[4]{y^3}+\sqrt{\sqrt{x}-1}}=$$
$$=\sqrt{\frac{x+y}{2}}\left(\sqrt{\sqrt[4]{\left(\frac{x+y}{2}\right)^3}-\sqrt{\sqrt{\frac{x+y}{2}}+1}}\right).$$

Mihály Bencze

PP. 15422. Solve the following equation
$\left(\sqrt[3]{\cos 2x}+\sqrt[3]{\cos 4x}-\sqrt[3]{\cos x}\right)^3=\frac{3}{2}\left(\sqrt[3]{9}-2\right).$

Mihály Bencze

PP. 15423. The triangle ABC with sides a, b, c is rectangle if and only if

$$2\left(a^2b^2-\left(a^2+b^2\right)c^2\right)^4=a^8\left(b^2+c^2\right)^4+b^8\left(a^2+c^2\right)^4+c^8\left(a^2-b^2\right)^4.$$

Mihály Bencze

PP. 15424. Solve the equation $\frac{tgx}{\sqrt{2}-\sqrt{tgx}}+\frac{tg5x}{\sqrt{2}+\sqrt{tg5x}}=\sqrt{2}.$

Mihály Bencze

PP. 15425. Determine all $x, y \geq 1$ such that

$$e\left(\frac{14x+5}{14y+12}+\frac{14y+5}{14x+12}\right)<\left(1+\frac{1}{x}\right)^y+\left(1+\frac{1}{y}\right)^x<e\left(\frac{12x+5}{12y+11}+\frac{12y+5}{12x+11}\right).$$

Mihály Bencze

PP. 15426. Solve in Z the following equations:
1). $x^4-y^4=2009z^2$ 2). $x^4-y^4=2009z^3$

Mihály Bencze

PP. 15427. If $a_k, b_k \in Z$ $(k=1,2,...,n)$, then solve in Z the following system: $a_1x_1^2-b_1x_2^2=a_2x_2^2-b_2x_3^2=...=a_nx_n^2-b_nx_1^2=1.$

Mihály Bencze

PP. 15428. Let be Ψ the digamma function, and $x_n = 2\ln n! - \sum\limits_{k=1}^{n} \Psi\left(k^2\right).$
Prove that:
1). $\frac{3n}{(n+1)(2n+1)} < x_n < 2 - \frac{1}{n}$ for all $n \geq 1$
2). $\lim\limits_{n\to\infty} x_n \in \left(\frac{\pi^2}{12}, \frac{\pi^2}{6}\right)$ 3). Compute $\lim\limits_{n\to\infty} x_n$

Mihály Bencze

PP. 15429. In all triangle ABC holds
$\sum \left(\cos\frac{A}{2}\cos\frac{B}{2}\right)^{\alpha} \leq 3\left(\frac{\left(s^2+r^2+4Rr\right)^2}{96R^2r^2} - \frac{s^2}{12Rr} + \frac{2R-r}{12R}\right)^{\alpha}$ for all $\alpha \in [0,1]$.

Mihály Bencze

PP. 15430. In all triangle ABC holds $\frac{2\sqrt{3}s}{R} \leq \left(\sum\cos\frac{A}{2}\right)\left(\sum\frac{1}{\cos\frac{A}{2}}\right) \leq \frac{9R}{r}.$

Mihály Bencze

PP. 15431. In all triangle ABC holds
$\sum\left(\frac{m_a}{a}\right)^2 \geq \frac{1}{3}\left(\frac{a}{b} + \frac{a}{c} + 2\sqrt{\frac{b}{c}} + 2\sqrt{\frac{b}{a}} + 3\sqrt[3]{\frac{c}{a}} + 3\sqrt[3]{\frac{c}{b}}\right) - \frac{17}{2}.$

Mihály Bencze

PP. 15432. In all triangle ABC holds $\sum\left(\frac{\cos\frac{A}{2}\cos\frac{B}{2}}{\sin\frac{C}{2}}\right)^{\alpha} \geq 3\left(\frac{s^2+r^2+4Rr}{12Rr}\right)^{\alpha}$
for all $\alpha \in (-\infty, 0] \cup [1, +\infty)$. If $\alpha \in (0,1)$, then holds the reverse inequality.

Mihály Bencze

PP. 15433. In all triangle ABC holds
$\frac{4R+7r}{16s^3R^3r^2} \leq \sum\frac{1}{a^2(a-b)(a-c)} \leq \frac{27R^2+4r^2+4Rr}{256s^3R^3r^3}.$

Mihály Bencze

PP. 15434. In all triangle ABC holds $2\sum\cos\frac{A}{4} + \sum\cos\left(\frac{\pi}{4} + \frac{A}{4}\right) > \frac{9}{2}.$

Mihály Bencze

PP. 15435. If $a, b, c \in C$, then
$4\left(|a|^3 + |b|^3 + |c|^3\right) \leq |a+b|^3 + |b+c|^3 + |c+a|^3 + |a-b|^3 + |b-c|^3 + |c-a|^3.$

Mihály Bencze

PP. 15436. 1). If $\alpha \geq 2$, then for all $z_1, z_2 \in C$ holds
$2\left(|z_1|^\alpha + |z_2|^\alpha\right) \leq |z_1 + z_2|^\alpha + |z_1 - z_2|^\alpha$
2). If $\alpha \geq 2$ and $z_1, z_2, z_3 \in C$ then
$\sqrt[3]{\prod\left(|z_1|^\alpha + |z_2|^\alpha\right)} \leq \frac{1}{6}\left(\sum|z_1 + z_2|^\alpha + \sum|z_1 - z_2|^\alpha\right).$

Mihály Bencze

PP. 15437. If $A_k \in M_2(R)$ $(k = 1, 2, ..., n)$, then

$$\sum_{cyclic}\left(\det\left(A_1^2 + A_2^2\right)\right)^{2n} \geq \left(\det\left(\prod_{cyclic}(A_1A_2 - A_2A_1)\right)\right)^2.$$

Mihály Bencze

PP. 15438. Prove that

$$\sum_{k=1}^{n}\frac{1}{k\,(k!)^{\frac{1}{k}}\,((k+1)!)^{\frac{1}{k+1}}} \leq \frac{n\,(n+3)}{(n+1)\,(n+2)}.$$

Mihály Bencze

PP. 15439. If $z_1, z_2, z_3 \in C$ then
$(|1 + z_1z_2| + |z_1 + z_2|)(|1 + z_2z_3| + |z_2 + z_3|)(|1 + z_3z_1| + |z_3 + z_1|) \geq$
$\geq \left|\left(z_1^2 - 1\right)\left(z_2^2 - 1\right)\left(z_3^2 - 1\right)\right|.$

Mihály Bencze

PP. 15440. If $a, b, c \in C$ such that $a|bc| + b|ca| + c|ab| = 0$, then
$|(a-b)(b-c)(c-a)| \leq \left(|a|^2 + |b|^2 + |c|^2\right)^{\frac{3}{2}}.$

Mihály Bencze

PP. 15441. In all triangle ABC $(a \neq b \neq c)$ we have
$3r\sqrt{3} \leq \sum\frac{a^3}{(a-b)(a-c)} \leq \frac{3\sqrt{3}}{2}R$ (A refinement of Euler's inequality).

Mihály Bencze

PP. 15442. In all triangle ABC $(a \neq b \neq c)$ we have
$2r(13r - 2R) \leq \sum\frac{a^4}{(a-b)(a-c)} \leq \frac{1}{4}\left(27R^2 - 4r^2 - 16Rr\right).$

Mihály Bencze

PP. 15443. If $a > 1$, then solve the following system:

$\begin{cases} a^{[x_1]} + a^{\{x_2\}} = a^{x_3+1} \\ a^{[x_2]} + a^{\{x_3\}} = a^{x_4+1} \\ ------------ \\ a^{[x_n]} + a^{\{x_1\}} = a^{x_2+1} \end{cases}$, where $[\cdot]$ and $\{\cdot\}$ denote the integer respective the fractional part.

Mihály Bencze

PP. 15444. 1). If $a, b, c \in (0,1)$ or $a, b, c > 1$ then $\sum \log_{a^2 b} a \leq 1$.

2). What happen's if $a, b > 1$ and $c \in (0,1)$?

Mihály Bencze

PP. 15445. If $a, b, c > 1$, then $\sum \frac{\ln^2 a \ln b \ln c}{\ln^2 b \left(\ln^2 c + \ln a \ln b\right)} \geq \frac{3}{2}$.

Mihály Bencze

PP. 15446. If $a, b, c > 1$, then $2\left(\sum \log_{ac} b\right) \leq \sum \frac{\ln b}{\sqrt{\ln a \ln c}}$.

Mihály Bencze

PP. 15447. If $z \in C$, $|z| = 1$ and $\mathrm{Im}(z) > 0$, then

$$|z+1|^2 + |z-1|^2 + \sqrt{2}\left|z^2 + i\right| = \left|z^2 + 1\right| + 2\left|z + i\right|^2 .$$

Mihály Bencze

PP. 15448. Let ABC be a triangle with sides a, b, c. Compute the integer part of the expression: $\frac{\left(a^k + b^k + c^k\right)\left(a^n + b^n + c^n\right)}{a^{k+n} + b^{k+n} + c^{k+n}}$.

Mihály Bencze

PP. 15449. If $a_k, b_k, x_k \in R$ $(k = 1, 2, ..., n)$ and $\sum\limits_{k=1}^{n} a_k^2 = \sum\limits_{k=1}^{n} b_k^2 = 1$, then $\sum\limits_{k=1}^{n} \left(a_k + b_k \sin x_k\right)^2 + \sum\limits_{k=1}^{n} \left(a_k + b_k \cos x_k\right)^2 \leq 4$.

Mihály Bencze

PP. 15450. Determine all functions $f, g, h : N^* \to N^*$ such that

$$\begin{cases} f(ab) = (a, g(b)) \cdot [h(a), b] \\ g(ab) = (a, h(b)) \cdot [f(a), b] \\ h(ab) = (a, f(b)) \cdot [g(a), b] \end{cases}, \text{ for all } a, b \in N^*.$$

Mihály Bencze

PP. 15451. If $z_1, z_2, z_3 \in C$ and $|z_1| = |z_2| = |z_3| = 1$, then $2\sum |z_1 + 1| + 3\sum |z_1 z_2 + 1| + \sum \left|z_1^2 z_2 z_3 + 1\right| \geq 12$.

Mihály Bencze

PP. 15452. Let ABC be a triangle and $a \leq b \leq c$. Prove that for all $x \geq 0$ holds $\frac{1}{a+bx} + \frac{1}{b+cx} + \frac{1}{a+cx} + x\left(\frac{1}{ax+b} + \frac{1}{bx+c} + \frac{1}{ax+c}\right) \geq \frac{5s^2+r^2+4Rr}{s(s^2+r^2+2Rr)}$.

Mihály Bencze

PP. 15453. Prove that for all $x \in (0,1)$ exist $n_k \in Z$ $(k = 1, 2, ..., 6m)$ such that $2m \leq \sum\limits_{k=1}^{6m} \{n_k x\} \leq 3m$, where $\{\cdot\}$ denote the fractional part.

Mihály Bencze

PP. 15454. Let ABC be a triangle and $a \leq b \leq c$. Prove that $\left(\sqrt{\frac{a}{b+1}} + \sqrt{\frac{b}{a+1}}\right)\left(\sqrt{\frac{b}{c+1}} + \sqrt{\frac{c}{b+1}}\right)\left(\sqrt{\frac{a}{c+1}} + \sqrt{\frac{c}{a+1}}\right) \leq$
$\leq \frac{(1+2s)^3 - 2s(1+2s)^2 + (1+2s)\left(s^2+r^2+4Rr\right) - 4sRr}{(a+1)^2(b+1)}$.

Mihály Bencze

PP. 15455. Let ABC be a triangle and $A_1 \in (BC)$, $B_1 \in (CA)$, $C_1 \in (AB)$ such that $\frac{BA_1}{A_1C} = \frac{c^2}{b^2}$, $\frac{CB_1}{B_1A} = \frac{a^2}{c^2}$, $\frac{AC_1}{C_1B} = \frac{b^2}{a^2}$. Prove that
$\frac{AA_1}{bc} + \frac{BB_1}{ca} + \frac{CC_1}{ab} \leq \frac{5s^2+r^2+4Rr}{s(s^2+r^2+2Rr)}$.

Mihály Bencze

PP. 15456. In all triangle ABC holds $\sum tg^3 \frac{A}{2} tg^3 \frac{B}{2} + \frac{8r(R+r)}{s^2} \leq 1$.

Mihály Bencze

PP. 15457. If $n_i \in N\ (i = 1, 2, ..., k)$, then $\prod\limits_{i=1}^{k} \left(2^{3^{n_i}} + 1\right)$ is divisible by $3^{k+\sum\limits_{i=1}^{k} n_i}$.

Mihály Bencze

PP. 15458. If $a_k > 0\ (k = 1, 2, ..., n)$ and $\prod\limits_{k=1}^{n} a_k = 1$, then

$$\left(\sum_{cyclic} \frac{1}{\sqrt{a_n (a_1+a_n)(a_2+a_n)...(a_{n-1}+a_n)}}\right)^2 \le \left(\sum_{k=1}^{n} \frac{1}{a_k}\right)\left(\sum_{k=1}^{n} \frac{a_k}{\left(1+a_k^{\frac{n}{n-1}}\right)^{n-1}}\right).$$

Mihály Bencze

PP. 15459. Let be $a_k, x > 0\ (k = 1, 2, ..., n)$ and

$$f(x) = \frac{1}{a_1+a_2x+...+a_nx^{n-1}} + \frac{1}{a_2+a_3x+...+a_1x^{n-1}} + ... + \frac{1}{a_n+a_1x+...+a_{n-1}x^{n-1}} - \frac{n}{\sum\limits_{k=1}^{n} a_k}$$

1). Solve the equation $f(x) = 0$
2). Solve the inequation $f(x) \le 0$
3). Solve the inequation $f(x) \ge 0$

Mihály Bencze

PP. 15460. If $x_k > 0\ (k = 1, 2, ..., n)$, then

$$\left(\prod_{k=1}^{n} x_k + n - 1\right)\left(\sqrt[n]{\frac{x_1}{x_2}} + \sqrt[n]{\frac{x_2}{x_3}} + ... + \sqrt[n]{\frac{x_n}{x_1}}\right) \ge$$
$$\ge n\left(\sqrt[n]{x_1^2x_3x_4...x_n} + \sqrt[n]{x_2^2x_4...x_1} + ... + \sqrt[n]{x_n^2x_2x_3...x_{n-1}}\right).$$

Mihály Bencze

PP. 15461. Let be $f : R \to R$, where $f(x) = \sum\limits_{k=1}^{n} \{kx\}$. Prove that f is periodical and determine his principial period. Solve the inequality $f(x) \le 1$, where $\{\cdot\}$ denote the fractional part.

Mihály Bencze

PP. 15462. Determine all $x, y, z \in R$ such that

$$\begin{cases} xyz = 1 \\ \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = -1 \\ \frac{1}{2+x+y} + \frac{1}{2+y+z} + \frac{1}{2+z+x} = 1 \end{cases}.$$

Mihály Bencze

PP. 15463. Determine all $x_k > 0$ $(k = 1, 2, ..., n)$ such that $\sum\limits_{k=1}^{n} x_k^4 = \frac{3n^2+3n-1}{5} \left(\sum\limits_{k=1}^{n} x_k^2 \right)$.

Mihály Bencze

PP. 15464. Determine all $x_k > 0$ $(k = 1, 2, ..., n)$ such that $\sum\limits_{k=1}^{n} x_k^5 = \frac{2n^2+2n-1}{3} \left(\sum\limits_{k=1}^{n} x_k^3 \right)$.

Mihály Bencze

PP. 15465. In all triangle ABC holds:
1). $\left(s^2 + r^2 - 2Rr\right)^2 \geq 4\left(s^2 - r^2 - 4Rr\right)Rr$
2). $4R^2 + 4Rr + 3r^2 \geq s^2$
3). $\left((2R - r)\left(s^2 + r^2 - 8Rr\right) - 6Rr^2\right)^2 \geq 16(2R - r)\left(8R^2 + r^2 - s^2\right)Rr^2$
4). $\left((4R + r)\left((4R + r)^2 + s^2\right) - 6Rs^2\right)^2 \geq$
$\geq 16(4R + r)\left((4R + r)^2 - s^2\right)Rs^2$.

Mihály Bencze

PP. 15466. Let be $a_k \in Q^+$ $(k = 1, 2, ..., n)$ such that $\prod\limits_{k=1}^{n} a_k = 1$. If $\sum\limits_{k=1}^{n} \{xa_k\} = 1$ and $x > 0$, then $x \notin Q$.

Mihály Bencze

PP. 15467. In all triangle ABC holds:
1). $\sqrt{tg\frac{A}{2}} + \sqrt{tg\frac{B}{2}} - \sqrt{tg\frac{C}{2}} < 3\sqrt{\frac{s}{r}}$ and its permutations
2). $\sqrt{ctg\frac{A}{2}} - 2\sqrt{ctg\frac{B}{2}} + \sqrt{ctg\frac{C}{2}} \leq \sqrt{\frac{6s}{r}}$ and its permutations.

Mihály Bencze

PP. 15468. If $a_p \in Q$ $(p = 1, 2, ..., m)$ and $\sum\limits_{p=1}^{m} a_p = 0$, then
$\sum\limits_{k=1}^{n} \sum\limits_{p=1}^{m} |x + ka_p| \geq \frac{mn(n+1)}{2}$ for all $x \in R$.

Mihály Bencze

PP. 15469. Determine all $x, y, z, t > 0$ such that in all triangle ABC holds $x\left(a^2 + b^2\right) + yc^2 \geq zc\left(a + b\right) + tab$ (A solution is $x = 9$, $y = 17$, $z = 14$, $t = 6$).

Mihály Bencze

PP. 15470. Solve in R the equation $\left[\sqrt[3]{\frac{17x+7}{3}}\right] + \left[\sqrt{\frac{x+7}{2}}\right] = \frac{25x-1}{6}$, where $[\cdot]$ denote the integer part.

Mihály Bencze

PP. 15471. 1). If $a, b, c > 0$, then $\sum \frac{1}{a^2(b+c)} \geq \frac{3}{2abc}$

2). If $x_k > 0$ $(k = 1, 2, ..., n)$ and $S = \sum\limits_{k=1}^{n} a_k$, then $\sum\limits_{k=1}^{n} \frac{1}{x_k^2(s-x_k)} \geq \frac{n}{(n-1)\prod\limits_{k=1}^{n} x_k}$.

Mihály Bencze

PP. 15472. 1). If $x_k > 0$ $(k = 1, 2, ..., n)$ then

$$\sqrt{\frac{x_1}{x_2}} + \sqrt{\frac{x_2}{x_3}} + ... + \sqrt{\frac{x_n}{x_1}} \geq 2\left(\frac{x_1}{x_1x_2+1} + \frac{x_2}{x_2x_3+1} + ... + \frac{x_n}{x_nx_1+1}\right)$$

2). Determine all $x_k > 0$ $(k = 1, 2, ..., n)$ for which
$\frac{x_1}{x_1x_2+1} + \frac{x_2}{x_2x_3+1} + ... + \frac{x_n}{x_nx_1+1} \geq \frac{n}{2}$

Mihály Bencze

PP. 15473. If $a, b, c > 0$, then

$$(abc + 2)\left(\sqrt[3]{\frac{a}{b}} + \sqrt[3]{\frac{b}{c}} + \sqrt[3]{\frac{c}{a}}\right) \geq 3\left(\sqrt[3]{a^2c} + \sqrt[3]{b^2a} + \sqrt[3]{c^2b}\right).$$

Mihály Bencze

PP. 15474. In all triangle ABC holds

$$1).\ \max\left\{\sqrt{3};\frac{9r}{\sqrt{3(s^2-2r^2-8Rr)}};\frac{27sr}{(4R+r)^2}\right\}\le\sum tg\frac{A}{2}\le$$
$$\le\min\left\{\frac{s}{3r};\frac{1}{s}\sqrt{3\left((4R+r)^2-2s^2\right)}\right\}$$

$$2).\ \max\left\{3\sqrt{3};\frac{27r}{s};\frac{9s}{\sqrt{3\left((4R+r)^2-2s^2\right)}}\right\}\le\sum ctg\frac{A}{2}\le$$
$$\le\min\left\{\frac{(4R+r)^2}{3sr};\frac{1}{r}\sqrt{3\left(s^2-2r^2-8Rr\right)}\right\}.$$

Mihály Bencze

PP. 15475. Solve the following system:

$$\begin{cases}(x+2)\left(y^3+2\right)=3\left(2z^2+1\right)\\(y+2)\left(z^3+2\right)=3\left(2x^2+1\right)\\(z+2)\left(x^3+2\right)=3\left(2y^2+1\right)\end{cases}.$$

Mihály Bencze

PP. 15476. In all triangle ABC holds:

$$1).\ 3\sqrt[3]{\frac{r}{4R}}\le\sum\sin\frac{A}{2}\le\frac{3}{2}\le\sqrt{\frac{3(2R-r)}{2R}}\qquad 2).\ 6\le\sum\frac{1}{\sin\frac{A}{2}}\le\frac{3R}{r}.$$

Mihály Bencze

PP. 15477. In all triangle ABC holds:

$$1).\ \frac{s}{R}\le3\sqrt[3]{\frac{s}{4R}}\le\sum\cos\frac{A}{2}\le\sqrt{\frac{3(4R+r)}{2R}}\le\frac{3\sqrt{3}}{2}\le\frac{s}{2r}$$

$$2).\ 2\sqrt{3}\le\sum\frac{1}{\cos\frac{A}{2}}\le\frac{9R}{s}$$

Mihály Bencze

PP. 15478. In all acute triangle ABC holds $\sum\frac{1}{\cos^2A(\cos B+\cos C)^2}\ge12$.

Mihály Bencze

PP. 15479. Solve the following equation

$$a^{\frac{n(n+1)}{2}\lg a}\prod_{k=1}^{n} x_k^{\lg x_k}\prod_{1\le i<j\le n} x_j^{\lg x_i}=\left(\prod_{k=1}^{n} x_k\right)^{(n+1)\lg a}, \text{ where } a>0.$$

Mihály Bencze

PP. 15480. Solve the equation

$$4\sum_{k=1}^{n}\lg^2 x_k-8\lg a\lg\prod_{k=1}^{n} x_k+4n\lg^2 a=\lg^2\tfrac{x_1}{x_2}+\lg^2\tfrac{x_2}{x_3}+...+\lg^2\tfrac{x_n}{x_1}.$$

Mihály Bencze

PP. 15481. Prove that $\int\limits_0^{\arcsin\left(\frac{\sqrt{6}-\sqrt{2}}{4}tgt\right)}\frac{dx}{\sqrt{1-\cos^2 t\cos^2 x}}=\int\limits_0^{\frac{??}{12}}\frac{dx}{\sqrt{\cos^2 t-\sin^2 x}}.$

Mihály Bencze

PP. 15482. If $x,y,z>0$, then

$$\left(\sum x\right)\left(\sum \tfrac{1}{x}\right)-9\ge 2\left(\sqrt{\tfrac{(\sum x)(\sum x^2)}{xyz}}-3\right)\ge 0.$$

Mihály Bencze

PP. 15483. Determine the best constant $\lambda>0$ such that in all triangle ABC holds: $2R-r\ge\sqrt{s^2-8Rr-2r^2}+\lambda\left((a-b)^2+(b-c)^2+(c-a)^2\right).$

Mihály Bencze

PP. 15484. If $f(x,y)=\{x\}+[y]$, where $\{\cdot\}$ and $[\cdot]$ denote the fractional respective the integer part, then solve the following system

$$\begin{cases} f(x_1,x_2)f(x_2,x_3)...f(x_{n-1},x_n)=x_1x_2...x_{n-1}\\ f(x_2,x_3)f(x_3,x_4)...f(x_n,x_1)=x_2x_3...x_n\\ ----------------------\\ f(x_n,x_1)f(x_1,x_2)...f(x_{n-2},x_{n-1})=x_nx_1...x_{n-2}\end{cases}$$

Mihály Bencze

PP. 15485. If $x_p > 0$ $(p = 1, 2, ..., n)$, then

$$\sum_{p=1}^{n} \frac{x_p+2}{2x_p^2+1} \geq 3\max\left\{ \frac{\left(\sum_{p=1}^{n} x_p^k\right)^2}{\sum_{p=1}^{n} x_p^{2k+3} + 2\sum_{p=1}^{n} x_p^{2k}} | k \in N \right\}.$$

Mihály Bencze

PP. 15486. If $g(x, y, z) = x + [y] + \{z\}$, where $[\cdot]$ and $\{\cdot\}$ denote the integer, respective the fractional part. Solve the following system:

$$\begin{cases} g(x_1, x_2, x_3) g(x_2, x_3, x_4) ... g(x_{n-1}, x_n, x_1) = 2^{n-1} x_1 x_2 ... x_{n-1} \\ g(x_2, x_3, x_4) g(x_3, x_4, x_5) ... g(x_n, x_1, x_2) = 2^{n-1} x_2 x_3 ... x_n \\ - \\ g(x_n, x_1, x_2) g(x_1, x_2, x_3) ... g(x_{n-2}, x_{n-1}, x_n) = 2^{n-1} x_n x_1 ... x_{n-2} \end{cases}.$$

Mihály Bencze

PP. 15487. If $x_p > 0$ $(p = 1, 2, ..., n)$, then

$$\sum_{p=1}^{n} \frac{x_p^3+2}{2x_p^2+1} \geq 3\max\left\{ \frac{\left(\sum_{p=1}^{n} x_p^k\right)^2}{\sum_{p=1}^{n} x_p^{2k+1} + 2\sum_{p=1}^{n} x_p^{k}} | k \in N \right\}.$$

Mihály Bencze

PP. 15488. Solve the following system: $\begin{cases} (x + ty)(x + tz) = a \\ (y + tz)(y + tx) = b \\ (z + tx)(z + ty) = c \end{cases}$, where $a, b, c > 0$.

Mihály Bencze

PP. 15489. In all triangle ABC holds:

1). $\sum \left|\sin\frac{A}{2}\sin\frac{B-C}{2}\right| ctg(IOI_a\measuredangle) = \frac{r}{2R} - \frac{1}{4}$

2). $\sum (2\cos A - 1)^2 tg^2(IOI_a\measuredangle) = \frac{2\left(s^2-3r^2-12Rr\right)}{R^2}$

Mihály Bencze

PP. 15490. Prove that $\frac{5\left(25^{n+1}-1\right)}{8} + \frac{16n(n+1)(2n+1)}{3} + 60n^2 + 45n - 15$ is divisible by 128 for all $n \in N$.

Mihály Bencze

PP. 15491. If $x_k \in (0,1)$ $(k=1,2,...,n)$ and $\sum\limits_{k=1}^{n} x_k = 1$, then $\prod\limits_{k=1}^{n} \frac{1-\cos\frac{\pi x_k}{2}}{1+\cos\frac{\pi x_k}{2}} \leq \left(tg\frac{\pi}{4n}\right)^{2n}$.

Mihály Bencze

PP. 15492. Prove that $\sum\limits_{k=2}^{\infty} \left(\frac{\sin\left(\frac{\pi}{k^2}-\frac{\pi}{4}\right)}{\sin\left(\frac{\pi}{k^2+1}-\frac{\pi}{4}\right)} - 1 \right)^2 > \frac{\pi^2}{6} - 1$.

Mihály Bencze

PP. 15493. In all triangle ABC holds:

1). $\sum \sqrt{(s-a)}\cos\frac{A}{2} \geq 3\sqrt{\frac{sr}{2R}}$ 2). $\sum \sqrt{\sin\frac{A}{2}} \geq \sqrt{\frac{r}{2R}}$

Mihály Bencze

PP. 15494. In all triangle ABC holds:

1). $\sum \frac{tg\frac{A}{2}}{2(4R+r)-s\cdot tg\frac{A}{2}} \geq \frac{3}{5\cdot s}$ 2). $\sum \frac{1}{2ctg\frac{A}{2}ctg\frac{B}{2}-1} \geq \frac{3}{5}$

Mihály Bencze

PP. 15495. In all triangle ABC holds:

$2\prod\left(tg^2\frac{A}{2} + tg\frac{A}{2}tg\frac{B}{2} + tg^2\frac{C}{2}\right) \geq 1 + \frac{r(4R+r)^3}{s^4}$.

Mihály Bencze

PP. 15496. In all triangle ABC holds:

1). $\prod\left(1 + tg^2\frac{A}{2}tg^2\frac{B}{2}\right) \geq \frac{72\sqrt{5}}{125}$ 2). $\prod\left(1 + tg^2\frac{A}{2}\right) \geq \frac{72\sqrt{5}(4R+r)}{125s}$

Mihály Bencze

PP. 15497. In all triangle ABC holds:

1). $\sum \frac{\sin\frac{A}{2}\sin\frac{B}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}+\sin\frac{C}{2}} \geq \frac{3}{5}$ 2). $\sum \frac{tg\frac{A}{2}}{1-tg\frac{B}{2}tg\frac{C}{2}} \geq \frac{3s}{2r}$

Mihály Bencze

PP. 15498. In all triangle ABC holds:

1). $\sum \frac{\sin\frac{A}{2}}{\cos^3\frac{A}{2}} \geq \frac{6R}{s}$ 2). $\sum \frac{ctg^2\frac{A}{2}}{tg\frac{B}{2}+tg\frac{C}{2}} \geq \frac{3s}{2r}$

Mihály Bencze

PP. 15499. In all triangle ABC holds: $\sum \frac{ctg\frac{A}{2}}{1+tg\frac{B}{2}tg\frac{C}{2}} \geq \frac{s}{2r}$.

Mihály Bencze

PP. 15500. In all triangle ABC holds:
1). $\sum \frac{ctg\frac{C}{2}}{1+\sqrt{ctg\frac{A}{2}tg\frac{C}{2}}} \geq \frac{s}{2r}$ 2). $\sum \frac{tg^3\frac{A}{2}}{tg\frac{A}{2}+tg\frac{B}{2}} \geq \frac{1}{2}\left(\frac{4R+r}{s}\right)^2 - 1$.

Mihály Bencze

PP. 15501. In all triangle ABC holds:
1). $\sum \frac{tg\frac{A}{2}ctg\frac{C}{2}}{tg\frac{A}{2}+tg\frac{C}{2}} \geq \frac{s}{2r}$ 2). $\sum \frac{tg^2\frac{A}{2}}{tg\frac{A}{2}+tg\frac{B}{2}} \geq \frac{4R+r}{2s}$

Mihály Bencze

PP. 15502. In all triangle ABC holds:
1). $\frac{9r}{2s} \leq \sum \frac{tg\frac{B}{2}tg\frac{C}{2}}{tg\frac{A}{2}+tg\frac{C}{2}} \leq \frac{s^2-8Rr-2r^2}{2sr}$
2). $\frac{9r}{2(4R+r)} \leq \sum \frac{tg\frac{A}{2}tg^2\frac{B}{2}}{tg\frac{A}{2}+tg\frac{B}{2}} \leq \frac{1}{2}\left(\frac{4R+r}{s}\right)^2 - 1$

Mihály Bencze

PP. 15503. If $x_k > 0$ $(k = 1, 2, ..., n)$ and $\alpha \in [0, 1]$, then
$\sum_{cyclic} (x_1x_2)^\alpha \leq \frac{\left(\sum_{k=1}^{n} x_k\right)^{2\alpha}}{4^\alpha n^{\alpha-1}}$.

Mihály Bencze

PP. 15504. In all triangle ABC holds:
1). $27r(4R-r) \leq 7s^2$ 2). $s^2 + r^2 + 2Rr \leq 8R^2$ 3). $4r(4R+r) \leq s^2$

Mihály Bencze

PP. 15505. In all triangle ABC holds:
1). $\sum ctg^3\frac{A}{2} \leq \frac{s\left(s^2-8Rr-8r^2\right)}{r^3}$ 2). $\sum ctg^3\frac{A}{2} \geq \frac{3s(8R-9r)}{7r^2}$

Mihály Bencze

PP. 15506. In all triangle ABC holds: $\sum \frac{a(a+1)+b+1}{(a+1)(a^2+b)} \leq \frac{s^2+r^2+4Rr}{4sRr}$.

Mihály Bencze

PP. 15507. In all triangle ABC holds: $\sum \sqrt[6]{tg\frac{A}{2}tg^2\frac{B}{2}} \le \sqrt{\frac{s}{r}}$.

Mihály Bencze

PP. 15508. In all triangle ABC holds:

$\prod \left(x + yctg\frac{A}{2}ctg\frac{B}{2}\right) + \prod \left(y + xctg\frac{A}{2}ctg\frac{B}{2}\right) \ge 8\left(x+y\right)^2$ for all $x, y > 0$.

Mihály Bencze

PP. 15509. In all triangle ABC holds:

1). $\sum \frac{1}{ctg\frac{A}{2}+ctg\frac{B}{2}} \le \frac{\sqrt{3}}{2}$ 2). $\sum \frac{tg\frac{A}{2}}{ctg\frac{B}{2}+ctg\frac{C}{2}} \le \frac{\sqrt{3r(4R+r)}}{2s}$.

Mihály Bencze

PP. 15510. In all triangle ABC holds:

1). $\sum \frac{tg^2\frac{A}{2}ctg\frac{B}{2}}{tg^2\frac{A}{2}+tg\frac{A}{2}tg\frac{B}{2}+tg^2\frac{B}{2}} \ge \frac{3s}{4R+r}$ 2). $\sum \frac{tg^2\frac{A}{2}ctg\frac{B}{2}ctg\frac{C}{2}}{tg^2\frac{A}{2}+tg\frac{A}{2}tg\frac{C}{2}+tg^2\frac{C}{2}} \ge 3$

Mihály Bencze

PP. 15511. In all triangle ABC holds: $\sum \frac{\left(tg\frac{A}{2}tg\frac{B}{2}\right)^{\lambda+2}}{1+tg^2\frac{C}{2}} \ge \frac{1}{4\cdot 3^{\lambda-1}}$, for all $\lambda \ge 1$.

Mihály Bencze

PP. 15512. In all triangle ABC holds: $\prod \left(tg\frac{A}{2}tg\frac{B}{2}\right)^{tg\frac{C}{2}} \le 3^{-\frac{s}{3r}}$.

Mihály Bencze

PP. 15513. In all triangle ABC holds:
1). $\sum \sqrt{tg^2\frac{A}{2} + tg\frac{A}{2}tg\frac{B}{2} + tg^2\frac{B}{2}} \ge \frac{3\sqrt{3}s}{4R+r}$
2). $\sum tg\frac{A}{2}\sqrt{tg^2\frac{B}{2} + tg\frac{B}{2}tg\frac{C}{2} + tg^2\frac{C}{2}} \ge \frac{3\sqrt{3}r(4R+r)}{s^2}$

Mihály Bencze

PP. 15514. In all triangle ABC holds: $\sum tg^2\frac{A}{2}ctg\frac{C}{2}\left(tg\frac{B}{2} + tg\frac{C}{2}\right) \ge 2$.

Mihály Bencze

PP. 15515. If $n \geq 2\,(n \in N)\,, x_i \geq 0\,(i = 1, 2, ..., k)$ and $\sum\limits_{i=1}^{k} x_i^2 = 1$, then $\sum\limits_{i=1}^{k} n^{x_i} \geq n + k - 1$.

Mihály Bencze

PP. 15516. Let ABC be a triangle and $x = rctg\frac{C}{2}\left(s + rctg\frac{A}{2}\right)\left(s + rctg\frac{B}{2}\right)$ and y, z his permutations, then $\sum \frac{1}{x+y+s(s^2-3r^2-4Rr)} \geq \frac{1}{s(s^2-3r^2-4Rr)}$.

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PP. 15517. In all triangle ABC holds: $tg\frac{A}{2}tg^2\frac{B}{2} \geq \frac{s^3}{432r^3}$ and his permutations.

Mihály Bencze

PP. 15518. If $a, b, c > 0$ and $abc = 1$, then $\sum a + \sum \sqrt{a} + \sum \sqrt[3]{a} > \frac{9}{2}$.

Mihály Bencze

PP. 15519. In all triangle ABC holds:
$\sum \frac{1}{3r^2+4Rr-s^2-\left(r+stg\frac{A}{2}\right)\left(4R+3r-stg\frac{A}{2}\right)} \geq \frac{1}{3r^2+4Rr-s^2}$.

Mihály Bencze

PP. 15520. In all triangle ABC holds: $\sum \left(3 + ctg\frac{A}{2}ctg\frac{B}{2}\right)^2 \geq 98$.

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PP. 15521. In all triangle ABC holds: $\left(tg\frac{A}{2} + tg\frac{C}{2}\right)\left(tg\frac{B}{2} + tg\frac{C}{2}\right) \geq 2tg\frac{C}{2}$ and his permuations.

Mihály Bencze

PP. 15522. In all triangle ABC holds: $\sum \sqrt{ctg\frac{A}{2} + 2ctg\frac{B}{2} + 3ctg\frac{C}{2}} \leq 3\sqrt{\frac{2s}{r}}$.

Mihály Bencze

PP. 15523. If $1 \geq a > b > c > d > e \geq -1$ and $\alpha \in [0, 1]$, then
$\frac{1}{(a-b)^\alpha} + \frac{1}{(b-c)^\alpha} + \frac{1}{(c-d)^\alpha} + \frac{1}{(d-e)^\alpha} \geq 8$

Mihály Bencze and Titu Andreescu

PP. 15524. Let ABC be a triangle and $\alpha \geq 1$. Prove that $\sum \left(ctg\frac{A}{2}\right)^{\alpha} \geq 3\left(\sqrt{3}\right)^{\alpha}$.

Mihály Bencze

PP. 15525. Let ABC be a triangle. Prove that:
1). $\sum tg^2\frac{A}{2}tg^2\frac{B}{2} + \frac{2\sqrt{3}r}{s} \leq 1$ 2). $\sqrt{3}s \leq 4R + r$

Mihály Bencze

PP. 15526. In all triangle ABC holds $\sqrt{ctg\frac{A}{2}ctg\frac{B}{2}} + \sqrt{ctg\frac{B}{2}ctg\frac{C}{2}} < \frac{3s}{5r} + \sqrt{ctg\frac{C}{2}ctg\frac{A}{2}}$ and his permutations.

Mihály Bencze

PP. 15527. In all triangle ABC holds $\sum \frac{tg\frac{A}{2}}{1+tg\frac{B}{2}tg\frac{C}{2}} \leq \frac{s}{4r}$.

Mihály Bencze and Shanhe Wu

PP. 15528. Let ABC be a triangle. Prove that
1). $\frac{1}{bc}\sqrt{\frac{s-a}{a}}, \frac{1}{ca}\sqrt{\frac{s-b}{b}}, \frac{1}{ab}\sqrt{\frac{s-c}{c}}$
2). $\frac{1}{(s-b)(s-c)}\sqrt{\frac{a}{s-a}}, \frac{1}{(s-c)(s-a)}\sqrt{\frac{b}{s-b}}, \frac{1}{(s-a)(s-b)}\sqrt{\frac{s}{s-c}}$ are the sides of a triangle.

Mihály Bencze

PP. 15529. In all triangle ABC holds $\sum \sqrt[3]{tg\frac{A}{2}tg\frac{B}{2}} + \sqrt[3]{\left(\frac{s}{r}\right)^2} \geq 3 + \sqrt[3]{9}$.

Mihály Bencze and Shanhe Wu

PP. 15530. In all triangle ABC holds $\left|\sum tg\frac{A-B}{2}tg\frac{C}{2}\right| < \frac{1}{8}$.

Mihály Bencze

PP. 15531. In all triangle ABC the following statements
1). $\prod\left(1 + ctg^2\frac{A}{2}\right) \geq 64$ 2). $R \geq 2r$ are equivalent.

Mihály Bencze

PP. 15532. In all triangle ABC holds $\sum \sqrt{\frac{5-4\cos A}{1+\cos A}} \geq 3\sqrt{2}$.

Mihály Bencze and Zhao Changjian

PP. 15533. In all triangle ABC holds $\sum \frac{\sqrt{3-2\cos A+\cos 2A}}{1+\cos A} \geq \sqrt{6}$.

Mihály Bencze

PP. 15534. In all triangle ABC holds $\sum \frac{\sqrt{15-16\cos A+5\cos 2A}}{1+\cos A} \geq 3\sqrt{2}$.

Mihály Bencze

PP. 15535. In all triangle ABC holds $\sum \sqrt{tg\frac{A}{2}tg\frac{B}{2}} + \frac{s}{r} \geq 4\sqrt{3}$.

Mihály Bencze and Shanhe Wu

PP. 15536. If ζ denote the Riemann zeta function, then for all $s > 1$ holds
1). $\prod\limits_{k=2}^{\infty}\left(1-\frac{1}{k^s}\right) \leq \frac{1}{\zeta(s)}$ 2). $\sum\limits_{k=1}^{\infty}\frac{1}{k^s+1} \geq \frac{\zeta(s)}{1+\zeta(s)}$ 3). $\sum\limits_{k=1}^{n}\frac{2k-1}{k^{2s}} \leq \zeta^2(s)$

Mihály Bencze

PP. 15537. If $a, b, c > 0$, then
1). $\sum \frac{\left(a^3+b^3\right)(b+c)}{a^2+ab+b^2} \geq \frac{1}{3}\sum(a+b)(b+c)$
2). $\sum \frac{\left(a^3+b^3\right)(b+c)(c+a)}{a^2+ab+b^2} \geq (a+b)(b+c)(c+a)$.

Mihály Bencze and Zhao Changjian

PP. 15538. Prove that:
1). $\sum\limits_{k=1}^{n}\frac{1}{k^2+k+1} \geq \frac{n}{2n+1}$ 2). $\sum\limits_{k=1}^{n}\frac{2k-1}{k^2(k+1)^2} \leq \left(\frac{n}{n+1}\right)^2$
3). $(n!)^2 \leq \prod\limits_{k=1}^{n}\left(k^2+k-1\right) \leq \frac{(n+1)^2(n!)^2}{2n+1}$

Mihály Bencze

PP. 15539. Prove that $\sum\limits_{k=0}^{n-1}\sqrt{\frac{n-k}{k+1}} \geq \sqrt{(n-1)\left(n-\sum\limits_{k=1}^{n}\frac{1}{k+1}\right)}$.

Mihály Bencze

PP. 15540. In all triangle ABC holds
$2\sqrt{2}\sum\cos\frac{a-B}{2}\cos\frac{C}{2} \geq \sum\sqrt{2\sin A\sin B\cos C+\sin^2 C}+\sqrt{2}\sum\sqrt{\sin A\sin B}$.

Mihály Bencze

PP. 15541. If $a, b, c > 0$, then $\sum \frac{a}{b} \geq \sum \frac{c+a}{c+b} + \frac{a+b+c}{3abc} \sum \frac{c^2(a-b)^2}{(c+a)(c+b)}$.

Mihály Bencze

PP. 15542. If $a_k > 0$ $(k = 1, 2, ..., n)$, then

$$\left(a_1 - \sqrt{a_1 a_2} + a_2\right)\left(a_2 - \sqrt{a_2 a_3} + a_3\right) ... \left(a_n - \sqrt{a_n a_1} + a_1\right) \geq$$
$$\geq \sqrt{\frac{1}{2^n}\left(a_1^2 + a_2^2\right)\left(a_2^2 + a_3^2\right) ... \left(a_n^2 + a_1^2\right)} \geq \frac{1}{2^n}\left(a_1 + a_2\right)\left(a_2 + a_3\right) .. \left(a_n + a_1\right) \geq$$
$$\geq \prod_{k=1}^{n} a_k.$$

Mihály Bencze

PP. 15543. In quarilateral $ABCD$ $\left(\widehat{A} = \widehat{C} = 90^\circ\right)$ denote E and F the proiection of A and C to BD. Prove that

$$AB + BC + CD + DA \geq \left(\sqrt{AE} + \sqrt{2BD} + \sqrt{CF}\right)\sqrt{BD}.$$

Mihály Bencze

PP. 15544. If $e(n) = \left(1 + \frac{1}{n}\right)^n$, then in all triangle ABC holds

$\sum \frac{1}{a^2 + e(n)ab + b^2} \geq \frac{9}{(2 + e(n))(s^2 + r^2 + 4Rr)}$, for all $n \in N^*$.

Mihály Bencze

PP. 15545. If $a_k, b_k \in R$ $(k = 1, 2, ..., n)$ such that the equation $\sum\limits_{k=1}^{n} |x - a_k| = \sum\limits_{k=1}^{n} |x - b_k|$ have p solutions, then $\max S = n - 1$, where $p \in \{1, 2, ..., n\}$.

Mihály Bencze

PP. 15546. Let $ABCD$ be a tetrahedron inscribed in a sphere. The altitudes AM, BN, CK, DL $(AM \perp (BCD), M \in (BCD), BN \perp (ACD), N \in (ACD), CK \perp (ABD),$ $K \in (ABD), DL \perp (ABC), L \in (ABC))$ meet the circumsphere at A_1, B_1, C_1, D_1 respectively. Prove that $\frac{AA_1}{AM} + \frac{BB_1}{BN} + \frac{CC_1}{CK} + \frac{DD_1}{DL} = 5$.

Mihály Bencze

PP. 15547. Let $ABCD$ be a tetrahedron in which is inscribed a sphere with center I. Denote E and F the tangent point of insphere with faces ABD and ADC. For a point M on the line segment EF, show that $Vol\,[MABD]$ and $Vol\,[MADC]$ are equal if and only if $MI \perp (BDC)$.

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PP. 15548. If $x_k > 0$ $(k = 1, 2, ..., n)$ such that $\sum\limits_{k=1}^{n} x_k = 4n$, then
$\sum\limits_{j=1}^{n} \left(\sum\limits_{\substack{i=1 \\ i \neq j}} \frac{x_j}{\sqrt[3]{x_i^4 + 87}} \right) \geq \frac{4n(n-1)}{7}$.

Mihály Bencze

PP. 15549. If $a, b, c \geq 0$, then
$2\sum a^4 + 33\sum a^2 + abc\sum a \geq \left(\sum a^3\right)\left(\sum a\right) + 2\sum ab.$

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PP. 15550. If $a, b, c \geq 0$ and $x, y \in R$, then
$2\sum a^4 + abc\sum a + 2\left(x^2 + xy + y^2\right)\sum a^2 \geq$
$\geq \left(\sum a^3\right)\left(\sum a\right) + 2\left(x^2 + xy + y^2\right)\sum ab.$

Mihály Bencze

PP. 15551. 1). If $x, y, z > 0$, then $\sum (-x + y + z)\left(\sqrt{x} - \sqrt{y}\right)^2 \geq 0$
2). Determine all $\alpha > 0$ such that $\sum (-x + y + z)\left(x^{\frac{1}{\alpha}} - y^{\frac{1}{\alpha}}\right)^{\alpha} \geq 0$
3). Determine all $a, b, c \in R$ such that $\sum (ax + by + cz)\left(\sqrt{x} - \sqrt{y}\right)^2 \geq 0.$

Mihály Bencze

PP. 15552. If $x, y, z > 0$ then determine all $a, b \in R$ for which
$a\left(\sum x^2\right)^2 + b\left(\sum x^2\right)\left(\sum xy\right) \geq 3(a + b)\,xyz\sum x.$

Mihály Bencze

PP. 15553. If $0 \leq a_1 \leq a_2 \leq ... \leq a_n$ and $0 < b_1 \leq b_2 \leq ... \leq b_n$, then
$a_1^{\frac{1}{1+b_2} + \frac{b_1}{1+b_1}} a_2^{\frac{1}{1+b_3} + \frac{b_2}{1+b_2}} ... a_n^{\frac{1}{1+b_1} + \frac{b_n}{1+b_n}} \geq a_1 a_2 ... a_n.$

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PP. 15554. If $0 < a_1 \le a_2 \le ... \le a_n$, then
$\frac{1}{(n-1)!}\sum\limits_{k=1}^{n}\left(a_1+(k+1)\,a_2\right)\left(a_2+(k+2)\,a_3\right)...\left(a_n+ka_1\right)\ge$
$\ge\left(\binom{n}{1}+\binom{n+1}{2}+...+\binom{2n-1}{n}\right)a_1a_2...a_n.$

Mihály Bencze

PP. 15555. In all triangle ABC holds $\sum\frac{A}{s-a}\ge\frac{9\pi}{2s^2}.$

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PP. 15556. If $x, y, z > 0$, then $\sum\frac{x^3}{x+2y}\ge\frac{(x+y+z)^2}{27}.$

Mihály Bencze

PP. 15557. If $m \in N, m \ge 2$, then:
1). $\sum\limits_{k=1}^{\infty}\left(1+\frac{1}{k^2}\right)^{m+1}\ge\frac{(m+1)^{m+1}\pi^2}{6m^m}$
2). $((n+1)!)^{m+1}\,m^{nm}\ge n!\,(m+1)^{n(m+1)}$ for all $n \in N$
3). $\sum\limits_{k=1}^{m}\left(\frac{k+1}{m+1}\right)^{m+1}\ge\frac{m+1}{2m^{m-1}}$

Mihály Bencze

PP. 15558. If $a, b, c > 0$, then
$5+\sum\frac{a}{b}+\left(\sum a^2\right)^{\frac{1}{2}}\left(\sum\frac{1}{a^2}\right)^{\frac{1}{2}}\ge\frac{5}{3}\left(\sum a\right)\left(\sum\frac{1}{a}\right).$

Mihály Bencze

PP. 15559. Let $A_1A_2...A_n$ be a simplex inscribed in a sphere. The a altitudes A_kB_k, $A_kB_k \perp (A_1...A_{k-1}A_{k+1}...A_n)$, $B_k \in (A_{k-1}A_{k+1}...A_n)$ $(k = 1, 2, ..., n)$ meet the circumsphere at C_k $(k = 1, 2, ..., n)$ respectively. Prove that $\sum\limits_{k=1}^{n}\frac{A_kC_k}{A_kB_k}=n+1.$

Mihály Bencze

PP. 15560. Solve in $(0, +\infty)$ the equation $x^n+[x^m]=x^m+[x^n]$, when $n, m \in N$ are given and $[\cdot]$ denote the integer part.

Mihály Bencze

PP. 15561. If $x, a_k > 0$ $(k = 1, 2, ..., n)$ and $\sum\limits_{k=1}^{n} a_k^2 = x \prod\limits_{k=1}^{n} a_k$, then
$\sum\limits_{k=1}^{n} \frac{a_k}{a_1^2 ... a_{k-1}^2 a_{k+1}^2 ... a_n^2} \geq \frac{x^2}{\sum\limits_{k=1}^{n} a_k}$.

Mihály Bencze

PP. 15562. In all triangle ABC holds:
1). $\sum \sqrt{a^4 + b^4} \leq \sqrt{2}\left(3s^2 - 5r^2 - 20Rr\right)$
2). $\sum \sqrt{(s-a)^4 + (s-b)^4} \leq \sqrt{2}\left(2s^2 - 5r^2 - 20Rr\right)$
3). $\sum \sqrt{h_a^4 + h_b^4} \leq \sqrt{2}\left(\frac{1}{2}\left(\frac{s^2+r^2+4Rr}{R}\right)^2 - \frac{10s^2r}{R}\right)$
4). $\sum \sqrt{r_a^4 + r_b^4} \leq \sqrt{2}\left(2\left(4R+r\right)^2 - 5s^2\right)$
5). $\sum \sqrt{\sin^8 \frac{A}{2} + \sin^8 \frac{B}{2}} \leq \frac{\sqrt{2}\left(32R^2+8Rr+3r^2-5s^2\right)}{16R^2}$
6). $\sum \sqrt{\cos^8 \frac{A}{2} + \cos^8 \frac{B}{2}} \leq \frac{\sqrt{2}\left(3(4R+r)^2-5s^2\right)}{16R^2}$

Mihály Bencze

PP. 15563. If $a_k > 0$ $(k = 1, 2, ..., n)$ and $\sum\limits_{k=1}^{n} a_k \leq 1$, then
$\sum\limits_{k=1}^{n} \frac{a_k}{1+a_k+a_k^2+...+a_k^m} \leq \frac{(n-1)n^m}{n^{m+1}-1}$.

Mihály Bencze

PP. 15564. If $a_k > 0$ $(k = 1, 2, ..., n)$ and $\prod\limits_{k=1}^{n} a_k \leq 1$, then
$\sum\limits_{cyclic} \frac{a_1}{a_2+a_2^2+a_2^3+...+a_2^m} \geq \frac{n}{m}$.

Mihály Bencze

PP. 15565. Determine all triangle ABC for which $ABC \geq \frac{27\pi^3 R r^3}{2s^4}$.

Mihály Bencze

PP. 15566. If $a, b, c > 0$ and $x \in \left(-\infty, -\frac{1}{\sqrt{2}}\right] \cup \left[\frac{1}{\sqrt{2}}, +\infty\right)$, then
$\sum\limits_{cyclic} \sqrt{\frac{a^2+(2x^2-1)bc}{b^2+c^2}} \geq 3\left|x\right|$.

Mihály Bencze

PP. 15567. In all triangle ABC holds:

1). $\sum ctg\frac{A}{2}\sqrt{ctg^2\frac{B}{2}+ctg\frac{B}{2}ctg\frac{C}{2}+ctg^2\frac{C}{2}} \geq \frac{\sqrt{3}(4R+r)}{r}$
2). $\sum\sqrt{ctg^2\frac{A}{2}+ctg\frac{A}{2}ctg\frac{B}{2}+ctg^2\frac{B}{2}} \geq \frac{\sqrt{3}s}{r}$.

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PP. 15568. In all triangle ABC holds:
$\max\left\{\sum\cos\frac{A}{2}\sqrt{tg\frac{B}{2}tg\frac{C}{2}};\sum\sin\frac{A}{2}\right\} \leq \frac{3}{2}$.

Mihály Bencze

PP. 15569. In all triangle ABC holds:
1). $\sum\sqrt[3]{tg\frac{A}{2}} \leq \frac{5}{3}\sqrt[3]{\frac{s}{r}}$ 2). $\sum\sqrt[3]{tg\frac{A}{2}tg\frac{B}{2}} \leq 1+\frac{2(4R+r)}{3s}$.

Mihály Bencze

PP. 15570. In all triangle ABC holds:
1). $\sum\frac{tg\frac{A}{2}}{tg^2\frac{A}{2}+tg^2\frac{B}{2}} \leq \frac{s}{2r}$ 2). $\sum\frac{tg\frac{A}{2}ctg\frac{B}{2}}{tg^2\frac{A}{2}+tg^2\frac{C}{2}} \leq \frac{4R+r}{2r}$

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PP. 15571. In all triangle ABC holds:
1). $\sum\frac{ctg\frac{A}{2}}{ctg^2\frac{A}{2}+ctg^2\frac{B}{2}} \leq \frac{4R+r}{2s}$ 2). $\sum\frac{ctg\frac{A}{2}tg\frac{B}{2}}{ctg^2\frac{A}{2}+ctg^2\frac{C}{2}} \leq \frac{1}{2}$

Mihály Bencze

PP. 15572. In all triangle ABC holds:
1). $\sum tg\frac{A}{2}\sqrt{tg^2\frac{A}{2}+tg\frac{B}{2}tg\frac{C}{2}+tg^2\frac{C}{2}} \geq \sqrt{3}$
2). $\sum\sqrt{tg^2\frac{A}{2}+tg\frac{A}{2}tg\frac{B}{2}+tg^2\frac{B}{2}} \geq \frac{\sqrt{3}(4R+r)}{s}$

Mihály Bencze

PP. 15573. In all triangle ABC holds:
1). $\sum\sin^2\frac{A}{2}\cos^2\frac{B}{2} \geq \frac{4R+r}{8R}$ 2). $\sum\frac{tg^2\frac{A}{2}tg\frac{B}{2}}{tg\frac{A}{2}+tg\frac{C}{2}} \geq \frac{1}{2}$

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PP. 15574. In all triangle ABC holds:
1). $\sum\frac{\cos^2\frac{A}{2}\sin^2\frac{B}{2}}{s-c}\geq\frac{s}{2r}$ 2). $\sum\frac{ctg^2\frac{A}{2}ctg\frac{B}{2}}{ctg\frac{A}{2}+ctg\frac{C}{2}}\geq\frac{4R+r}{2r}$

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PP. 15575. In all triangle ABC holds:
1). $\sum\frac{tg\frac{A}{2}}{tg^2\frac{A}{2}-tg\frac{A}{2}tg\frac{B}{2}+tg^2\frac{B}{2}}\leq\frac{s}{r}$ 2). $\sum\frac{tg\frac{A}{2}ctg\frac{B}{2}}{tg^2\frac{A}{2}-tg\frac{A}{2}tg\frac{C}{2}+tg^2\frac{C}{2}}\leq\frac{4R+r}{r}$
3). $\sum\frac{ctg\frac{A}{2}}{ctg^2\frac{A}{2}-ctg\frac{A}{2}ctg\frac{B}{2}+ctg^2\frac{B}{2}}\leq\frac{4R+r}{s}$ 4). $\sum\frac{ctg\frac{A}{2}tg\frac{B}{2}}{ctg^2\frac{A}{2}-ctg\frac{A}{2}ctg\frac{C}{2}+ctg^2\frac{C}{2}}\leq 1.$

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PP. 15576. In all triangle ABC holds:
1). $\sum\frac{tg\frac{A}{2}}{\left(\frac{4R+r}{s}\right)^2-2-tg^2\frac{C}{2}}\leq\frac{s}{2r}$ 2). $\sum\frac{tg\frac{A}{2}ctg\frac{B}{2}}{\left(\frac{4R+r}{s}\right)^2-2-tg^2\frac{B}{2}}\leq\frac{4R+r}{2r}$
3). $\sum\frac{ctg\frac{A}{2}}{\frac{s^2-8Rr}{r^2}-2-ctg^2\frac{C}{2}}\leq\frac{4R+r}{2s}$ 4). $\sum\frac{ctg\frac{A}{2}tg\frac{B}{2}}{\frac{s^2-8Rr}{r^2}-2-ctg^2\frac{B}{2}}\leq\frac{1}{2}.$

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PP. 15577. In all triangle ABC holds
$2\max\left\{\sum tg\frac{A}{2}ctg\frac{B}{2};\sum ctg\frac{A}{2}tg\frac{B}{2}\right\}+3\leq\frac{\left((4R+r)^2-2s^2\right)\left(s^2-8Rr-2r^2\right)}{s^2r^2}.$

Mihály Bencze

PP. 15578. In all triangle ABC holds $\sum\frac{ctg^2\frac{A}{2}}{2s-r\left(ctg\frac{A}{2}-ctg\frac{C}{2}\right)}\geq\frac{s}{6r^2}.$

Mihály Bencze

PP. 15579. If $a_k>0\ (k=1,2,...,n)$, then
$$\left(\sum_{k=1}^{n}a_k^2\right)\left(\sum_{k=1}^{n}\frac{1}{a_k^2}\right)\geq\frac{4}{(n-1)^2}\left(\sum_{1\leq i<j\leq n}a_ia_j\right)\left(\sum_{1\leq i<j\leq n}\frac{1}{a_ia_j}\right)\geq n^2.$$

Mihály Bencze

PP. 15580. In all triangle ABC holds:
1). $\max\left\{tg\frac{A}{2},tg\frac{B}{2},tg\frac{C}{2}\right\}-\min\left\{tg\frac{A}{2},tg\frac{B}{2},tg\frac{C}{2}\right\}\leq\frac{4}{3}\sqrt{\left(\frac{4R+r}{s}\right)^2-3}$
2). $\max\left\{ctg\frac{A}{2},ctg\frac{B}{2},ctg\frac{C}{2}\right\}-\min\left\{ctg\frac{A}{2},ctg\frac{B}{2},ctg\frac{C}{2}\right\}\leq$
$\leq\frac{4}{3r}\sqrt{s^2-3r_2-12Rr}.$

Mihály Bencze

PP. 15581. 1). If ABC is a nonisoscelle, nonequilateral triangle, then $(AB+BC+CA)\left(\frac{1}{AB}+\frac{1}{BC}+\frac{1}{CA}\right)\geq 10$
2). If $A_1A_2...A_n$ is a convex polygon ($A_iA_j \neq A_kA_p$), then determine
$\min\left(\sum\limits_{1\leq i<j\leq n} A_iA_j\right)\left(\sum\limits_{1\leq i<j\leq n}\frac{1}{A_iA_j}\right)$.

Mihály Bencze

PP. 15582. In all triangle ABC holds:
1). $\sum\frac{tg\frac{A}{2}}{\sqrt{1+tg^4\frac{A}{2}-\frac{2r(4R+r)}{s^2}}}\leq\frac{s^2+(4R+r)^2}{4sR}$
2). $\sum\frac{1}{\sqrt{s^2\left(1+tg^4\frac{A}{2}\right)-2r(4R+r)}}\leq\frac{1}{4R}\sum\frac{1}{\sin\frac{A}{2}\cos\frac{B}{2}}$.

Mihály Bencze

PP. 15583. In all triangle ABC holds:
1). $\frac{(4R+r)^3-12s^2R}{(4R+r)^2-2s^2}\geq\frac{(4R+r)^2-2s^2}{4R+r}\geq\frac{4R+r}{3}$ 2). $\frac{s^2-12Rr}{s^2-8Rr-2r^2}\geq\frac{s^2-8Rr-2r^2}{s^2}\geq\frac{1}{3}$.

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PP. 15584. In all triangle ABC holds:
1). $\left(\frac{4R+r}{s}\right)^2+3\sqrt[3]{\left(\frac{r}{s}\right)^2}\geq 4$ 2). $s^2+3r\sqrt[3]{s^2r}\geq 4r(4R+r)$.

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PP. 15585. In all triangle ABC holds:
1). $\sum\sqrt[4]{tg\frac{A}{2}}\leq\frac{4R+r}{\sqrt[3]{s^3r}}$ 2). $\sum\sqrt[4]{tg\frac{A}{2}tg\frac{B}{2}}\leq\sqrt{\frac{s}{r}}$

Mihály Bencze

PP. 15586. In all triangle ABC holds:
1). $\prod(15-8\cos A+\cos 2A)\leq\frac{2\left(16R^2-r^2\right)^3}{R^4s^2}$
2). $\prod(15+8\cos A+\cos 2A)\leq\frac{2\left(16R^2-s^2\right)^3}{R^4r^2}$.

Mihály Bencze

PP. 15587. In all triangle ABC holds:
1). $\sum\frac{tg\frac{A}{2}}{4R+r+s\cdot tg\frac{B}{2}}\geq\frac{3}{4s}$ 2). $\sum\frac{ctg\frac{A}{2}}{1+tg\frac{A}{2}tg\frac{C}{2}}\geq\frac{3s}{4r}$

Mihály Bencze

PP. 15588. In all triangle ABC holds:
1). $\sum \frac{1}{\cos^2\frac{A}{2}} \le \frac{2R(2R+5r)}{r(4R+r)}$ 2). $\sum \frac{1}{\sin\frac{A}{2}\cos\frac{B}{2}} \le \frac{2R(2R+5r)}{sr}$
3). $r\left(rs^2+(4R+r)^3\right) \le 2R(2R+3r)s^2$

Mihály Bencze

PP. 15589. In all triangle ABC holds: $\sum \frac{1}{ctg^6\frac{A}{2}+ctg^6\frac{B}{2}+4} \le \frac{1}{6}$.

Mihály Bencze

PP. 15590. In all triangle ABC holds:
1). $\prod\left(5+tg^2\frac{A}{2}\right) \ge \frac{72(4R+r)}{s}$ 2). $\prod\left(5s^2+r^2ctg^2\frac{A}{2}\right) \ge 72s^6$.

Mihály Bencze

PP. 15591. In all triangle ABC holds:

1). $\prod\left(2+tg^2\frac{A}{2}\right) \ge 3\left(\frac{4R+r}{s}\right)^2$ 2). $\prod\left(2+tg^2\frac{A}{2}tg^2\frac{B}{2}\right) \ge 3$.

Mihály Bencze

PP. 15592. In all triangle ABC holds:
1). $\sum(s-a)\cos\frac{A}{2} \ge \frac{3\sqrt{2}sr}{4R}$ 2). $\sum tg\frac{A}{2}\sqrt{tg\frac{B}{2}+tg\frac{C}{2}} \ge 3\sqrt{\frac{2r}{s}}$

Mihály Bencze

PP. 15593. In all triangle ABC holds:
1). $\sum\sqrt{1+ctg\frac{A}{2}tg\frac{B}{2}} \ge 3\sqrt{2}$ 2). $\sum\sqrt{1+ctg\frac{A}{2}tg\frac{C}{2}} \ge 3\sqrt{2}$

Mihály Bencze

PP. 15594. In all triangle ABC holds: $\sum tg\frac{A}{2}\sqrt{1+\frac{2\sin\frac{A}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}}} \le \frac{s^2-12Rr}{s}$.

Mihály Bencze

PP. 15595. If $x_k > 0$ $(k = 1, 2, ..., n)$, then $\sum \frac{x_1\sqrt{x_2}+x_2\sqrt{x_1}}{x_1+x_2} \le \frac{n}{4} + \sum\limits_{k=1}^{n} x_k$.

Mihály Bencze

PP. 15596. In all triangle ABC holds:

$$\sum \frac{\left(tg\frac{A}{2}+tg\frac{B}{2}\right)^3}{\left(\sqrt{tg\frac{A}{2}}+\sqrt{tg\frac{B}{2}}\right)^2} \ge 2 \ge 4\sum \frac{\left(\sqrt{ctg\frac{A}{2}}+\sqrt{ctg\frac{B}{2}}\right)^2}{\left(ctg\frac{A}{2}+ctg\frac{B}{2}\right)^3}.$$

Mihály Bencze

PP. 15597. In all triangle ABC holds:

1). $\prod \left(\sqrt{a}+\sqrt{b}\right)^2 \le \frac{s\left(s^2+r^2+2Rr\right)^3}{16R^2r^2}$

2). $\prod \left(\sqrt{s-a}+\sqrt{s-b}\right)^2 \le \frac{8sR^3}{r}$

3). $\prod \left(\sqrt{h_a}+\sqrt{h_b}\right)^2 \le \frac{s^2\left(s^2+r^2+2Rr\right)^3}{R^4r}$

4). $\prod \left(\sqrt{r_a}+\sqrt{r_b}\right)^2 \le \frac{8s^2R^3}{r^2}$

Mihály Bencze

PP. 15598. In all triangle ABC holds:

1). $\sum \frac{a}{b+c} \le \frac{s^2+r^2}{2Rr} - \frac{11}{2}$ 2). $\sum \frac{h_a}{h_b+h_c} \le \frac{s^2+r^2}{2Rr} - \frac{11}{2}$

3). $\sum \frac{r_a}{r_b+r_c} \le \frac{4R}{r} - \frac{13}{2}$ 4). $\sum \frac{tg\frac{A}{2}}{tg\frac{B}{2}+tg\frac{C}{2}} \le \frac{4R}{r} - \frac{9}{2}$

5). $\sum \frac{\sin^2\frac{A}{2}}{\sin^2\frac{B}{2}+\sin^2\frac{C}{2}} \le \frac{(2R-r)\left(s^2+r^2-8Rr\right)}{2Rr} - \frac{17}{2}$

6). $\sum \frac{\cos^2\frac{A}{2}}{\cos^2\frac{B}{2}+\cos^2\frac{C}{2}} \le \frac{(4R+r)^3+s^2(2R+r)}{2Rs^2} - \frac{9}{2}$

Mihály Bencze

PP. 15599. 1). If $a,b,c>0$, then $\left(\frac{a}{b}\right)^k+\left(\frac{b}{c}\right)^k+\left(\frac{c}{a}\right)^k \ge \frac{a}{b}+\frac{b}{c}+\frac{c}{a}$ for all $k\in N$.

2). If $a_i>0$ $(i=1,2,...,n)$, then

$\left(\frac{a_1}{a_2}\right)^k+\left(\frac{a_2}{a_3}\right)^k+...+\left(\frac{a_n}{a_1}\right)^k \ge \frac{a_1}{a_2}+\frac{a_2}{a_3}+...+\frac{a_n}{a_1}$, for all $k\in N$.

Mihály Bencze

PP. 15600. In all triangle ABC holds:

1). $\prod \left(stg^2\frac{A}{2}+2rctg\frac{A}{2}\right) \le \frac{(4R+r)^6}{27s^3}$ 2). $\prod \left(2stg^2\frac{A}{2}+rctg\frac{A}{2}\right) \le \frac{s^5}{27r^2}$

Mihály Bencze

PP. 15601. If $x_k > 0$ $(k = 1, 2, ..., n)$, then

$$\sqrt[n-1]{\frac{a_1a_2...a_{n-1}}{(a_1+a_n)(a_2+a_n)...(a_{n-1}+a_n)}}+$$

$$+\sqrt[n-1]{\frac{a_2a_3...a_n}{(a_2+a_1)(a_3+a_1)...(a_n+a_1)}}+...+\sqrt[n-1]{\frac{a_na_1...a_{n-2}}{(a_n+a_{n-1})(a_1+a_{n-1})...(a_{n-2}+a_{n-1})}}\leq\frac{n}{2}.$$

Mihály Bencze

PP. 15602. If $a_k > 0$ $(k = 1, 2, ..., n)$, then

$$\left(\sum_{k=1}^{n}\frac{1}{a_k}\right)\left(\sum\frac{a_1^2}{a_1+a_2}\right)\geq\left(\sum a_k\right)\left(\sum\frac{a_1}{a_1^2+a_2^2}\right).$$

Mihály Bencze

PP. 15603. In all triangle ABC holds

$$(2-\sin A)(2-\sin B)(2-\sin C)\geq\frac{sr}{2R^2}.$$

Mihály Bencze

PP. 15604. If $a_i > 0$ $(i = 1, 2, ..., n)$ and $k \in \{1, 2, ..., n\}$, then

$$\frac{1}{a_1^k+...+a_k^k+a_1a_2...a_k}+\frac{1}{a_2^k+...+a_{k+1}^k+a_2a_3...a_{k+1}}+...+\frac{1}{a_n^k+...+a_{k-1}^k+a_na_1...a_{k-1}}\leq$$

$$\leq\frac{\sum a_{k+1}...a_n}{(k+1)\prod\limits_{i=1}^{n}a_i}.$$

Mihály Bencze

PP. 15605. In all triangle ABC holds:

1). $\prod\left(a^2+b^2\right)\geq 256Rs^2r^3$ 2). $\prod\frac{m_a^2+m_b^2}{m_a+m_b-m_c}\geq 8m_am_bm_c$

Mihály Bencze

PP. 15606. In all triangle ABC holds:

1). $\sum\frac{a^2+b^2}{a+b-c}\geq 4s$ 2). $\sum\frac{a^2+b^2}{(a+b-c)c}\geq 6$ 3). $\sum\frac{a^2+b^2}{(a+b-c)c^2}\geq\frac{s^2+r^2+4Rr}{2sRr}$

Mihály Bencze

PP. 15607. In all triangle ABC holds: $\sum\frac{r_a}{\sqrt{\left(r_a^2+r_b^2\right)\left(r_a^2+r_c^2\right)}}\leq\frac{(4R+r)^2+s^2}{4s^2R}.$

Mihály Bencze

PP. 15608. If $a_k \in R$ $(k = 1, 2, ..., n)$, then
$\frac{n}{2} \min\left\{(a_1 - a_2)^2; (a_2 - a_3)^2, ..., (a_n - a_1)^2\right\} \le \sum\limits_{k=1}^{n} a_k^2 - \sum\limits_{cyclic} a_1 a_2 \le$
$\frac{n}{2} \max\left\{(a_1 - a_2)^2; (a_2 - a_3)^2, ..., (a_n - a_1)^2\right\}.$

Mihály Bencze

PP. 15609. If $a_k \in R$ $(k = 1, 2, ..., n)$, then
$\frac{n(n-1)}{2} \min\left\{(a_i - a_j)^2 | 1 \le i < j \le n\right\} \le (n-1) \sum\limits_{k=1}^{n} a_k^2 - 2 \sum\limits_{1 \le i < j \le n} a_i a_j \le$
$\frac{n(n-1)}{2} \max\left\{(a_i - a_j)^2 | 1 \le i < j \le n\right\}.$

Mihály Bencze

PP. 15610. In all triangle ABC holds:
$y(x+y) m_a^2 + x(x+y) m_b^2 - xy m_c^2 \ge 0$ for all $x, y \in R$.

Mihály Bencze

PP. 15611. In all triangle ABC holds:
$\sum \left(5 + \sqrt{6}\sin A - 2\cos 2A\right)\left(5 - \sqrt{6}\sin A - 2\cos 2B\right) \ge \frac{21\left(s^2 + r^2 + 4Rr\right)}{2R^2}.$

Mihály Bencze

PP. 15612. If $a, b, c > 0$ and $abc = 1$, then
$\sum \frac{a+b+c^n}{a^{2n+3} + b^{2n+3} + ab} \le a^{n+1} + b^{n+1} + c^{n+1}$ for all $n \in N$.

Mihály Bencze

PP. 15613. In all triangle ABC holds:
$\prod |a - b| \le \frac{2s\left(s^2 + r^2 + 4Rr\right)\left(s^2 - 7r^2 - 10Rr\right)}{3(s^2 + r^2 + 2Rr)}.$

Mihály Bencze

PP. 15614. In all triangle ABC holds: $\sum \left(s^2 + r^2 ctg^2 \frac{A}{2}\right) \sin^2 B \le s^2$.

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PP. 15615. In all triangle ABC holds: $\frac{1}{OI} \sum |a - b| \le 2\sqrt{6}$.

Mihály Bencze

PP. 15616. If $x_k, a_k > 0$ $(k = 1, 2, ..., n)$ and $\prod\limits_{k=1}^{n} a_k x_k = 1$, then $\prod\limits_{k=1}^{n} \left(a_k + x_k^{a_k}\right) \geq 2^n$.

Mihály Bencze

PP. 15617. If $a, x_k > 0$ $(k = 1, 2, ..., n)$ and $\sum\limits_{k=1}^{n} x_k = 1$, then $\sum\limits_{k=1}^{n} \frac{x_k^{a+1} + a x_k}{a x_k + 1} \geq 1$.

Mihály Bencze

PP. 15618. Prove that exist infinitely many prime numbers, where can be expressed in following way: $x^3 + y^3 + z^3 - 3xyz$, when $x, y, z \in N$.

Mihály Bencze

PP. 15619. In all acute triangle ABC holds

$$\prod \left(\frac{b}{c+a} + \frac{c}{a+b} - \frac{a}{b+c}\right) \geq 8 \left(\frac{Rr + \sqrt[3]{4s^2 r^2 \left(s^2 - (2R+r)^2\right)}}{s^2 + r^2 + 2Rr}\right)^3.$$

Mihály Bencze

PP. 15620. In all triangle ABC holds: $R - 2r \geq \frac{1}{8R} \sum (a-b)^2$.

Mihály Bencze

PP. 15621. If $x, y, z > 0$, then $\sum\limits_{k=1}^{\infty} \left(\sum\limits_{cyclic} \frac{x}{(k^4+1)x + y + z}\right) \leq \frac{\sqrt{3}\pi^2}{12}$.

Mihály Bencze

PP. 15622. If $\alpha \in (-\infty, 0] \cup [1, +\infty)$, then in all triangle ABC holds: $\sum \left(\frac{a^2}{r_b r_c}\right)^{\alpha} \geq 3 \left(\frac{4(R-r)}{3r}\right)^{\alpha}$.

Mihály Bencze

PP. 15623. Prove that the equation $xy + yz + zx + 2xyz = 1$ have infinitely many positive solutions $(x_k, y_k, z_k)_{k\in N}$ for which x_k, y_k, z_k $(k \in N)$ are the sides of a triangle.

Mihály Bencze

PP. 15624. If $F(x) = \frac{1}{a\sin^2 x+b\cos^2 x} + \frac{1}{b\sin^2 x+a\cos^2 x}$, when $a, b > 0$ and $x \in R$, then

1). $\frac{a+b-abF(x)}{(a+b)F(x)-4} = \frac{a+b}{4}tg^2 2x.$

2). $\frac{4ab}{a+b}\left(\frac{1}{a} + \frac{1}{b} - F(x)\right) + (a+b)\left(F(x) - \frac{4}{a+b}\right) =$
$= \frac{(a-b)^2}{\left(a\sin^2 x+b\cos^2 x\right)\left(b\sin^2 x+a\cos^2 x\right)}$

3). $\left(\frac{1}{a} + \frac{1}{b} - F(x)\right)\left(F(x) - \frac{4}{a+b}\right) \geq \frac{(a-b)^4}{ab(a+b)^4}\sin^2 4x.$

Mihály Bencze

PP. 15625. If $x_k > 0$ $(k = 1, 2, ..., n)$ and $\sum\limits_{k=1}^{n} x_k = 1$, then
$\sum\limits_{k=1}^{n} \frac{x_k(3-x_k)}{(1-x_k)(2-x_k)} \geq \frac{n(3n-2)}{(n-1)(2n-1)}.$

Mihály Bencze

PP. 15626. In all triangle ABC holds
$\sum \frac{(2-\sin A)\left(s^2+r^2 ctg^2\frac{A}{2}\right)}{1+\cos A} \geq 2\left(s^2 + sr + r^2\right).$

Mihály Bencze

PP. 15627. In all triangle ABC holds

1). $\left(s^2 + r^2 + 4Rr\right)^2 \geq 16s^2r\left(R + \sqrt{3}s\right)$

2). $sR \geq 6\sqrt{3}r^2$ 3). $(4R + r)^2 - 2s^2 \geq \frac{2\sqrt{3}(2R+r)sr}{3R}$

Mihály Bencze

PP. 15628. 1). If $x, y, z \in R$, then
$\sqrt{x^2 - xy + y^2}; \sqrt{y^2 - yz + z^2}; \sqrt{z^2 - zx + x^2}$ are the sides of a triangle.

2). Determine all $x, y, z \in R$ and $n \in N$ for which
$\sqrt[n]{\frac{x^{n+1}-y^{n+1}}{x-y}}; \sqrt[n]{\frac{y^{n+1}-z^{n+1}}{y-z}}; \sqrt[n]{\frac{z^{n+1}-x^{n+1}}{z-x}}$ are the sides of a triangle.

Mihály Bencze

PP. 15629. 1). If $x, y, z \in [-1, 1]$, then $\sum x\sqrt{(1-y^2)(1-z^2)} \leq 1 + xyz$

2). Determine all $x_k \in [-1, 1]$ $(k = 1, 2, ..., n)$ and $n \in N^*$ for which
$\sum\limits_{cyclic} x_1\sqrt{\left(1-x_2^2\right)\left(1-x_3^2\right)...\left(1-x_n^2\right)} \leq 1 + \prod\limits_{k=1}^{n} x_k.$

Mihály Bencze

PP. 15630. Prove that
$2\sum\limits_{k=1}^{\infty} \frac{2k^2+2k+1}{2k^4+4k^3+6k^2+4k+1} < \frac{\pi^2}{3} - 1 < \sum\limits_{k=1}^{\infty} \frac{2k^6+6k^5+15k^4+20k^3+15k^2+6k+1}{(k^2+k)^4}.$

Mihály Bencze

PP. 15631. If $a > b > 1$, then determine all $c \in R$ for which

$\frac{a^n-c^n}{a^{n+1}-c^{n+1}} < \frac{b^n-c^n}{b^{n+1}-c^{n+1}}$, for all $n \in N^*$.

Mihály Bencze

PP. 15632. In all triangle ABC holds:

1). $\sum \frac{ctg\frac{A}{2}}{1+tg\frac{B}{2}} \geq \frac{3s}{s+r}$ 2). $\sum \frac{ctg\frac{A}{2}ctg\frac{B}{2}}{1+tg\frac{B}{2}tg\frac{C}{2}} \geq \frac{3s^2}{s^2+r^2}$

Mihály Bencze

PP. 15633. If $a, x_k > 0$ $(k = 1, 2, ..., n)$, then

$$\sum\limits_{cyclic} \frac{x_1^3}{(x_1+ax_2)\left(\sqrt{3a+2}x_1+\sqrt{a}x_2\right)\left(\sqrt{\sqrt{3a+2}x_1}+\sqrt{\sqrt{a}x_2}\right)} \geq \frac{\sqrt[4]{3a+2}-\sqrt[4]{a}}{2(a+1)^2}\sum\limits_{k=1}^{n}\sqrt{x_k}.$$

Mihály Bencze

PP. 15634. If $a, x_k > 0$ $(k = 1, 2, ..., n)$, then

$$\sum\limits_{cyclic} \frac{x_1^3}{(x_1+ax_2)\left(\sqrt[3]{(3a+2)^2x_1^4}+\sqrt[3]{(3a+2)ax_1^2x_2^2}+\sqrt[3]{a^2x_2^4}\right)} \geq \frac{\sqrt[3]{3a+2}-\sqrt[3]{a}}{2(a+1)^2}\sum\limits_{k=1}^{n}\sqrt[3]{x_k}.$$

Mihály Bencze

PP. 15635. If $x_k > 0$ $(k = 1, 2, ..., n)$, then

1). $\sum\limits_{cyclic} \frac{x_1^3}{(x_1+x_2)\left(\sqrt{5}x_1+x_2\right)} \geq \frac{\sqrt{5}-1}{8} \sum\limits_{k=1}^{n} x_k$

2). $\sum\limits_{cyclic} \frac{x_1^3}{(x_1+x_2)\left(\sqrt{5}x_1+x_2\right)\left(\sqrt{\sqrt{5}x_1}+\sqrt{x_2}\right)} \geq \frac{\sqrt[4]{5}-1}{8} \sum\limits_{k=1}^{n} \sqrt{x_k}$

3). $\sum\limits_{cyclic} \frac{x_1^3}{(x_1+x_2)\left(\sqrt[3]{25x_1^4}+\sqrt[3]{5x_1^2x_2^2}+\sqrt[3]{x_2^4}\right)} \geq \frac{\sqrt[3]{5}-1}{8} \sum\limits_{k=1}^{n} \sqrt[3]{x_k^2}$

Mihály Bencze

PP. 15636. If $x_k, a > 0$ $(k = 1, 2, ..., n)$, then $\sum\limits_{cyclic} \frac{x_1^3}{x_1+ax_2} \geq \frac{1}{a+1} \sum\limits_{k=1}^{n} x_k^2.$

Mihály Bencze

PP. 15637. If $a, x_k > 0$ $(k = 1, 2, ..., n)$, then
$\sum\limits_{cyclic} \frac{x_1^3}{(x_1+ax_2)\left(\sqrt{3a+2}x_1+\sqrt{a}x_2\right)} \geq \frac{\sqrt{3a+2}-\sqrt{a}}{2(a+1)^2} \sum\limits_{k=1}^{n} x_k.$

Mihály Bencze

PP. 15638. If $0 \leq x, y, z \leq 1$, then $\sum (x+1)\, yz^2 \geq \sum x^2y^2 + 3xyz.$

Mihály Bencze

PP. 15639. If $x, y, z > 0$, then
$\prod \sqrt{x^2 - xy + y^2} \geq \frac{1}{16\sqrt{2}} \prod \left(|2x - y| + \sqrt{3}y\right).$

Mihály Bencze

PP. 15640. If $x_k > 0$ $(k = 1, 2, ..., n)$, then

1). $\sum\limits_{cyclic} \frac{x_1^3}{\left(x_1^2+x_1x_2+x_2^2\right)\left(\sqrt{2x_1}+\sqrt{x_2}\right)} \geq \frac{\sqrt{2}-1}{3} \sum\limits_{k=1}^{n} \sqrt{x_k}$

2). $\sum\limits_{cyclic} \frac{x_1^3}{\left(x_1^2+x_1x_2+x_2^2\right)\left(\sqrt[3]{4x_1^2}+\sqrt[3]{2x_1x_2}+\sqrt[3]{x_2^2}\right)} \geq \frac{\sqrt[3]{2}-1}{3} \sum\limits_{k=1}^{n} \sqrt[3]{x_k}.$

Mihály Bencze

PP. 15641. Prove that $\sum\limits_{k=1}^{\infty} \frac{(k+1)^4}{k^2(3k^4+6k^3+7k^2+4k+1)} > \frac{\pi^2}{18} + \frac{1}{3}.$

Mihály Bencze

PP. 15642. Prove that $\sum\limits_{k=2}^{\infty} \frac{k^s}{k^{2s}-1} > \frac{3\sqrt{3}}{2}\left(\zeta\left(2s\right)-1\right)$ for all $s>1$, where ζ denote the Riemann zeta function.

Mihály Bencze

PP. 15643. In all triangle ABC holds $\frac{s}{4R}\sum \frac{1}{\cos\frac{A-B}{2}\cos\frac{C}{2}} \le \sum \frac{ctg\frac{A}{2}}{tg\frac{B}{2}+3tg\frac{C}{2}}$.

Mihály Bencze

PP. 15644. If $x_k>0$ $(k=1,2,...,n)$ and $\prod\limits_{k=1}^{n} x_k = 1$, then determine all $n\in N^*$ for which $\sum\limits_{k=1}^{n} x_k\left(x_k-1\right)\ge 0$.

Mihály Bencze

PP. 15645. Prove that:

1). $\sum\limits_{k=1}^{\infty} \frac{k+1}{k^2(2k+1)} > \frac{\pi^2}{12}+\frac{1}{8}$

2). $\sum\limits_{k=1}^{\infty} \frac{(k+1)^s}{k^{2s}(k^s+(k+1)^s)} > \frac{1}{2}\zeta\left(2s\right)+\frac{1}{8}$, for all $s>1$, where ζ denote the Riemann zeta function.

Mihály Bencze

PP. 15646. In all triangle ABC holds:

1). $\sum \frac{ctg^4\frac{A}{2}}{s+rctg^3\frac{A}{2}} \ge \frac{s}{2r^2}$ 2). $\sum \frac{tg\frac{A}{2}}{s+rctg^3\frac{A}{2}} \ge \frac{4R+r}{2s^2}$

Mihály Bencze

PP. 15647. If $x,y,z>0$, then $\sum\limits_{cyclic} \frac{x^3(ax+by)}{x^2+xy+y^2} \ge \frac{2a-b}{3}\sum x^2+\frac{2b-a}{3}\sum xy$ for all $a,b\ge 0$.

Mihály Bencze

PP. 15648. If $x,y,z>0$, then $\sum\limits_{cyclic} \frac{x^3\left(ax^2+by^2\right)}{x+y} \ge \frac{5a-b}{8}\sum x^4+\frac{5b-a}{8}\sum x^2y^2$ for all $a,b\ge 0$.

Mihály Bencze

PP. 15649. In all triangle ABC holds:

1). $\sum \frac{\sin^2 \frac{A}{2}}{4R\sin^4 \frac{A}{2}+r} \le \frac{1}{2r}$ 2). $\sum \frac{\sin A}{4R\sin^4 \frac{A}{2}+r} \le \frac{4R+r}{sr}$
3). $\sum \frac{1}{ctg^2 \frac{A}{2}+ctg\frac{B}{2}ctg\frac{C}{2}} \le \frac{1}{2}$.

Mihály Bencze

PP. 15650. In all triangle ABC holds:
1). $\sum \frac{\sin^4 \frac{A}{2}}{\left(4R\sin^4 \frac{A}{2}+r\right)\cos^2 \frac{A}{2}} \ge \frac{4R+r}{2s^2}$ 2). $\sum \frac{ctg\frac{A}{2}}{r+s\cdot tg^3 \frac{A}{2}} \ge \frac{r^2}{2s^3}$

Mihály Bencze

PP. 15651. In all triangle ABC holds:
1). $\sum \frac{1}{\cos^2 \frac{A}{2}} \le \frac{2R}{r}$ 2). $\sum \frac{1}{\sin A} \le \frac{R(4R+r)}{sr}$

Mihály Bencze

PP. 15652. In all triangle ABC holds:
1). $\sum \frac{1}{1-\sin A\sin B+\cos C} \le \frac{2R}{r}$ 2). $\sum \frac{ctg\frac{B}{2}}{1-\sin A\sin C+\cos B} \le \frac{2R(4R+r)}{sr}$
3). $\sum \frac{s-c}{1-\sin A\sin B+\cos C} \le \frac{2R(4R+r)}{s}$.

Mihály Bencze

PP. 15653. In all triangle ABC holds:
1). $\sum \frac{1}{tg^2\frac{A}{2}+tg\frac{A}{2}tg\frac{B}{2}+tg^2\frac{B}{2}} \le \frac{4R+r}{3r}$ 2). $\sum \frac{ctg^2\frac{A}{2}}{tg^2\frac{A}{2}+tg\frac{A}{2}tg\frac{B}{2}+tg^2\frac{B}{2}} \le \frac{s^2}{3r^2}$

Mihály Bencze

PP. 15654. In all triangle ABC holds:
1). $\sum \frac{1}{ctg^2\frac{A}{2}+ctg\frac{A}{2}ctg\frac{B}{2}+ctg^2\frac{B}{2}} \le \frac{1}{3}$ 2). $\sum \frac{tg^2\frac{A}{2}}{ctg^2\frac{A}{2}+ctg\frac{A}{2}ctg\frac{B}{2}+ctg^2\frac{B}{2}} \le \frac{r(4R+r)}{3s^2}$

Mihály Bencze

PP. 15655. In all triangle ABC holds:
1). $1+\left(\frac{4R+r}{s}\right)^2 \le \frac{2R}{r}$ 2). $s^2+r^2+2Rr \le 8R^2$

Mihály Bencze

PP. 15656. In all triangle ABC holds:
1). $\sum \sqrt[3]{ctg\frac{A}{2}} \le \frac{8R+5r}{3\sqrt[3]{sr^2}}$ 2). $\sum \sqrt[3]{ctg\frac{A}{2}ctg\frac{B}{2}} \le 1+\frac{2s}{3r}$

Mihály Bencze

PP. 15657. In all triangle ABC holds:
1). $\sum \sqrt[k]{tg\frac{A}{2}} \le \frac{3k-4}{k}\sqrt[k]{\frac{s}{r}}$ 2). $\sum \sqrt[k]{tg\frac{A}{2}tg\frac{B}{2}} \le \frac{3(k-2)}{k}+\frac{2(4R+r)}{ks}$,
for all $k \in N$, $k \ge 2$.

Mihály Bencze

PP. 15658. In all triangle ABC holds:
1). $\sum \sqrt[k]{ctg\frac{A}{2}} \le \left(\frac{3(k-2)}{k}+\frac{2(4R+r)}{kr}\right)\sqrt[k]{\frac{r}{s}}$
2). $\sum \sqrt[k]{ctg\frac{A}{2}ctg\frac{B}{2}} \le \frac{3(k-2)}{k}+\frac{2s}{kr}$, for all $k \ge 2$, $k \in N$.

Mihály Bencze

PP. 15659. If $a \ge e$, then $1+\sum\limits_{k=1}^{\infty}\left(\frac{\ln\left(k^2+a\right)}{\ln a}-1\right) \le \frac{\pi^2}{6}$.

Mihály Bencze

PP. 15660. Determine all $x, y \ge 0$ such that
$\ln\left(\left(1+x^2\right)\left(1+y^2\right)\right) \le xarctgy + yarctgx \le x^2+y^2$.

Mihály Bencze

PP. 15661. If $x_k > 0$ $(k=1,2,...,n)$, then
$\left(\frac{1}{n}\sum\limits_{k=1}^{n}x_k\right)^2 \le \ln\sqrt[n]{\prod\limits_{k=1}^{n}\left(1+x_k^2\right)} \le \frac{1}{n}\sum\limits_{k=1}^{n}x_k arctgx_k \le \frac{1}{n}\sum\limits_{k=1}^{n}x_k^2$.

Mihály Bencze

PP. 15662. In all acute triangle ABC holds:
1). $\frac{5s^2+3r^2-12R^2}{2sr} \le \exp\left(\frac{s^2-4Rr-r^2}{2sr}\right)$ 2). $\frac{(s+2R+r)^2}{s^2-(2R+r)^2} \le \exp\frac{2sr}{s^2-(2R+r)^2}$

Mihály Bencze

PP. 15663. Prove that $\sum\limits_{k=1}^{n}k^{\frac{2k+1}{k+1}} \ge \frac{n(n+1)(n+2)}{6}$.

Mihály Bencze

PP. 15664. If $x > 0$, then
1). $\ln\left(1+\sqrt{1+e^{-2x}}\right)\ln\left(1+\sqrt{1+e^{2x}}\right) < \left(e^{-x}+x\right)\left(e^{x}-x\right)$
2). $\left(1+\sqrt{1+e^{-2x}}\right)\left(1+\sqrt{1+e^{2x}}\right) \le e^{2chx}$

Mihály Bencze

PP. 15665. Determine all $a, b > 0$ for which the function
$f(x) = \log_{x+a}(x+b)$ is increasing and convex, for all $x > 1$.

Mihály Bencze

PP. 15666. Prove that:
1). $\int\limits_{1}^{\sqrt{3}} \ln^2\left(1+\frac{1}{x}\right)dx < \frac{\pi}{12}$ 2). $\int\limits_{\sqrt{3}}^{\sqrt{5}} x\ln^2\left(1+\frac{1}{x}\right)dx < \frac{1}{2}\ln\frac{3}{2}$

Mihály Bencze

PP. 15667. If $x > 0$, then
$\ln^2\left(1+\frac{1}{x}\right)+\ln\left(1+\sqrt{x^4+2x^2+2}\right) \le \frac{2}{x^2+1}+\ln\left(x^2+1\right).$

Mihály Bencze

PP. 15668. Prove that $\prod\limits_{k=1}^{n}\frac{\ln(2k-1)}{\ln k} \ge \frac{2^{2n-2}(n!)^2}{(2n)!}$.

Mihály Bencze

PP. 15669. If $x \ge 0$, then $\sum\limits_{k=1}^{n}\ln\left(e^k+x\right) \ge \frac{n(n+1)}{2}+\frac{(e^n-1)x}{e^n(e-1)}$.

Mihály Bencze

PP. 15670. If $x \ge 1$, then:
1). $x^4+3x^2+6+12x^2\ln x \ge 2x\left(3x^2+2\right)$
2). $2x^3+3x^2+7 \ge 11x+e\ln x$
3). $x^4+7x+12x^2\ln x+(x+1)\ln(x+1) \ge 8x^3+1$.

Mihály Bencze

PP. 15671. If $x \ge 1$, then $x^2\left(x^3+8+12x\ln x\right) \ge \left(8x^3+1\right)arctgx$.

Mihály Bencze

PP. 15672. If $x \geq 0$, then $\ln(e^x - x)\ln(e^x + x) + (e + x)\ln(e - x) \leq \ln\left(1 + x^2e^x\right)\ln\left(1 + 2x + x^2e^x\right) + (e - x)\ln(e + x)$.

Mihály Bencze

PP. 15673. Prove that:
1). $\prod\limits_{k=1}^{n}\left(1 + \frac{1}{k^2}\right) > \exp\left(\frac{2n-1}{en}\right)$ 2). $\prod\limits_{k=1}^{n}\left(1 + \frac{k}{a}\right) \leq \exp\left(\frac{n(n+1)\ln a}{2a}\right)$,
where $a > 1$.

Mihály Bencze

PP. 15674. If $f(x) = e^x + x^2 + x$ and $g(x) = e^x + ex$, then determine the minimum and the maximum of the expression: $\sum\limits_{k=1}^{n}\frac{f^{-1}(x_k)}{g^{-1}(x_k)}$, where $x_k > 0$ $(k = 1, 2, ..., n)$.

Mihály Bencze

PP. 15675. If $x \in \left[0, \frac{\pi}{2}\right]$, then $3x\sin 2x \leq 2(2 + \cos x)\sin^2 x$.

Mihály Bencze

PP. 15676. Prove that:
1). $\sum\limits_{k=1}^{n}\left(\frac{\ln k}{n}\right)^n < \frac{n(n+1)}{2}$ 2). $\sum\limits_{k=1}^{n}\left(\frac{\ln n}{k}\right)^k < n^2$

Mihály Bencze

PP. 15677. Prove that $\arcsin\left(\sin\frac{\pi\sqrt{e}}{180} + \sin\frac{\pi e}{180} + \sin\frac{\pi e^2}{180}\right) > \frac{67\pi}{180}$.

Mihály Bencze

PP. 15678. If $f(x) = \frac{2x}{1+x^2}$ and $x_k > 0$ $(k = 1, 2, ..., n)$, then determine the minimum and the maximum of the expression $\sum\limits_{k=1}^{n} f^{-1}(x_k)$.

Mihály Bencze

PP. 15679. Prove that
$\sqrt{\ln(e^\pi - 1)\ln(e^\pi + 1)} + \sqrt{\ln(\pi^e - 1)\ln(\pi^e + 1)} < 6,252$.

Mihály Bencze

PP. 15680. If $x \in [0,1]$, then $1+\frac{x}{4}+\frac{x}{2(x+2)}-\frac{x^2}{16} \le \sqrt{1+x} \le 1+\frac{x}{2}$.

Mihály Bencze

PP. 15681. Prove that $\left(\ln\left(e^{\pi}-1\right)\right)^{\frac{1}{e}}+\left(\ln\left(\pi^{e}-1\right)\right)^{\frac{1}{\pi}}<0,943$.

Mihály Bencze

PP. 15682. If $x \ge 1$, then $\prod\limits_{k=1}^{n}(x+k) \le \exp\left(nx+\frac{n(n-1)}{2}\right)$.

Mihály Bencze

PP. 15683. If $a>0$,then determine all $x_k>0$ $(k=1,2,...,n)$ such that $\sum\limits_{k=1}^{n} a^{x_k}=\sum\limits_{k=1}^{n} x_k^a$.

Mihály Bencze

PP. 15684. Determine all a_k $(k=1,2,...,n)$ such that $\frac{x\sum\limits_{k=1}^{n} a_k}{1+x^2} \le n \cdot arctgx+\frac{nx^2}{2}$ for all $x \in R$.

Mihály Bencze

PP. 15685. If $x \ge 0$, then $\frac{n(nx+1)}{n+x} \le (1+x)^{\frac{n^2-1}{n}}$ for all $n \in N^*$.

Mihály Bencze

PP. 15686. If $x \ge 1$, then:
1). $\left(1+\frac{1}{x}\right)^x \le e^{x^2-x+1}$ 2). $(x-1)\ln(x+1) \le x^2\ln x$
3). $\frac{x^3+x^2-2}{2x} \le e^x\ln x \le \frac{1}{2}\left(x^3e^x+\left(2-e^x\right)x^2-2\right)$.

Mihály Bencze

PP. 15687. If $x \in [0,1]$, then $2+\frac{3x}{2}+\frac{3x^2}{8} \le e^x+\sqrt{1+x} \le 2+\frac{3x}{2}+\frac{x^2e^x}{2}$.

Mihály Bencze

PP. 15688. Prove that $\int\limits_2^3 \ln x\ln\left(x^2-1\right)dx \le \frac{35}{8}+\ln\frac{3}{2}$.

Mihály Bencze

PP. 15689. Prove that $1+\sum\limits_{k=2}^{n}\frac{(k-1)\ln(k+1)}{\ln k}\le\frac{n(n+1)(2n+1)}{6}$.

Mihály Bencze

PP. 15690. If $x\in[0,1]$, then
$9\left(5+8x-8x^2\right)\ge\left(5+12x-16x^2\right)\left(1+20x-16x^2\right)$.

Mihály Bencze

PP. 15691. Prove that $\int\limits_{\frac{\pi}{6}}^{\frac{\pi}{3}}ctgx\ln\left(1+\sin^2x\right)dx\le\frac{1}{\sqrt{5}+\sqrt{7}}$.

Mihály Bencze

PP. 15692. 1). If $x_n=\sum\limits_{k=2}^{n}\frac{1}{\sqrt{1+k}\ln k\ln(k+1)}$, then $x_n\ge\frac{n-1}{n}$ for all $n\ge2$.
2). Determine $\max\left\{x_n|n\in N^*\right\}$
3). Prove that $\left(x_n\right)_{n\ge1}$ is convergent and compute its limit.

Mihály Bencze

PP. 15693. 1). Determine all $n,k\in N$ such that
$\sqrt[k]{n+2}+\sqrt[k]{n+5}<\sqrt[k]{n+3}+\sqrt[k]{n+4}$
2). Determine all $n,k\in N$ such that
$\sqrt[k]{(n+2)!}+\sqrt[k]{(n+5)!}<\sqrt[k]{(n+3)!}+\sqrt[k]{(n+4)!}$

Mihály Bencze

PP. 15694. If $x\ge0$, then $\left(1+\frac{x}{2}-\frac{x^2}{8}\right)\ln(1+x)\le x$.

Mihály Bencze

PP. 15695. If $0<a\le b$, then $\left(\sqrt{1+a^2}+\sqrt{1+b^2}\right)\int\limits_a^b\frac{\ln\left(1+x^2\right)}{x}dx\le b^2-a^2$.

Mihály Bencze

PP. 15696. If $a,x\in R$, $2b\le1$ and $4c\ge b$, then $a+bx-cx^2\le\sqrt{1+x}$ for all $x\ge0$.

Mihály Bencze

PP. 15697. If $x \geq 0$, then
$\sum\limits_{k=1}^{n} \ln^k \left(1 + x^{\frac{1}{k}} + \frac{1}{2}x^{\frac{2}{k}}\right) \leq nx \leq \sum\limits_{k=1}^{n} \ln^k \left(1 + x^{\frac{1}{k}} + \frac{1}{2}x^{\frac{2}{k}} \exp\left(x^{\frac{1}{k}}\right)\right).$

Mihály Bencze

PP. 15698. In all triangle ABC holds:
1). $\sum \frac{r_a}{r_a^2+s^2} \leq \frac{1}{4r}$ 2). $\sum \frac{h_a}{Rh_a^2+2s^2r} \leq \frac{1}{4Rr}$

Mihály Bencze

PP. 15699. Prove that
$\log_{2009}^2 4019 + \log_{2010}^2 4021 > \log_{2009} 8036 + \log_{2010} 8040$

Mihály Bencze

PP. 15700. If $x \in R$, then $\frac{\sin^2 x}{1+4\left(1+4\sin^2 x\right)\cos^2 x} + \frac{\cos^2 x}{1+4(1+4\cos^2 x)\sin^2 x} > \frac{1}{9}$.

Mihály Bencze

PP. 15701. In all triangle ABC holds
$\min\left\{\prod (r+r_a)^{\frac{1}{r_a}}; \prod (r+h_a)^{\frac{1}{h_a}}\right\} \geq (4r)^{\frac{1}{r}}.$

Mihály Bencze

PP. 15702. If $a_1, a_2, ..., a_n, ... > 0$ is an arithmetical progression, $c > 0$, $b_1, b_2, ..., b_n, ... > 0$ is a geometrical progression, then
$\max\left\{\prod\limits_{k=1}^{n} \frac{c^{a_{k+3}}+c^{a_{k+2}}+c^{a_{k+1}}}{c^{a_{k+4}}+c^{a_k}}; \prod \frac{b_{k+3}+b_{k+2}+b_{k+1}}{b_{k+4}+b_k}\right\} \leq \left(\frac{3}{2}\right)^n.$

Mihály Bencze

PP. 15703. If $a, b, c > 0$, then $\sum \frac{a^2+b^2c}{a(1+b)} + \sum \frac{a^3+c^2}{ab+c} \geq \left(-1 + \sum \frac{a+1}{b+1}\right)\left(\sum a\right).$

Mihály Bencze

PP. 15704. Let $(A, +, \cdot)$ be a ring. Determine all $a, b, c, d, e \in N$ for which from $\left(x^a + y^b\right)\left(x^c + y^d\right) = (x+y)^e$ for all $x, y \in A$ holds $x^2 = x$ for all $x \in A$.

Mihály Bencze

PP. 15705. If $a_1, a_2, ..., a_n, ... > 0$ is an arithmetical progression with ratio $r > 0$ and $k \in \{2, 3, ..., n-1\}$, then
$\left(r + \sqrt[k]{a_1 a_2 ... a_k}\right)\left(r + \sqrt[k]{a_2 a_3 ... a_{k+1}}\right) ... \left(r + \sqrt[k]{a_n a_1 ... a_{k-1}}\right) \leq a_2 a_3 ... a_{n+1}.$

Mihály Bencze

PP. 15706. If $b_1, b_2, ..., b_n, ... > 0$ is a geometrical progression with ratio $q > 0$ and $k \in \{2, 3, ..., n-1\}$, then
$\left(1 + q\sqrt[k]{b_1 b_2 ... b_k}\right)\left(1 + q\sqrt[k]{b_2 b_3 ... b_{k+1}}\right) ... \left(1 + q\sqrt[k]{b_n b_1 ... b_{k-1}}\right) \leq$
$\leq (1 + b_2)(1 + b_3) ... (1 + b_{n+1}).$

Mihály Bencze

PP. 15707. In all triangle ABC holds $r(4R + r) \geq sr\sqrt{3} \geq s^2 + 2r^2 + 8Rr$.

Mihály Bencze

PP. 15708. In all triangle ABC holds
1). $\sum \frac{r_a}{r_b + r_c} \geq 2 - \frac{r}{R}$ 2). $\sum \frac{h_a}{h_b + h_c} \geq 2 - \frac{8Rr}{s^2 + r^2 + 2Rr}$

Mihály Bencze

PP. 15709. Determine all $n, k \in N$ such that
$\left(3^{n+1} + 2k + 1\right)\left(3^{k+1} + 2n + 1\right)$ is divisible by 16.

Mihály Bencze

PP. 15710. If $x, y, z > 0$ $(x \neq y \neq z)$, then
$\sum_{cyclic} \frac{x^2 y^2}{z^2 (z-x)(z-y)} - 6 \geq \frac{1}{2x^2 y^2 z^2} \max\left\{(xy + yz)^3; (yz + zx)^3; (zx + xy)^3\right\}.$

Mihály Bencze

PP. 15711. If $a, b, c > 0$ and $a + b + c = 1$, then determine all $x > 0$, $y > 0$ such that $\left(\sum \frac{x^3 ab}{x - c}\right)^2 \leq y \sum (1 - a)^3$.

Mihály Bencze

PP. 15712. Determine all $x, y \in R^*$ such that $\frac{\arcsin x}{\{x\}} + \frac{\arcsin y}{\{y\}} = \frac{2(x+y)\pi}{3}$, where $\{\cdot\}$ denote the fractional part.

Mihály Bencze

PP. 15713. If $x \in \left(0, \frac{\pi}{2}\right)$, then $\frac{4(2+\sin 2x)}{4+5\sin 2x} \le \ln\left(2 + \frac{2}{\sin 2x}\right) \le \sin x + \cos x$.

Mihály Bencze

PP. 15714. In all triangle ABC holds
$\sum\left(\sqrt[3]{\sin A} + \frac{1}{\sqrt[3]{\sin A}}\right)^3 \le 12 + \frac{s^2(R+2r)+(4R+r)Rr}{sRr}$.

Mihály Bencze

PP. 15715. Let $(F_n)_{n\ge 0}$ be the Fibonacci sequence. Prove that
$(F_{n+2}-1)\left((n-1)F_{n+1} + (n+1)F_{n-1}\right)^2 \le 25n\left(F_0^2 + F_1^2 + ... + F_n^2\right)^3$.

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PP. 15716. In all triangle ABC holds $\sum \frac{1-tg\frac{A}{2}tg\frac{B}{2}}{tg\frac{A}{2}+tg\frac{B}{2}} \ge \sqrt{3}$.

Mihály Bencze

PP. 15717. In all triangle ABC holds $\left(a^2+b^2+c^2\right)^3 \ge \left(\frac{x}{x^2+x+1}\right)^3 (4sRr)^4$ for all $x > 0$.

Mihály Bencze

PP. 15718. Prove that $\frac{16}{9}\ln\frac{5}{3} - \frac{8}{9}\ln\frac{7}{3} - \frac{4}{63} < \int\limits_1^2 \frac{\ln(x+1)-\ln x}{x^4+x^2+1}dx < \frac{4-\sqrt{10}}{2\sqrt{5}}$.

Mihály Bencze

PP. 15719. In all triangle ABC holds $\sum\left(4 + \left(\sqrt{\frac{b}{c}} + \sqrt{\frac{c}{b}}\right)^2\right) w_a^2 \le 8s^2$.

Mihály Bencze

PP. 15720. Prove that $\frac{1}{\cos 1}\sum\limits_{k=1}^{n}\cos\frac{1}{k(k+1)} \ge \frac{n}{n+1} \ge \frac{1}{\sin 1}\sum\limits_{k=1}^{n}\sin\frac{1}{k(k+1)}$.

Mihály Bencze

PP. 15721. If $A \in M_3(R)$ is simmetric and invertable, then
$\det\left(A^2+I_3\right)\det\left(A^{-2}+I_3\right) \ge 55$.

Mihály Bencze

PP. 15722. In all triangle ABC holds $1+\sum\frac{a^2+b^2}{c^2}\geq\frac{s^2+r^2}{2Rr}$.

Mihály Bencze

PP. 15723. In all triangle ABC holds

$$\max\left\{\left(\frac{a}{b}-\frac{b}{c}\right)^2;\left(\frac{b}{c}-\frac{c}{a}\right)^2;\left(\frac{c}{a}-\frac{a}{b}\right)^2\right\}+$$
$$+\max\left\{\left(\frac{a}{c}-\frac{b}{a}\right)^2;\left(\frac{b}{a}-\frac{c}{b}\right)^2;\left(\frac{c}{b}-\frac{a}{c}\right)^2\right\}\leq$$
$$\leq\frac{\left(s^2-r^2-4Rr\right)\left(\left(s^2+r^2+4Rr\right)^2-16s^2Rr\right)}{6s^2R^2r}-\frac{s^2+r^2}{6Rr}-\frac{3}{8}.$$

Mihály Bencze

PP. 15724. 1). If $x_k\in R$ $(k=1,2,...,n)$ and $\prod\limits_{k=1}^{n}(s-x_k)\neq 0$, where $S=\sum\limits_{k=1}^{n}x_k$, then $\sum\limits_{k=1}^{n}\left|\frac{x_k}{S-x_k}\right|>1$

2). Determine all $x_k\in C$ $(k=1,2,...,n)$ for which $\sum\limits_{k=1}^{n}\left|\frac{x_k}{S-x_k}\right|>1$

Mihály Bencze

PP. 15725. In all triangle ABC holds

1). $\sum\frac{\sin^2\frac{A}{2}}{\left(1-\frac{r}{2R}-\sin^2\frac{A}{2}\right)^2}\geq\left(1-\frac{r}{2R}\right)^2\sum\frac{\sin\frac{A}{2}\sin\frac{B}{2}}{\left(1-\frac{r}{2R}-\sin^2\frac{A}{2}\sin^2\frac{B}{2}\right)^2}$

2). $\sum\frac{\cos^2\frac{A}{2}}{\left(2+\frac{r}{2R}-\cos^2\frac{A}{2}\right)^2}\geq\left(2+\frac{r}{2R}\right)^2\sum\frac{\cos\frac{A}{2}\cos\frac{B}{2}}{\left(2+\frac{r}{2R}-\cos^2\frac{A}{2}\cos^2\frac{B}{2}\right)^2}$

Mihály Bencze

PP. 15726. Compute $\int\frac{(2x-\sin 2x)dx}{(x+tgx)(x-tgx)}$.

Mihály Bencze

PP. 15727. In all triangle ABC holds $\sum\left(\frac{r_br_c}{w_a^2}\right)^{\alpha}\geq 3\left(\frac{5}{12}+\frac{s^2+r^2}{24Rr}\right)^{\alpha}$ for all $\alpha\in(-\infty,0]\cup[1,+\infty)$.

Mihály Bencze

PP. 15728. Let $a \in (-1,1)$ and $x_n = \sum\limits_{k=1}^{n} y_k a^k$. Study the convergence of sequence $(x_n)_{n\geq 1}$ and compute its limit in following cases:

1). $y_k = y_1 + (k-1)\,r$ 2). $y_k = y_1 q^{k-1}$
3). $y_k = \sqrt[k+1]{(k+1)!} - \sqrt[k]{k!}$ 4). $y_k = 1 + \frac{1}{2} + ... + \frac{1}{k} - \ln k$

Mihály Bencze

PP. 15729. Solve in Z the equations:
1). $(xyz-3)^2 = x^2+y^2+z^2$ 2). $(xy+yz+zx-3)^3 = x^3+y^3+z^3$

Mihály Bencze

PP. 15730. In all triangle ABC holds
$$\sum \sqrt{\left(m_a^2 + \frac{a^2}{4}\right)\left(m_b^2 + \frac{b^2}{4}\right)} \geq s^2 + r^2 + 4Rr.$$

Mihály Bencze

PP. 15731. In all triangle ABC holds
$$\sum \frac{h_a}{\min(b,c)} \geq 3\sqrt[3]{\frac{2s^2r^2}{R}} \max\left\{\frac{1}{\sqrt[3]{ab^2}}; \frac{1}{\sqrt[3]{bc^2}}; \frac{1}{\sqrt[3]{ca^2}}\right\}.$$

Mihály Bencze

PP. 15732. In all triangle ABC holds
$$\sum \frac{\max(a,b)}{r_c} \geq \frac{3}{\sqrt[3]{s^2 r}} \max\left\{\sqrt[3]{ab^2}; \sqrt[3]{bc^2}; \sqrt[3]{ca^2}\right\}.$$

Mihály Bencze

PP. 15733. If $a_k, \lambda > 0$ $(k = 1, 2, ..., n)$, then
$$\left(1+\frac{\lambda}{a_2}\right)^{a_1}\left(1+\frac{\lambda}{a_3}\right)^{a_2} ... \left(1+\frac{\lambda}{a_1}\right)^{a_n} \geq \left(1 + \frac{\lambda \sum\limits_{k=1}^{n} a_k}{a_1a_2 + a_2a_3 + ... + a_na_1}\right)^{\sum\limits_{k=1}^{n} a_k}.$$

Mihály Bencze

PP. 15734. If $x \in R$, then
$$\left(1 - 4\left|\cos\frac{x}{2}\right|\right)^2 \leq 5 + 2\cos 4x + 4\cos 3x + 6\cos 2x + 8\cos x.$$

Mihály Bencze

PP. 15735. In all triangle ABC holds
$27\left(\sum m_a\right)^4 \geq 512r\sum m_a\left(m_a+m_b\right)\left(m_a+m_c\right).$

Mihály Bencze

PP. 15736. In all triangle ABC holds

1). $\sum \frac{a}{h_a-2r} \geq \frac{4sR}{s^2+r^2-8Rr}$ 2). $\sum \frac{a^2}{h_a-2r} \geq \frac{12\left(s^2-r^2-4Rr\right)R}{s^2+r^2-8Rr}$
3). $\sum \frac{a^3}{h_a-2r} \geq \frac{12sR\left(s^2-3r^2-6Rr\right)}{s^2+r^2-8Rr}$

Mihály Bencze

PP. 15737. If $a_k > 0$ $(k=1,2,...,n)$, then
$\left(\sum\limits_{k=1}^{n} \frac{1}{a_k}\right)\left(\sum\limits_{k=1}^{n} a_k^k\right) + \left(\sum\limits_{k=1}^{n} a_k\right)\left(\sum\limits_{k=1}^{n} \frac{1}{a_k^{k-2}}\right) \geq 2\sum\limits_{k=1}^{n} a_k.$

Mihály Bencze

PP. 15738. In all triangle ABC holds $\sum \frac{m_a^2}{h_a} \geq 4R+r.$

Mihály Bencze

PP. 15739. Let be $z_k \in C$ $(k=1,2,...,n)$ such that
$\left(z_1-z_2\right)^n = \left(z_2-z_3\right)^n = ... = \left(z_n-z_1\right)^n$. Determine all $n \in N$ for which
$\left|nz_1-S\right| = \left|nz_2-S\right| = ... = \left|nz_n-S\right|$ where $S = z_1+z_2+...+z_n.$

Mihály Bencze

PP. 15740. If $a_k \in N^*$ $(k=1,2,...,n)$, then determine all $x_k \in R^*$ $(k=1,2,...,n)$, such that $\frac{\left[\sum\limits_{k=1}^{n} a_k x_k\right]}{\left\{\sum\limits_{k=1}^{n} a_k x_k\right\}} = \sum\limits_{k=1}^{n} \frac{a_k\left[x_k\right]}{\left\{x_k\right\}}$, where $[\cdot]$ and $\{\cdot\}$ denote the integer, respective the fractional part.

Mihály Bencze

PP. 15741. Compute $\lim\limits_{n\to\infty} n\left(\frac{1}{96} - \sum\limits_{k=1}^{n} \frac{13^{k-1}}{13^{2k-1}+98\cdot 13^{k-1}+49}\right).$

Mihály Bencze

PP. 15742. Compute:
1). $\int \frac{\cos x dx}{\prod\limits_{k=1}^{n} \cos(x+k)}$ 2). $\int \frac{\sin x dx}{\prod\limits_{k=1}^{n} \sin(x+k)}$

Mihály Bencze

PP. 15743. If $x, y, z > 0$ and $\lambda \in (-\infty, 0] \cup [1, +\infty)$, then
$\sum \left(\frac{x}{y+z}\right)^{\lambda} \geq \sum \left(\frac{x+y}{x+y+2z}\right)^{\lambda}$.

Mihály Bencze

PP. 15744. In all triangle ABC holds: $\sum \frac{ctg\frac{A}{2} ctg\frac{B}{2}}{1+9tg^2\frac{A}{2}tg^2\frac{B}{2}} \geq \frac{9}{2}$.

Mihály Bencze

PP. 15745. In all triangle ABC holds:
$\sum\limits_{k=2}^{\infty} \left(\sum \frac{tg^2\frac{A}{2}tg^2\frac{B}{2}ctg\frac{C}{2}}{tg\frac{B}{2}+(k^2-1)tg\frac{A}{2}}\right) \geq 27\left(\frac{\pi^2}{6}-1\right)\left(\frac{r}{s}\right)^2$.

Mihály Bencze

PP. 15746. If $z \in C$, then
$2(|z-1|+|z|+|z+1|)+\left|z^4-7z^3+19z^2-23z+11\right|+$
$+\left|z^4-3z^3+4z^2-2z+1\right|+\left|z^4+z^3+z^2+z+1\right| \geq 3$.

Mihály Bencze

PP. 15747. If $a, b, c > 0$ and $a+b+c=1$, then
$\sum \frac{a\left(3b^2+2bc^2+2b^2c+2abc+ac^3+bc^3+ab^2c\right)}{b^2c(b+ac)(c+1)(b+c^2)} \geq 27\left(\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}\right)$.

Mihály Bencze

PP. 15748. If $x \in R$, then $\sum\limits_{k=0}^{2n} \frac{4k^2+6k+3}{(k+1)(2k+1)}x^k \geq \frac{3(n+1)}{2n+1}$.

Mihály Bencze

PP. 15749. In all triangle ABC holds $\min\left\{\sum \frac{b}{\sin^2\frac{A}{2}}; \sum \frac{c}{\sin^2\frac{A}{2}}\right\} \geq \frac{2R}{r}$.

Mihály Bencze

PP. 15750. If $a, b, c > 0$ and $a + b + c = abc$, then:
1). $\frac{1}{2}\left(\frac{a}{a^2+1}+\frac{b}{b^2+1}+\frac{c}{c^2+1}\right)\le$
$\le \min\left\{\frac{1}{a(b+c)\left(\sqrt{b}+\sqrt{c}\right)};\frac{1}{b(c+a)\left(\sqrt{c}+\sqrt{a}\right)};\frac{1}{c(a+b)\left(\sqrt{a}+\sqrt{b}\right)}\right\}$
2). $\frac{1}{a^2+1}+\frac{1}{b^2+1}+\frac{1}{c^2+1}\ge 1-\frac{2}{\left(1+(abc)^{\frac{2}{3}}\right)^{\frac{3}{2}}}$.

Mihály Bencze

PP. 15751. In all triangle ABC holds:
$\sum\left(\frac{m_a}{a}\right)^2+\frac{3}{4}\ge\frac{1}{16}\left(\frac{s^2+r^2-2Rr}{Rr}\right)^2-\frac{s^2-r^2-Rr}{2Rr}$.

Mihály Bencze

PP. 15752. In all triangle ABC holds:
$\sum\frac{b+c}{\sin^2\frac{A}{2}}\ge\frac{2s^4-6s^2r^2-28s^2Rr-2\left(s^2-r^2-4Rr\right)^2+\left(s^2+r^2+4Rr\right)^2}{2s^2r^2}$.

Mihály Bencze

PP. 15753. Determine all $x, y, z \in R$ such that
$ch^4xch^4ych^4z = 8\left(sh^2x + sh^2y\right)\left(sh^2y + sh^2z\right)\left(sh^2z + sh^2x\right)$.

Mihály Bencze

PP. 15754. If $x, y, z \in R$, then
$\frac{\left(1+\sin^2 x\right)^2\left(1+\sin^2 y\right)^2\left(1+\sin^2 z\right)^2}{\left(\sin^2 x+\sin^2 y\right)\left(\sin^2 y+\sin^2 z\right)\left(\sin^2 z+\sin^2 x\right)}+\frac{\left(1+\cos^2 x\right)^2\left(1+\cos^2 y\right)^2\left(1+\cos^2 z\right)^2}{(\cos^2 x+\cos^2 y)(\cos^2 y+\cos^2 z)(\cos^2 z+\cos^2 x)}\ge$
≥ 16.

Mihály Bencze

PP. 15755. In all triangle ABC holds: $\sum\left(\frac{\cos\frac{A}{2}\cos\frac{B}{2}}{\sin\frac{C}{2}}\right)^{\alpha}\ge 3\left(\frac{s^2+r^2+4Rr}{12Rr}\right)^{\alpha}$
for all $\alpha\in(-\infty, 0]\cup[1, +\infty)$.

Mihály Bencze

PP. 15756. Denote $s(n)$ the sum of digits of number n. Prove that
$\sum\limits_{k,p\ge 1}\sqrt{\left[\frac{n}{10^k}\right]\left[\frac{m}{10^p}\right]}\le\frac{1}{9}\sqrt{(n-s(n))(m-s(m))}$, where $[\cdot]$ denote the integer part.

Mihály Bencze

PP. 15757. If $a, b > 0$ then solve in R the equation:

$$\begin{cases} \left(a+x_1^{2k}\right)\left(a+x_2^{2k}\right)...\left(a+x_{n-1}^{2k}\right)=bx_n \\ \left(a+x_2^{2k}\right)\left(a+x_3^{2k}\right)...\left(a+x_n^{2k}\right)=bx_1 \\ ------------------ \\ \left(a+x_n^{2k}\right)\left(a+x_1^{2k}\right)...\left(a+x_{n-2}^{2k}\right)=bx_{n-1} \end{cases}.$$

Mihály Bencze

PP. 15758. If $z_1, z_2, z_3 \in C$, then

$$|z_1|^2+|z_2|^2+|z_3|^2 \geq \mathrm{Re}(\overline{z_1}\cdot z_2)+\mathrm{Re}(\overline{z_2}\cdot z_3)+\mathrm{Re}(\overline{z_3}\cdot z_1)+$$
$$+\frac{1}{6}\left(|z_1-z_2|+|z_2-z_3|+|z_3-z_1|\right)^2.$$

Mihály Bencze

PP. 15759. If $a, b, c > 0$, then $\sum\limits_{k=1}^{n}\left(\sum\limits_{cyclic}\frac{a}{(k+1)(k+2)a+b+c}\right)^2 \leq \frac{3n}{4(n+1)}$.

Mihály Bencze

PP. 15760. Prove that:

1). $\sum\limits_{k=1}^{n}\left(\frac{k}{k^4+k^2+1}\right)^{\alpha} \geq n\left(\frac{n+1}{2(n^2+n+1)}\right)^{\alpha}$

2). $\sum\limits_{k=1}^{n}\left(\frac{k}{(4k^2-1)^2}\right)^{\alpha} \geq n\left(\frac{n+1}{2(2n+1)^2}\right)^{\alpha}$, for all $\alpha \in (-\infty, 0] \cup [1, +\infty)$.

Mihály Bencze

PP. 15761. 1). If $n \geq 6$, then $n \geq d^2(n)$

2). Determine all $p, k \in N$ such that from $n \geq k$ holds $n \geq d^p(n)$.

Mihály Bencze

PP. 15762. Solve the following system:

$$x_1+x_2^2+x_3^3+x_4^4 = x_1^2+x_2^3+x_3^4+x_4^5 = x_1^3+x_2^4+x_3^5+x_4^6 = x_1^4+x_2^5+x_3^6+x_4^7 = 1.$$

Mihály Bencze

PP. 15763. Determine all $k \in N$, such that

$$\prod_{p=1}^{n}\left(1+\frac{1}{p}+\frac{1}{p^2}+...+\frac{1}{p^k}\right) \leq n+1.$$

Mihály Bencze

PP. 15764. If $x, y \in R$ and $z > 1$, then
$\sqrt[3]{\log_3(z-1) \cdot \sin^2 x \cos^2 y} + \sqrt[3]{\log_3 z \cdot \cos^2 x \sin^2 y} + \sqrt[3]{\log_3(z+1)} \leq$
$\leq \sqrt[3]{12 \log_3 \frac{z}{3}}$.

Mihály Bencze

PP. 15765. If $b, c \in (0,1)$ and $y_n = b \cdot \arcsin x_n + c \cdot arctg x_{n-1}$ for all $n \geq 1$. Prove that the sequence $(x_n)_{n \geq 1}$ is convergent if and only if the sequence $(y_n)_{n \geq 1}$ is convergent.

Mihály Bencze

PP. 15766. If $a, b \in R$ and $a^2 + 8 \leq 4b$, then $\left(x^2 + b\right) chx + axshx \geq b$ for all $x \in R$.

Mihály Bencze

PP. 15767. Determine all $A, B \in M_n(R)$ such that
$\begin{cases} \det\left(A^2 + B + 2009 I_n\right) = -1 \\ \det\left(B^2 + A - 2009 I_n\right) = 1 \end{cases}$.

Mihály Bencze

PP. 15768. Let $A, B \in M_{2n}(R)$ invertable such that
$A^i - (B^*)^i + A^{-i} = B^i + (A^*)^i + B^{-i}$ for $i \in \{n, k, p\}$, where n, k, p are different given positive integers, then $A = B$.

Mihály Bencze

PP. 15769. Solve in Z the following equation $y^3 = x^3 + x^2 + x + 6$.

Mihály Bencze

PP. 15770. Let AA_1, BB_1, CC_1 be concurent line in triangle ABC, where $A_1 \in (BC), B_1 \in (CA), C_1 \in (AB)$. Prove that
$\sum \frac{a}{BB_1 + CC_1} \leq \frac{R\left(5s^2 + r^2 + 4Rr\right)}{s(s^2 + r^2 + 2Rr)}$.

Mihály Bencze

PP. 15771. If $x, y, z, t > 0$, then $\frac{x+y}{z} + \frac{y+z}{t} + \frac{z+t}{x} + \frac{t+x}{y} \geq \frac{2(x+y+z+t)}{\sqrt[4]{xyzt}} \geq 8$.

Mihály Bencze

PP. 15772. Let be $x_n(p) = \frac{1}{n}\sum\limits_{k=1}^{n} \sqrt[p]{k(k+1)\ldots(k+p-1)}$
1). Prove that $[x_n(2)] = \left[\frac{n+1}{2}\right]$
2). Compute $[x_n(p)]$ for all $p \geq 3$, $p \in N$ where $[\cdot]$ denote the integer part.

Mihály Bencze and Ferenc Kacsó

PP. 15773. If $x_0 = a, x_1 = a + \frac{1}{a}$, and $ax_n + a^2x_{n-1} + a^2 + 1 = ax_{n-1}x_{n-2}^2$ for all $n \geq 2$. Determine all $a \in R$ such that $2x_n = ch\left(\frac{a^n-(-1)^n}{2}\ln a\right)$ for all $n \geq 0$. ($a = 2$, is a solution).

Mihály Bencze

PP. 15774. Determine all n-time differentiable functions $f, g : R \to R$ such that $\begin{cases} (e^x f(x))^{(n)} = \left(\sqrt{2}\right)^n e^x g\left(x + \frac{n\pi}{4}\right) \\ (e^x g(x))^{(n)} = \left(\sqrt{2}\right)^n e^x f\left(x + \frac{n\pi}{4}\right) \end{cases}$.

Mihály Bencze

PP. 15775. Determine all $\alpha_n \in R$ $(n \in N)$ such that
$5 + 4ch\left(\left(2^{n-1} + \alpha_n\right)\ln 2\right) = 8ch\left(\left(2^{n-2} + \alpha_{n-1}\right)\ln 2\right) ch\left(\left(2^{n-2} + 2\alpha_{n-2}\right)\ln 2\right)$
for all $n \in N$.

Mihály Bencze

PP. 15776. 1). If n is odd, then for all $A \in M_n(R)$ holds $\det\left(A - A^T\right) = 0$
2). Determine all $A \in M_n(R)$ such that $\det\left(A - A^T\right) = 1$.

Mihály Bencze

PP. 15777. Prove that:
1). $\prod\limits_{n=1}^{m} tg\left(\sum\limits_{k=1}^{n} arctg\frac{2}{8k^2-4k-1}\right) = \frac{4^m(m!)^2}{(2m+1)!}$
2). $\sum\limits_{n=1}^{\infty}\left(1 - tg\left(\sum\limits_{k=1}^{n} arctg\frac{2}{8k^2-4k-1}\right)\right)^2 = \frac{\pi^2}{8}$.

Mihály Bencze

PP. 15778. 1). If $\varepsilon = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$, $n \in N$, $n \geq 3$, then
$\left|\sum\limits_{k=1}^{n-1} k\varepsilon^{k-1}\right| = \frac{n}{2\sin\frac{\pi}{n}}$
2). Prove that $\sin\frac{\pi}{n} \geq \frac{1}{n-1}$ for all $n \geq 2$

3). Determine all $z \in C$ such that $\left|\sum\limits_{k=1}^{n-1} kz^{k-1}\right| = \frac{n}{2\sin\frac{\pi}{n}}$

4). Determine all $z \in C$ such that $\left|\sum\limits_{k=1}^{n-1} kz^{k-1}\right| = \frac{n}{2\cos\frac{\pi}{n}}$

Mihály Bencze

PP. 15779. 1). Prove that exist infinitely many convex functions which graphics meet in infinitely many points
2). Determine all $f : R \to R$ convex and $g : R \to R$ concave functions for which $card\left(G_f \cap G_g\right) = +\infty$.

Mihály Bencze

PP. 15780. Solve in R^+ the following system $\begin{cases} \sum\limits_{k=1}^{n} x_k^3 = 1 \\ \sum\limits_{k=1}^{n} 3^{x_k} = n+1 \end{cases}$.

Mihály Bencze

PP. 15781. Solve in Z the equation $x^4 + y^4 = 2009z^4$.

Ferenc Kacsó

PP. 15782. If $x + a_1^2 + a_1a_2^2 + ... + a_1a_2...a_{n-1}a_n^2 = a_1a_2...a_n\left(x + a_n\right)$ for all $n \in N^*$, then determine the general term of the given sequence. If x is prime, then how many term of the given sequence are prime?

Mihály Bencze

PP. 15783. If $I_n = \int\limits_0^{\frac{\pi}{2}} (\sin x)^n \, dx$, then

1). $\sum\limits_{k=1}^{n} I_{k-1}I_k^2I_{k+1} = \frac{n\pi^2}{4(n+1)}$ 2). $\sum\limits_{n=1}^{\infty} I_n^2I_{n+1}^2 = \frac{\pi^4}{24}$

Mihály Bencze

PP. 15784. 1). Prove that $((2n)!)^{n-1}$ is divisible by $(2!4!...(2n-2)!)^2$ for all $n \geq 2$.

2). Prove that $((2n-1)!)^{n-1}$ is divisible by $\prod\limits_{k=1}^{2n-2} k!$ for all $n \geq 2$.

Mihály Bencze

PP. 15785. In all triangle ABC holds:
1). $\sum\left(\frac{r_a}{I_aB}\right)^4=\frac{(4R+r)^2-s^2}{8R^2}$ 2). $\sum\frac{1}{I_aB^2+\cos^2\frac{B}{2}}\leq\frac{1}{2r}$

Mihály Bencze

PP. 15786. Solve the following equation
$\sum\limits_{k=1}^{n}\left(\sqrt{x^2+2k-1}-\sqrt{x+2k}\right)=0.$

Ferenc Kacsó

PP. 15787. Prove that for each $n\in N$, there exist infinitely many $m\in N$ for which $5^{2m}-3^{3n}$ is divisible by 23.

Ferenc Kacsó

PP. 15788. Let $b_k=\frac{1}{k}\sum\limits_{p=1}^{k}a_p, c_k=\sqrt[k]{\prod\limits_{p=1}^{k}a_p}$, $d_k=\frac{k}{\sum\limits_{p=1}^{k}\frac{1}{a_p}}$, where $a_p>0$ $(p=1,2,...,k)$
1). Prove $(a_n)_{n\geq1}$ is arithmetical progression if and only if $(b_n)_{n\geq1}$ is arithmetical progression
2). Prove $(a_n)_{n\geq1}$ is geometrical progression if and only if $(c_n)_{n\geq1}$ is geometrical progression
3). Prove $(a_n)_{n\geq1}$ is harmonical progression if and only if $(d_n)_{n\geq1}$ is harmonical progression

Mihály Bencze

PP. 15789. 1). Solve the equation $\left[\frac{ax+b}{na+b}\right]=\frac{cx+d}{nc+d}$, where $[\cdot]$ denote the integer part, $a,b,c,d\in Z$ and $n\in N$ is given
2). How many prime solution have the given equation if a,b,c,d are prime?

Mihály Bencze

PP. 15790. 1). If $a,b>0$ then
$-\frac{a^2+b^2}{2b}\leq a(\sin x+\cos x)+b\sin x\cos x\leq\frac{2a\sqrt{2}+b}{2}$
2). What happend if $a,b\in R$?

Mihály Bencze

PP. 15791. If $x_k \in [0,1]$ $(k = 1, 2, ..., n)$, then one from $x_1(1 - x_2), x_2(1 - x_3), ..., x_n(1 - x_1), \prod\limits_{k=1}^{n} x_k + \prod\limits_{k=1}^{n}(1 - x_k)$ is greather or equal than $\left(\frac{n}{n+1}\right)^2$.

Mihály Bencze

PP. 15792. In all triangle ABC holds:

1). $\sum \frac{1}{1+\sin A+\sin B} \le 3 - \frac{s}{R} + \frac{s^2+r^2+4Rr}{12R^2}$

2). $\sum \frac{1}{1+\cos A+\cos B} \le \frac{5}{3} - \frac{r}{R} + \frac{s^2+r^2}{3R^2}$ (acute)

3). $\sum \frac{1}{1+\sin^2 \frac{A}{2}+\sin^2 \frac{B}{2}} \le 2 + \frac{r}{2R} + \frac{s^2+r^2-8Rr}{48R^2}$

4). $\sum \frac{1}{1+\cos^2 \frac{A}{2}+\cos^2 \frac{B}{2}} \le \frac{4}{3} - \frac{r}{2R} + \frac{s^2+r^2+8Rr}{48R^2}$

Mihály Bencze

PP. 15793. Determine all $n, m \in N^*$ for which $\sum\limits_{d|n} d \sum\limits_{k|d} \frac{\Phi(k)\Phi\left(\frac{d}{k}\right)}{k} + \sum\limits_{d|m} d \sum\limits_{k|d} \frac{\Phi(k)\Phi\left(\frac{d}{k}\right)}{k}$ is a perfect cube.

Mihály Bencze

PP. 15794. If $x_k > 0$ $(k = 1, 2, ..., n)$ and $\sum\limits_{k=1}^{n} x_k = 1$, then $\sum\limits_{cyclic} \frac{1}{1+x_1+x_2} \le n - 1 + \frac{1}{3} \sum\limits_{cyclic} x_1 x_2$.

Mihály Bencze

PP. 15795. In all triangle ABC holds $\sum \cos^2 \frac{A-B}{2} \ge \frac{1}{2}\left(\left(\frac{s}{R}\right)^2 - \frac{3}{4}\right)$.

Mihály Bencze

PP. 15796. The triangle ABC is equilateral if and only if $\sum \sin A \cos(B - C) = \frac{s^4+r^4+16R^2r^2-6s^2r^2-8s^2Rr+8Rr^3}{4sR^2r}$.

Mihály Bencze

PP. 15797. Let be $x_{n+1} = a_1 x_n + b_1 y_n$ and $y_{n+1} = a_2 x_n + b_2 y_n$. Determine all $x_1, y_1, a_1, b_1, a_2, b_2 \in R$ such that $x_{2n} = 6y_n^2 + 1$ and $y_{n+1}^2 - y_n^2 = y_{2n+1}$ for all $n \in N^*$.

Mihály Bencze

PP. 15798. In all triangle ABC holds
$8\left(1+\frac{r}{R}\right) \leq 9+\sum \cos(A-B) \leq \left(\frac{s^2+r^2+Rr}{2sr}\right)^2 - \frac{4R}{r}$.

Mihály Bencze

PP. 15799. In all acute triangle ABC holds $\sum \left(\frac{A}{\sin A}+\frac{B}{\sin B}\right)^2 \leq \frac{32R(4R+r)}{s^2}$.

Mihály Bencze

PP. 15800. In all triangle ABC holds $9R^2+r^2+4Rr \geq s^2+2\sqrt{3}sr$.

Mihály Bencze

PP. 15801. Let be $x_n = tg\left(\sum\limits_{k=1}^{n} arctg \frac{k^2+k-1}{(k^2+k+1)(k^2+k+2)}\right)$ for all $n \geq 1$. Prove that the sequence $(x_n)_{n\geq 1}$ is convergent, and compute its limit.

Mihály Bencze

PP. 15802. If $a_1, a_2, ..., a_n, ... > 0$ is an arithmetical progression, and $c > 0$, and $b_1, b_2, ..., b_n > 0$ is a geometrical progression, then
$\min\left\{\prod\limits_{k=1}^{n} \frac{c^{a_{k+4}}+3c^{a_{k+2}}+c^{a_k}}{c^{a_{k+3}}+c^{a_{k+1}}}; \prod\limits_{k=1}^{n} \frac{b_{k+4}+3b_{k+2}+b_k}{b_{k+3}+b_{k+1}}\right\} \geq \left(\frac{5}{2}\right)^n$.

Mihály Bencze

PP. 15803. If $x > 0$, then
$\frac{2x}{(x+1)(2x+1)} \leq \ln(x+1)\ln\left(1+\frac{1}{x}\right) \leq \sqrt{\frac{x^2+1}{x+1}} arctg x$.

Mihály Bencze

PP. 15804. Let be $x_n = \sum\limits_{k=1}^{n} k^{-\frac{2k+1}{k+1}}$
1). Prove that $x_n \leq \frac{n}{2(n+1)}$ for all $n \geq 1$
2). Determine $\min\{x_n | n \in N^*\}$
3). Prove that the sequence $(x_n)_{n\geq 1}$ is convergent, and compute its limit.

Mihály Bencze

PP. 15805. In all triangle ABC holds
1). $s^2+r^2 \geq 4R^2+Rr$ 2). $s^2+r^2+4Rr+4R^2 \geq 4sR$

Mihály Bencze

PP. 15806. In all acute triangle ABC holds
1). $\sum \frac{\sin A}{1+2\cos\frac{A}{2}\cos\frac{B-C}{2}} + 8\prod \sin\frac{B+C-A}{4}\cos\left(\frac{\pi+2A}{4}\right) \le 1$
2). if ABC is acute, then $\sum \frac{\cos A}{1+2\sin\frac{A}{2}\cos\frac{B-C}{2}} \le 1-\frac{r^2}{2R^2}$.

Mihály Bencze

PP. 15807. If $x>0$ then $\frac{2x^3+3x^2+2}{(2x+1)(x^2+1)} \le \ln\left(x^2+x+1+\frac{1}{x}\right) \le \sqrt{x^2+1}$.

Mihály Bencze

PP. 15808. If $x, a>0$, then $\ln\left(x^a+x^{a-1}\right) \le \frac{ax^2+x+1}{ex}$.

Mihály Bencze

PP. 15809. If $x>0$, then
1). $\frac{\sqrt{1+x^2}}{1+\sqrt{1+x^2}} \le \ln\left(1+\sqrt{1+x^2}\right) \le \frac{1}{x}+\ln x$
2). $\ln\left(1+\sqrt{2+x^2+2\sqrt{1+x^2}}\right) \le \frac{\sqrt{1+x^2}-1}{x^2}+\ln\left(1+\sqrt{1+x^2}\right) \le$
$\le \frac{x-1+\sqrt{1+x^2}}{x^2}+\ln x$.

Mihály Bencze

PP. 15810. If $x, y \ge 0$, then
$2+\frac{x+y}{2}-\frac{1}{4}\left(\frac{x+y}{2}\right)^2 \le \sqrt{1+x}+\sqrt{1+y} \le 2+\frac{x+y}{2}$.

Mihály Bencze

PP. 15811. If $x_k>0$ $(k=1,2,...,n)$ and $\sum\limits_{k=1}^{n} x_k^2=8$, then
$n-1 \le \sum\limits_{k=1}^{n} \sqrt{1+x_k}-\frac{1}{2}\sum\limits_{k=1}^{n} x_k \le n$.

Mihály Bencze

PP. 15812. If $x \ge 0$, then
1). $1-\frac{2x}{3}+\frac{5x^2}{9} \ge \frac{1}{\sqrt[3]{(x+1)^2}} \ge 1-\frac{2x}{3}$
2). $1-\frac{x}{2} \le \frac{1}{\sqrt{1+x}} \le 1$
3). $1+\frac{3x}{2}+\frac{3x^2}{8}-\frac{x^3}{16} \le (x+1)\sqrt{x+1} \le 1+\frac{3x}{2}+\frac{3x^2}{8}$

Mihály Bencze

PP. 15813. In all triangle ABC holds:
$\sqrt{2}\sum m_a m_b \geq \sum r_a w_a + \frac{1}{4}\sum \left|2\left(s^2 - r^2 - 4Rr\right) - 3ab\right|.$

Mihály Bencze

PP. 15814. Prove that $\sum\limits_{m=1}^{\infty} \frac{\sqrt[4]{m}\left(\sqrt[4]{mn}+1\right)}{\sqrt{n}(m+n)} \leq \left(\sqrt{2}+1\right)\pi$ for all $n \in N^*$.

Mihály Bencze

PP. 15815. Determine all $x \in \{0,1,2,...,9\}$ such that
$\overline{875x9x19} = \left(\overline{4x6}\right)^x + \left(\overline{167}\right)^x = \left(\overline{42x}\right)^x + \left(\overline{218}\right)^x = \left(\overline{414}\right)^x + \left(\overline{255}\right)^x.$

Mihály Bencze

PP. 15816. If $a_1 = 1$ and $a_{n+1}a_n = n + a_n$ for all $n \geq 1$, then
$\lim\limits_{n\to\infty} \frac{1}{n^2} \sum\limits_{k=1}^{n} \left(a_k - \frac{1}{2}\right)^2 = \frac{1}{2}$. Determine all $\alpha \in R$ for which
$\lim\limits_{n\to\infty} \frac{1}{n^2} \sum\limits_{k=1}^{n} \left(a_k - \alpha\right)^2 = \alpha.$

Mihály Bencze

PP. 15817. If $x \in [0, e-1]$, then $3\left(e^{2x} - 1\right) \leq 6x \leq e^{3x} + 3e^x - 4.$

Mihály Bencze

PP. 15818. Let be $x_1 = 1$, $x_n x_{n+1} = n + x_n$, $y_n = 4^n \prod\limits_{k=1}^{n} \frac{\left(x_k - \frac{1}{2}\right)^2}{4k+1}$ for all $n \geq 1$. Prove that the sequence $\left(y_n\right)_{n\geq 1}$ is convergent, and compute its limit.

Mihály Bencze

PP. 15819. If $a > b > 0$, then
$\left(\frac{\pi e}{9} + 1\right)\sqrt{ab} + \left(2e - 1 - \frac{\pi e}{9}\right)\frac{a+b}{2} \leq 2b\left(\frac{a}{b}\right)^{\frac{a}{a-b}}.$

Mihály Bencze

PP. 15820. If $0 < a \leq b < \frac{\pi}{2}$, then
$\int\limits_a^b tgx \cdot arctgx dx \geq \frac{(b-a)\left(b^2+ab+a^2+6\right)}{9} - \frac{2}{3} arctg\frac{b-a}{1+ab}.$

Mihály Bencze

PP. 15821. If $x \geq 0$, then $2x \geq arctgx + \frac{1}{3}arctg^3x + arctg\left(x + \frac{x^3}{3}\right)$.

Mihály Bencze

PP. 15822. If $x_k \in [0,1]$ $(k = 1, 2, ..., n)$ and $\prod\limits_{k=1}^{n}(1 + x_k) = e^{\alpha}$, then $\sum\limits_{k=1}^{n}\left(x_k - \frac{x_k^2}{2}\right) < \alpha < \sum\limits_{k=1}^{n}\left(x_k - \frac{x_k^2}{2} + \frac{x_k^3}{3}\right)$.

Mihály Bencze

PP. 15823. If $a, b > 0$, then $(a + b)\left(1 + \frac{ab\left(3\left(a^2-ab+b^2\right)+1\right)}{(3a^3+b)(3b^3+a)}\right) \leq$ $\leq \sqrt[3]{a^3 + b} + \sqrt[3]{b^3 + a} \leq \frac{a+b}{2} + \sqrt{\frac{a^2}{4} + \frac{b}{3a}} + \sqrt{\frac{b^2}{4} + \frac{a}{3b}}$.

Mihály Bencze

PP. 15824. Denote $x_1 = 0; x_2 = 1; x_3 = 1,2; x_3 = 1,3; x_4 = 1,5; x_5 = 2$ and $436^{x_n} + 167^{x_{n+1}} = 423^{x_{n+2}} + 228^{x_{n+3}} = 414^{x_{n+4}} + 255^{x_{n+5}} = 87539319$ for all $n \geq 1$. Prove that the sequence $(x_n)_{n\geq 1}$ is convergent, and compute its limit.

Mihály Bencze

PP. 15825. If $a \geq \frac{1}{3}$, then $achx + \frac{1}{2}(1 - a)\left(1 + ch^2x\right) \leq \frac{1}{e}(chx)^{2cth^2x}$.

Mihály Bencze

PP. 15826. Compute $\sum\limits_{k=1}^{n} arctg\frac{2\left(k^4-4k^3-2k^2+5\right)}{k(k^5-2k^3+5k+16)}$.

Mihály Bencze

PP. 15827. If $a, b, c > 0$ then compute the integer part of the expression $\sqrt{a^2 + b^2 + c^2}\left(\frac{a}{a^2+b^2+c^2+bc} + \frac{b}{a^2+b^2+c^2+ca} + \frac{c}{a^2+b^2+c^2+ab}\right)$.

Mihály Bencze

PP. 15828. If $a, b, c > 0$ then $\sum \frac{\left(a^3+b^3\right)\left(a^5+b^5\right)}{a^2b^2(a+b)^2} \geq a^2 + b^2 + c^2$.

Mihály Bencze

PP. 15829. Compute $\lim\limits_{n\to\infty}\frac{1}{n+1}\prod\limits_{k=1}^{n}\left(\frac{k+2}{k+1}\right)^{\frac{2k+3}{2k+1}}$.

Mihály Bencze

PP. 15830. Let be 201 points in a square with sides 1 all three noncollineare, prove that exist three points which formed a triangle with area $\leq 0,005$.

Mihály Bencze

PP. 15831. In all triangle ABC holds $\sum\frac{\cos(OAI\measuredangle)}{\cos\frac{A}{2}}\leq\frac{3s^2-r^2-4Rr}{4sr}$.

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PP. 15832. If $f_k: P(M)\to P(M)$ $(M\neq\emptyset)$ such that for all $X\subset Y\subset M$ holds $f_k(X)\subset f_k(Y)\subset M$ $(k=1,2,...,n)$. Prove that exist $A_k\in M$ $(k=1,2,...,n)$ such that
$f_1(A_1)=M\setminus A_2, f_2(A_2)=M\setminus A_3,...,f_n(A_n)=M\setminus A_1$.

Mihály Bencze

PP. 15833. In all triangle ABC holds:
1). $\sum a\ln(bc)\leq 4s\ln\frac{2s}{3}$ 2). $\sum(s-a)\ln(s-b)(s-c)\leq 2s\ln\frac{s}{3}$
3). $\sum h_a\ln(h_bh_c)\leq\frac{s^2+r^2+4Rr}{R}\ln\frac{s^2+r^2+4Rr}{6R}$
4). $\sum r_a\ln(r_br_c)\leq 2(4R+r)\ln\frac{4R+r}{3}$
5). $\sum\sin^2\frac{A}{2}\ln\left(\sin\frac{B}{2}\sin\frac{C}{2}\right)\leq\frac{2R-r}{2R}\ln\frac{2R-r}{6R}$
6). $\sum\cos^2\frac{A}{2}\ln\left(\cos\frac{B}{2}\cos\frac{C}{2}\right)\leq\frac{4R+r}{2R}\ln\frac{4R+r}{6R}$

Mihály Bencze

PP. 15834. If $x_0=0, x_1=1, x_n=kx_{n-1}+x_{n-2}$ for all $n\geq 2$, then determine all $k\in N$ for which from $k^t|n$ result $k^t|x_n$ $(t\in N)$.

Mihály Bencze

PP. 15835. In all acute triangle ABC holds:
1). $\frac{2R-r}{2R}\geq\frac{1}{3}\left(\frac{5}{4}+\frac{r}{2R}\right)^2$
2). $\frac{8R^2+r^2-s^2}{8R^2}\geq\frac{1}{27}\left(\frac{5}{4}+\frac{r}{2R}\right)^4$
3). $\frac{(2R-r)\left((4R+r)^2-3s^2\right)+6Rr^2}{32R^3}\geq\frac{1}{243}\left(\frac{5}{4}+\frac{r}{2R}\right)^6$

Mihály Bencze

PP. 15836. In all triangle ABC holds $\sum\left(\frac{ctg\frac{A}{2}}{m_a}\right)^{\alpha} \leq 3\left(\frac{s}{9r^2}\right)^{\alpha}$, for all $\alpha \in [0,1]$.

Mihály Bencze

PP. 15837. In all triangle ABC holds:

1). $\frac{4R+r}{2R} \geq \frac{1}{3}\left(\frac{3\sqrt{3}}{4} + \frac{s}{2R}\right)^2$

2). $\frac{(4R+r)^2-s^2}{8R^2} \geq \frac{1}{27}\left(\frac{3\sqrt{3}}{4} + \frac{s}{2R}\right)^4$

3). $\frac{(4R+r)^3-3s^2(2R+r)}{32R^3} \geq \frac{1}{243}\left(\frac{3\sqrt{3}}{4} + \frac{s}{2R}\right)^6$

Mihály Bencze

PP. 15838. If $x_k > 0$ $(k = 1, 2, ..., n)$, then

$$\sum_{cyclic} \frac{x_1^4+x_2^4}{x_1^2+x_1x_2+x_2^2} \geq \sum_{cyclic} \frac{x_1x_2\left(x_1^2+x_2^2\right)}{x_1^2+x_1x_2+x_2^2} + \frac{1}{n}\left(\sum_{cyclic} |x_1 - x_2|\right)^2.$$

Mihály Bencze

PP. 15839. If $a, b, c \in C$, then

1). $16\left|\sum ab(a-b)\left(a^2+b^2\right)\right| \leq \left(\left|a^2+b^2+c^2\right| + \sum |a-b|\right)^4$

2). $\left|(a-b)(b-c)(c-a)\left(a^2+b^2+c^2\right)\right| \leq \sum \left|ab(a-b)\left(a^2+b^2\right)\right|$

Mihály Bencze

PP. 15840. In all acute triangle ABC holds $\sum\left(\frac{\cos A}{m_a}\right)^{\alpha} \leq \frac{3}{(6r)^{\alpha}}$ for all $\alpha \in [0,1]$.

Mihály Bencze

PP. 15841. In all triangle ABC holds:

1). $\left(s^2 - 3r^2 - 6Rr\right)\left(s^2 + r^2 + 4Rr\right) \geq 8s^2Rr$

2). $\left(s^2 - 12Rr\right)(4R + r) \geq s^2 r$

3). $(4R + r)^3 - 12s^2R \geq r(4R + r)^2$

Mihály Bencze

PP. 15842. If $a, b, c \in R$ and $a + b + c \geq 0$, then
$a^3 + b^3 + c^3 \geq 3abc + \frac{1}{6}(a + b + c)(|a - b| + |b - c| + |c - a|)^2$.

Mihály Bencze

PP. 15843. In all triangle ABC holds:
1). $3\sum ctgA \le \sum \sqrt{1+\left(\frac{2R^2}{sr}\right)^2+\frac{1}{\sin^2 A}}$
2). $3\sum tgA \le \sum \sqrt{1+\left(\frac{4R^2}{s^2-(2R+r)^2}\right)^2+\frac{1}{\cos^2 A}}$

Mihály Bencze

PP. 15844. If $x_k > 0$ $(k = 1, 2, ..., n)$ and $a \in R$, then
$\sum\limits_{cyclic} \frac{x_1^2}{x_2} \ge \left(2a - a^2\right) \sum\limits_{k=1}^{n} x_k.$

Mihály Bencze

PP. 15845. If $a, b, c \in C$, then
1). $\sum |a-b| \ge 3\sqrt[3]{\left|\sum a^2b - \sum ab^2\right|}$
2). $\left|(a-b)(b-c)(c-a)\right| \le \sum \left|ab(a-b)\right|.$

Mihály Bencze

PP. 15846. If $a_k > 0$ $(k = 1, 2, ..., n)$, then
$$\sum_{cyclic} \frac{a_1^3(a_2+1)+a_2^3(a_1+1)}{a_1a_2+a_1+a_2} \ge \sum_{cyclic} \frac{a_1^2a_2(a_2+1)+a_2^2a_1(a_1+1)}{a_1a_2+a_1+a_2} + \frac{1}{n}\left(\sum_{cyclic} |a_1-a_2|\right)^2.$$

Mihály Bencze

PP. 15847. If $x, y > 0$ then for all $n \in N$ holds the inequality
$\left(\frac{x}{y}\right)^n + \left(\frac{y}{x}\right)^n - 2 \ge n^2\left(\frac{x}{y}+\frac{y}{x}-2\right).$

Mihály Bencze

PP. 15848. If $x > 0$, then $x + \frac{1}{x} \ge 2 + \frac{1}{n}\sum\limits_{k=1}^{n} k^2\left(x^{\frac{1}{k}} + x^{-\frac{1}{k}} - 2\right).$

Mihály Bencze

PP. 15849. If $x > 0$, then $x + \frac{1}{x} \ge 2 + \sqrt[n]{(n!)^2 \prod\limits_{k=1}^{n}\left(x^{\frac{1}{k}} + x^{-\frac{1}{k}} - 2\right)}.$

Mihály Bencze

PP. 15850. If $A, B, C \in M_3(R)$ and $A^2+B^2+C^2=AB+BC+CA$, then

$\det((AB-BA)+(BC-CB)+(CA-AC)) \leq 0$.

Mihály Bencze

PP. 15851. In all triangle ABC holds
$\sum \frac{a}{b} \leq \sqrt{2\left(s^2-r^2-4Rr\right)\left(\left(\frac{s^2+r^2+4Rr}{4sRr}\right)^2-\frac{1}{Rr}\right)}$.

Mihály Bencze

PP. 15852. If $x \in (0,1) \cup (1,+\infty)$, then
$\frac{\left(x^{n+1}-1\right)\left(x^n-1\right)}{x^n(x-1)}-2n \geq \frac{n(n+1)(2n+1)}{6}\left(x+\frac{1}{x}-2\right)$.

Mihály Bencze

PP. 15853. If $A, B \in M_2(R)$, then
$(AB-BA)^2=(\det(AB+BA)-4\det AB)\, I_2$.

Mihály Bencze

PP. 15854. If $A \in M_2(C)$, $a \in (-1,1)$ and $Tr(A)=x$, $\det A=y$, then
$\det\left(A^4-aA^3-aA+I_2\right)=(y-1)^4-x(y-1)^3(2x-a)+$
$+y(y-1)^2\left(2x^2-2ax+a^2+4\right)+x^2(y-1)^2\left(x^2-ax-2\right)-$
$-xy(y-1)(2x-a)\left(x^2-ax-2\right)+y^2\left(x^2-ax-2\right)^2$.

Mihály Bencze

PP. 15855. If $A \in M_2(R)$, then $\det\left(A^2-A+I_2\right) \geq 0$.

Mihály Bencze

PP. 15856. If $A, B, C \in M_2(R)$, then
$(AB-BA)^2(BC-CB)^2(CA-AC)^2+\det\left(AB^2C^2A+ABCBAC+\right.$
$+BABCAC+BACBCA-AB^2CAC-ABCBCA-BABC^2A-$
$\left.-BACBAC\right) I_2=0$.

Mihály Bencze

PP. 15857. If $A \in M_2(R)$, then exists $B_k \in M_2(R)$ $(k = 1, 2, ..., n)$, such that $A = \sum\limits_{k=1}^{n} B_k^2$.

Mihály Bencze

PP. 15858. If $A, B \in M_2(R)$, then
$2(\det A)^2 + \det(AB + BA) + 2(\det B)^2 \geq \det\left(A^2 - B^2\right) + 4\det AB$.

Mihály Bencze

PP. 15859. If $A_k \in M_2(R)$ and $A_iA_j = A_jA_i$, $i, j \in \{1, 2, ..., n\}$ $(i \neq j)$, then $\det\left(\sum\limits_{k=1}^{n} A_k^2\right) \geq 0$.

Mihály Bencze

PP. 15860. If $A \in M_2(C)$ and $\det A = \alpha$, then
$\det\left(A^2 + A - \alpha I_2\right) + \det\left(A^2 + \alpha I_2\right) = \alpha(1 + 4\alpha)$.

Mihály Bencze

PP. 15861. If $A \in M_2(C)$ and $Tr(A) = \frac{\sqrt{2}}{2}$, then
$\det\left(A^2 + \frac{3\sqrt{2}}{2}A + 3I_2\right) - \det\left(A^2 - \frac{\sqrt{2}}{2}A\right) = 15$.

Mihály Bencze

PP. 15862. If $X, A_k \in M_2(C)$ $(k = 1, 2, ..., n)$, then
$\sum\limits_{k=1}^{n} \det(X + A_k) - \det\left(X + \sum\limits_{k=1}^{n} A_k\right) = \sum\limits_{k=1}^{n} \det(X - A_k) - \det\left(X - \sum\limits_{k=1}^{n} A_k\right)$.

Mihály Bencze

PP. 15863. Let be $\begin{cases} 4x_n = (x_{n-1} + y_{n-1})\sqrt{6} + (x_{n-1} - y_{n-1})\sqrt{2} \\ 4y_n = (y_{n-1} - x_{n-1})\sqrt{6} + (x_{n-1} + y_{n-1})\sqrt{2} \end{cases}$ for all $n \geq 1$. Prove that the sequences $(x_n)_{n\geq 1}$, $(y_n)_{n\geq 1}$ are periodical and compute its period.

Mihály Bencze

PP. 15864. Copute the limit of the sequence $(x_n)_{n\geq 0}$ defined in following way: $x_n(x_{n+1} - 1) = 1$ for all $n \geq 0$.

Mihály Bencze

PP. 15865. If $x, y, z \in [0,1]$ and $a, b, c \geq 0$, then
$0 \leq (xy + (1-x-y)z)^a + (yz + (1-y-z)x)^b + (zx + (1-z-x)y)^c \leq 3.$

Mihály Bencze

PP. 15866. Determine all $n \in N$ for which $\left[\left(\sqrt[3]{126} - 5\right)^{-n}\right]$ is not divisible by 7.

Mihály Bencze

PP. 15867. Let be $S(a,p,t) = \sum\limits_{k=0}^{p-2} (-1)^k a^{k^t}$ where $p > a > 1$ is a prime and $a \in N$. Determine all $t \in N$ such that $S(a,p,t) \equiv 0 \pmod{p}$ if and only if exist $2r+1$ $(r \in N)$ for which $a^{2r+1} \equiv 1 \pmod{p}.$

Mihály Bencze

PP. 15868. If $H_k = 1 + \frac{1}{2} + ... + \frac{1}{k}$, then
1). $\sum\limits_{k=1}^{n} \left(\sum\limits_{p=1}^{k} pH_p^2 \right) > \frac{n(n+1)(2n+1)}{12}$ 2). $\prod\limits_{k=1}^{n} (4k-1)(H_{2k} - H_k) \geq \frac{1}{n+1}.$

Mihály Bencze

PP. 15869. Prove that:
1). $\sum\limits_{k=1}^{n} \sqrt{L_{2k-1}L_{2k}} \leq \sqrt{(L_{2n}-1)(L_{2n+1}-2)}$
2). $\sum\limits_{k=1}^{n} \sqrt{F_k L_k} \leq \sqrt{(F_{n+2}-1)(L_{n+2}-3)}$

Mihály Bencze

Solutions

PP. 9455. Using the Cauchy-Rogers inequality we obtain

1). $\sum x^2 h_a = \sum \frac{x^2}{\frac{1}{h_a}} \geq \frac{\left(\sum x\right)^2}{\sum \frac{1}{h_a}} = r\left(\sum x\right)^2$ and
2). $\sum x^2 h_a = \sum \frac{x^2}{\frac{1}{r_a}} \geq \frac{\left(\sum x\right)^2}{\sum \frac{1}{r_a}} = \frac{r}{2}\left(\sum x\right)^2$

Traian Ianculescu

PP. 9821, PP. 9822, PP. 9825, PP. 9826, PP. 9827

If $e(n) = \left(1+\frac{1}{n}\right)^n$, $E(n) = \sum\limits_{k=0}^{n} \frac{1}{k!}$, $F(n) = \left(1+\frac{1}{n}\right)^{n+1}$, $G(n) = \left(1-\frac{1}{n}\right)^{n+1}$, $L_n = \sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}$ and $\alpha_1 = \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \frac{1}{n+e(k)}$, $\alpha_2 = \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \frac{1}{n+E(k)}$, $\alpha_3 = \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \frac{1}{n+F(k)}$, $\alpha_4 = \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \frac{1}{n+G(k)}$, $\alpha_5 = \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \frac{1}{n+L_k}$ then

$$\frac{n}{n+2} \geq \sum_{k=1}^{n} \frac{1}{n+e(k)} \geq \sum_{k=1}^{n} \frac{1}{n+E(k)} \geq \frac{n}{n+e} > \sum_{k=1}^{n} \frac{1}{n+F(k)} \geq \frac{n}{n+4}$$

therefore

$$\alpha_1 = \alpha_2 = \alpha_3 = 1$$

Because $0 \leq G(n) < \frac{1}{e}$ and $\frac{1}{F(n)} < L_n < \frac{1}{e(n)}$, then $1 \geq \sum\limits_{k=1}^{n} \frac{1}{n+G(k)} > \frac{ne}{ne+1}$ and $\frac{4n}{4n+1} > \sum\limits_{k=1}^{n} \frac{1}{n+L_k} > \frac{2n}{2n+1}$ finally $\alpha_4 = \alpha_5 = 1$.

Traian Ianculescu

PP. 10100. We start from $\frac{2}{3}n\sqrt{n+1} < \sum\limits_{k=1}^{n} \sqrt{k} < \frac{2}{3}(n+1)\sqrt{n}$, so we obtain

$$\frac{2}{3} < \frac{1}{n\sqrt{n+1}} \sum_{k=1}^{n} \sqrt{k} < \frac{2}{3}\sqrt{\frac{n+1}{n}} < \frac{5}{6} \text{ so}$$

$$1 < \frac{1}{3} + \frac{1}{n\sqrt{n+1}} \sum_{k=1}^{n} \sqrt{k} < 2$$

and finally we get

$$\left[\frac{1}{3}+\frac{1}{n\sqrt{n+1}}\sum_{k=1}^{n}\sqrt{k}\right]=1$$

Traian Ianculescu

PP. 10185. Using the Cauchy-Schwarz inequality, we get

$$\sqrt{F_nL_{n+1}}+\sqrt{L_nF_{n+1}}\leq\sqrt{\left(F_n+F_{n+1}\right)\left(L_n+L_{n+1}\right)}=\sqrt{F_{n+2}L_{n+2}}$$

in same way we obtain

$$\sqrt{L_nP_{n+1}}+\sqrt{P_nL_{n+1}}\leq\sqrt{L_{n+2}P_{n+2}}$$

Traian Ianculescu

PP. 10399. The function $f\left(x\right)=x\ln x$ for $x>0$ is convex, and from Jensen‘s inequality we obtain

$$f\left(\frac{F_n+F_{n+1}}{L_n+L_{n+1}}\right)\leq\frac{L_nf\left(\frac{F_n}{L_n}\right)+L_{n+1}f\left(\frac{F_{n+1}}{L_{n+1}}\right)}{L_n+L_{n+1}}$$

or

$$F_n\ln\frac{F_n}{L_n}+F_{n+1}\ln\frac{F_{n+1}}{L_{n+1}}\geq F_{n+2}\ln\frac{F_{n+2}}{L_{n+2}}$$

Traian Ianculescu

PP. 10717. Using the Kantorovici inequality for $x_k\in\left[m,M\right]$, $0<m<M$, $\lambda_k>0$ $\left(k=1,2,...,n\right)$, we obtain

$$\left(\sum_{k=1}^{n}\lambda_kx_k\right)\left(\sum_{k=1}^{n}\frac{\lambda_k}{x_k}\right)\leq\frac{\left(m+M\right)^2}{4mM}\left(\sum_{k=1}^{n}\lambda_k\right)^2$$

1). If $x_1=a,x_2=b,x_3=c,\lambda_1=A,\lambda_2=B,\lambda_3=C$, then

$$\left(\sum Aa\right)\left(\sum\frac{A}{a}\right)\leq\frac{\left(a+c\right)^2}{4ac}\left(\sum A\right)^2=\frac{\left(a+c\right)^2\pi^2}{4ac}$$

2). If $x_1=\lambda_1=a,x_2=\lambda_2=b,x_3=\lambda_3=c$, then

$$3\sum a^2 < \frac{(a+c)^2}{4ac}\left(\sum a\right)^2 = \frac{s^2(a+c)^2}{ac} \text{ or } \sum a^2 < \frac{s^2(a+c)^2}{3ac}$$

3). If $x_1 = a, x_2 = b, x_3 = c, \lambda_1 = \frac{1}{r_a}, \lambda_2 = \frac{1}{r_b}, \lambda_3 = \frac{1}{r_c}$, then

$$\left(\sum \frac{a}{r_a}\right)\left(\sum \frac{1}{ar_a}\right) < \frac{(a+c)^2}{4ac}\left(\sum \frac{1}{r_a}\right)^2 = \frac{(a+c)^2}{4acr^2}$$

Traian Ianculescu

PP. 10816. Using the Hermite identity we have:

$$\sum [3A] = \sum\left([A] + \left[A+\frac{1}{3}\right] + \left[A+\frac{2}{3}\right]\right)$$

but

$$\sum\left[A+\frac{1}{3}\right] \geq \sum [A] \text{ because } \left[A+\frac{1}{3}\right] \geq [A] + \left[\frac{1}{3}\right] = [A]$$

and

$$\sum\left[A+\frac{2}{3}\right] \geq \left[\sum\left(A+\frac{2}{3}\right)\right] - 2 = [\pi+2] - 2 = 3$$

therefore

$$[3A] \geq 2\sum [A] + 3$$

Traian Ianculescu

PP. 10817. We have

$$\sum\left[A+\frac{1}{4}\right] \geq \sum [A], \ \sum\left[A+\frac{2}{4}\right] \geq \sum [A],$$

$$\sum\left[A+\frac{3}{4}\right] \geq \left[\sum\left(A+\frac{3}{4}\right)\right] - 3 = [2\pi+3] - 3 = 6$$

therefore

$$\sum [4A] = \sum\left([A] + \left[A+\frac{1}{4}\right] + \left[A+\frac{2}{4}\right] + \left[A+\frac{3}{4}\right]\right) \geq 3\sum [A] + 6$$

The general case is, if $a_k \geq 0$ $(k = 1, 2, ..., p)$, then

$$\sum_{k=1}^{p}[pa_k] \geq (p-1)\sum_{k=1}^{n}[a_k] + \left[\sum_{k=1}^{p}a_k\right]$$

Traian Ianculescu

PP. 10820. In all convex polygon $A_1A_2...A_n$ $(n \geq 3)$ we have

$$[(n-2)\pi] - (n-1) \leq \sum_{k=1}^{n}[A_k] \leq [(n-2)\pi],$$

therefore $\alpha \leq 1 - \frac{1}{n}\sum\limits_{k=1}^{n}[A_k] \leq \alpha + \frac{n-1}{n}$, where $\alpha = 1 - \frac{[(n-2)\pi]}{n}$.
If $n = 3$, $\alpha = 0$, so we get

$$\left|1 - \frac{[A]+[B]+[C]}{3}\right| \leq \frac{2}{3}$$

Traian Ianculescu

PP. 10821. If in the prevoius inequality we take $n = 4$, $\alpha = -\frac{1}{2}$, then

$$\left|1 - \frac{1}{4}\left([A]+[B]+[C]+[D]\right)\right| < \frac{3}{4}$$

Traian Ianculescu

PP. 10822. If $n = 5$, $\alpha = -\frac{4}{5}$, then

$$\left|1 - \frac{1}{5}\left([A]+[B]+[C]+[D]+[E]\right)\right| < \frac{4}{5}$$

Traian Ianculescu

PP. 10826. From inequality

$$\sum_{k=1}^{n}[x_k] \leq \left[\sum_{k=1}^{n}x_k\right] \leq \sum_{k=1}^{n}[x_k] + n - 1$$

we have

$$\left[\sum_{k=1}^{n}x_k\right] - (n-1) \leq \sum_{k=1}^{n}[x_k] \leq \left[\sum_{k=1}^{n}x_k\right]$$

If $x_k = (2k-1)\pi$ $(k = 1, 2, ..., n)$, then

$$\left[n^2\pi\right] - (n-1) \le \sum_{k=1}^{n} [(2k-1)\pi] \le \left[n^2\pi\right]$$

but $n^2\pi - 1 < \left[n^2\pi\right] \le n^2\pi$ so

$$0 \le n\pi - \frac{1}{n}\sum_{k=1}^{n} [(2k-1)\pi] < 1$$

Traian Ianculescu

PP. 10827. From inequality

$$\left[\sum_{k=1}^{n} x_k\right] - (n-1) \le \sum_{k=1}^{n} [x_k] \le \left[\sum_{k=1}^{n} x_k\right]$$

we get

$$[(2k-1)\pi] - 2 \le [(2k-1)A] + [(2k-1)B] + [(2k-1)C] \le [(2k-1)\pi]$$

and

$$\sum_{k=1}^{n} [(2k-1)\pi] - 2n \le \sum_{k=1}^{n} \left([(2k-1)A] + [(2k-1)B] + [(2k-1)C]\right) \le$$

$$\le \sum_{k=1}^{n} [(2k-1)\pi]$$

but $n^2\pi - n < \sum\limits_{k=1}^{n} [(2k-1)\pi] \le n^2\pi$, therefore

$n^2\pi - 3n < \sum\limits_{k=1}^{n} \left([(2k-1)A] + [(2k-1)B] + [(2k-1)C]\right) \le n^2\pi$ and finally

$$0 \le n\pi - \frac{1}{n}\sum_{k=1}^{n} \left([(2k-1)A] + [(2k-1)B] + [(2k-1)C]\right) < 3$$

Traian Ianculescu

PP. 10828. From $\left[\sum\limits_{k=1}^{n} A_k\right] - (n-1) \leq \sum\limits_{k=1}^{n} [A_k] \leq \left[\sum\limits_{k=1}^{n} A_k\right]$ valid in all concex polygon $A_1A_2...A_n$ $(n \geq 3)$ we get

$$1 \leq [A] + [B] + [C] \leq 3,\ 3 \leq [A] + [B] + [C] + [D] \leq 6$$

and

$$5 \leq [A] + [B] + [C] + [D] + [E] \leq 9$$

Traian Ianculescu

PP. 10911. In triangle ABC we have

$$\sum a\,(b-c)^2 + 4abc > \sum a^3 \Leftrightarrow \sum a\,(a-b+c)\,(a+b-c) < 4abc \Leftrightarrow$$

$$\Leftrightarrow \sum \sin^2 \frac{A}{2} < 1$$

true because

$$\sum \sin^2 \frac{A}{2} = 1 - 2 \prod \sin \frac{A}{2} < 1$$

If $a \to m_a, b \to m_b, c \to m_c$, then
1). $\sum m_a\,(m_b - m_c)^2 + 4m_am_bm_c > \sum m_a^3$
2). $\sum a^2\,(-a+b+c) \leq 3abc \Leftrightarrow \left(\sum a\right)\left(\sum a^2\right) - 2\sum a^3 \leq 3abc \Leftrightarrow$
$\left(\sum a\right)\left(2\sum a^2 - \sum ab\right) \leq 3\sum a^3$, but $\sum ab = s^2 + r^2 + 4Rr$,
$\sum a^2 = 2s^2 - 2r^2 - 8Rr$, $\sum a^3 = 2s\left(s^2 - 3r^2 - 6Rr\right) \Leftrightarrow R \geq 2r.$
If $a \to m_a, b \to m_b, c \to m_c$, then

$$\sum m_a^2\,(-m_a + m_b + m_c) \leq 3m_am_bm_c$$

3). $\left(\sum a\right)^3 \leq 5\sum ab\,(a+b) - 3abc \Leftrightarrow \left(\sum a\right)\left(\sum a^2 - 3\sum ab\right) + 18abc \leq 0 \Leftrightarrow$
$s^2 + 5r^2 - 16Rr \geq 0 \Leftrightarrow IG^2 = \frac{1}{9}\left(s^2 + 5r^2 - 16Rr\right) \geq 0$
If $a \to m_a, b \to m_b, c \to m_c$, then

$$\left(\sum m_a\right)^3 \leq 5\sum m_am_b\,(m_a + m_b) - 3m_am_bm_c$$

Traian Ianculescu

PP. 10912. We have
1). $\sum \frac{a}{b+c-a} = \sum \frac{a}{2(s-a)} = \frac{1}{2}\left(s\sum \frac{1}{s-a} - 3\right) = \frac{s}{2}\sum \frac{1}{s-a} - \frac{3}{2} \geq \frac{s}{2}\cdot\frac{9}{s} - \frac{3}{2} = 3$
and if $a \to m_a, b \to m_b, c \to m_c$, then we get

$$\sum \frac{m_a}{m_b + m_c - m_a} \geq 3.$$

2). $\sum \frac{1}{a+b} \geq \frac{9}{\sum(a+b)} = \frac{9}{2\sum a}$. If $a \to m_a, b \to m_b, c \to m_c$, then

$$2\sum \frac{1}{m_a + m_b} \geq \frac{9}{\sum m_a}$$

3). $\sum a^4 = \sum \left(a^2\right)^2 \geq \sum (ab)^2 \geq abc\sum a$. If $a \to m_a, b \to m_b, c \to m_c$, then

$$\sum m_a^4 \geq m_a m_b m_c \sum m_a$$

Traian Ianculescu

PP. 10913. We have
1). $\sum a \leq \sum \frac{a^2+b^2}{2c}$ because $\frac{a^2+b^2}{2c} \geq \frac{ab}{c} = \frac{abc}{c^2}$ and
$\sum \frac{a^2+b^2}{2c} \geq abc\sum \frac{1}{c^2} = abc\sum \frac{1}{a^2} \geq abc\sum \frac{1}{ab} = 2s = \sum a$ and
$\sum \frac{a^2+b^2}{2c} \leq \sum \frac{a^3}{bc} \Leftrightarrow \sum ab\left(a^2+b^2\right) \leq 2\sum a^4$ but $\frac{a^4+b^4}{2} \geq \left(\frac{a^2+b^2}{2}\right)^2$ we get
$a^4 + b^4 \geq \left(a^2+b^2\right)\left(\frac{a^2+b^2}{2}\right) \geq ab\left(a^2+b^2\right)$ so $2\sum a^4 \geq \sum ab\left(a^2+b^2\right)$.
If $a \to m_a, b \to m_b, c \to m_c$, then

$$\sum m_a \leq \sum \frac{m_a^2 + m_b^2}{2m_c} \leq \sum \frac{m_a^3}{m_b m_c}$$

2). $\frac{abc\left(\sum a\right)^2}{\sum a^2} \geq 2abc + \prod (a+b-c) \Leftrightarrow \sum a^2 \geq \frac{2s^2R}{R+r}$ because
$\prod (a+b-c) = 8sr^2$ and
$\sum a^2 = 2s^2 - 2r^2 - 8Rr \leq \frac{2s^2R}{R+r} \Leftrightarrow s^2 \leq 4R^2 + 5Rr + r^2 \Leftrightarrow s^2 \leq 4R^2 + 4Rr + 3r^2$,
$\sum a^2 \leq 8R^2 + 4r^2$, $R \geq 2r$.
If $a \to m_a, b \to m_b, c \to m_c$, then

$$\frac{m_a m_b m_c \left(\sum m_a\right)^2}{\sum m_a^2} \geq 2m_a m_b m_c + \prod (m_a + m_b - m_c)$$

Traian Ianculescu

PP. 10915. We have $\sum a^2 (-a+b+c) \leq 3abc$ or $\sum a^2 (b+c) \leq 3abc + \sum a^3$ and $\sum a^3 \leq \sum a(b-c) + 4abc$, therefore after addition we get

$$\sum a^2 (b+c) \leq 7abc + \sum a(b-c)$$

Traian Ianculescu

PP. 10918. From previous problems we get $\frac{abc(\sum a)^2}{\sum a^2} \geq 2abc + \prod (a+b-c)$ and $\sum \frac{a^2+b^2}{2c} \geq \sum a$, therefore

$$\frac{abc\left(\sum a\right)^2}{\sum a^2} + \sum \frac{a^2+b^2}{2c} \geq 2abc + \prod (a+b-c) + \sum a$$

Traian Ianculescu

PP. 10990. If $S = \sum\limits_{k=1}^{n} F_k = F_{n+2} - 1$ then $\sum\limits_{k=1}^{n} \frac{F_k}{2(F_{n+2}-1)-F_k} = 2s \sum\limits_{k=1}^{n} \frac{1}{2S-F_k} - n \geq \frac{2n^2 S}{2nS-S} - n = \frac{n}{2n-1}$.

Traian Ianculescu

PP. 10991. We have $\sum\limits_{k=0}^{n} \frac{\binom{n}{k}}{2^{n+1}-\binom{n}{k}} = 2^{n+1} \sum\limits_{k=0}^{n} \frac{1}{2^{n+1}-\binom{n}{k}} - (n+1) \geq \frac{2^{n+1}(n+1)^2}{(n+1)2^{n+1}-2^n} - (n+1) = \frac{n+1}{2n+1}$.

Traian Ianculescu

PP. 11019. If $E(x) = \sum \frac{a}{xa+bc} = \sum \frac{a}{2s+(x-1)a}$, then $\frac{2}{x+1} \leq E(x) \leq \frac{3}{x+2}$ for all $x \geq 1$. For $x = 1$ we get the equality. If $x > 1$ then we show that

$$\frac{9}{x+2} \leq 2s \sum \frac{1}{2s+(x-1)a} < \frac{x+5}{x+1}$$

But $\sum \frac{1}{2s+(x-a)a} \geq \frac{9}{6s+2(x-1)s} = \frac{9}{2s(x+2)}$ so $2s \sum \frac{1}{2s+(x-a)a} \geq \frac{9}{x+2}$. In another way $\sum \frac{a}{2s+(x-1)a} > \sum \frac{a}{s(x+1)} = \frac{2}{x+1}$ and $\sum \frac{2s}{2s+(x-a)a} < 3 - \frac{2(x-1)}{x+1} = \frac{x+5}{x+1}$.
If $a \to m_a, b \to m_b, c \to m_c$, then

$$\frac{2}{x+1} < \sum \frac{m_a}{xm_a + m_b + m_c} \leq \frac{3}{x+2}$$

Traian Ianculescu

PP. 11020. We have from previous problem
$\left(\frac{2}{k+1}\right)^2 \leq \left(\sum \frac{a}{ka+b+c}\right)^2 \leq \left(\frac{3}{k+2}\right)^2$ and
$4\sum\limits_{k=1}^{n} \frac{1}{(k+1)^2} \leq \sum\limits_{k=1}^{n} \left(\sum \frac{a}{ka+b+c}\right)^2 \leq 9\sum\limits_{k=1}^{n} \frac{1}{(k+2)^2}$ and if $n \to \infty$ then we get

$$4\left(\zeta\left(2\right)-1\right) \leq \sum_{k=1}^{n} \left(\sum \frac{a}{ka+b+c}\right)^2 \leq 9\left(\zeta\left(2\right)-\frac{5}{4}\right)$$

Traian Ianculescu

PP. 11021. We have $2 \leq E\left(x\right) = \sum \frac{(x+1)a}{xa+b+c} \leq \frac{3(x+1)}{x+2} < 3$, therefore $\left[E\left(x\right)\right] = 2$ for all $x \geq 1$.

Traian Ianculescu

PP. 11040. Using the Chebyshev‘s inequality for sequences $\left(\frac{1}{F_1}, \frac{1}{F_2}, ..., \frac{1}{F_n}\right)$ and $(F_n, F_{n-1}, ..., F_1)$ we get
$n\left(\frac{F_n}{F_1} + \frac{F_{n-1}}{F_2} + ... + \frac{F_1}{F_n}\right) \geq \left(\sum\limits_{k=1}^{n} \frac{1}{F_k}\right)\left(\sum\limits_{k=1}^{n} F_k\right) = \left(F_{n+2}-1\right)\sum\limits_{k=1}^{n} \frac{1}{F_k}$,
therefore $\sum\limits_{k=1}^{n} \frac{1}{F_k} \leq \frac{n}{F_{n+2}-1}\left(\frac{F_n}{F_1} + ... + \frac{F_1}{F_n}\right)$.

Traian Ianculescu

PP. 11114. We have

$$\sum \log_{ab^2c^2} a = \sum \frac{x}{x+2y+2z} \geq \frac{3}{5}$$

where $x = \ln a, y = \ln b, z = \ln c$ (See: Inegalitati from L. Panaitopol etc., problem 2.24).

Traian Ianculescu

PP. 11130. We have

$$\frac{2}{3}n^2\sqrt{n^2+1} < \sum_{k=1}^{n^2}\left[\sqrt{k}\right] < \frac{2}{3}\left(n^2+1\right)n, \text{ so}$$

$$\left[\frac{2}{3}n\sqrt{n^2+1}\right] \leq \left[\frac{1}{n}\sum_{k=1}^{n^2}\sqrt{k}\right] \leq \left[\frac{2}{3}\left(n^2+1\right)\right] \text{ or}$$

$$\frac{1}{n^2}\left(\frac{2}{3}n\sqrt{n^2+1}-1\right) < \frac{1}{n^2}\left[\frac{1}{n}\sum_{k=1}^{n^2}\sqrt{k}\right] \le \frac{2}{3}\left(\frac{n^2+1}{n^2}\right)$$

and finally we get

$$\lim_{n\to\infty}\frac{1}{n^2}\left[\frac{1}{n}\sum_{k=1}^{n^2}\sqrt{k}\right] = \frac{2}{3}$$

Traian Ianculescu

PP. 11140. Using the Cauchy-Schwarz inequality we get:

$$\left(\sum_{k=1}^{n}\sigma(k)\right)^2 = \left(\sum_{k=1}^{n}k\left[\frac{n}{k}\right]\right)^2 \le \left(\sum_{k=1}^{n}k^2\right)\left(\sum_{k=1}^{n}\left(\left[\frac{n}{k}\right]\right)^2\right) =$$

$$= \frac{n(n+1)(2n+1)}{6}\sum_{k=1}^{n}\left(\left[\frac{n}{k}\right]\right)^2$$

Traian Ianculescu

PP. 11181. We have
1). $\sum \log_{b^3c} a = \sum \frac{x}{3y+z} \ge 1$, where $x = \ln a$, $y = \ln b$, $z = \ln c$, $t = \ln d$ (see: Inegalitati from L. Panaitopol, problem 2.48)
2). $\sum \log_{bc} a = \sum \frac{x}{y+z} \ge 2$ (see: Inegalitati from L. Panaitopol, problem 2.49)

Traian Ianculescu

PP. 11199. We have

$$\sum \log_{b^uc^v} a = \sum \frac{x}{uy+vz} \ge \frac{3}{u+v},$$

where $x = \ln a$, $y = \ln b$, $z = \ln c$. (see: Inegalitati from L. Panaitopol, problem 2.23)

Traian Ianculescu

PP. 11200. We have

$$\sum \log_{a^u bc} a = \sum \frac{x}{ux+y+z} \geq \frac{3}{u+2}$$

where $x = \ln a$, $y = \ln b$, $z = \ln c$. (see: Inegalitati from L. Panaitopol, problem 2.25)

Traian Ianculescu

PP. 11201. From PP. 11199 ($v = 2$) and PP. 11200 we get

$$\sum \log_{b^u c^2} a \geq \frac{3}{u+2} \geq \sum \log_{a^u bc} a$$

Traian Ianculescu

PP. 11224. We have the relation

$$|z_1\overline{z_2}+1|^2 + |z_1-z_2|^2 = \left(1+|z_1|^2\right)\left(1+|z_2|^2\right)$$

for all $z_1, z_2 \in C$, therefore

$$\prod_{cyclic}\left(|z_1\overline{z_2}+1|^2 + |z_1-z_2|^2\right) = \prod_{cyclic}\left(1+|z_1|^2\right)\left(1+|z_2|^2\right) = \prod_{k=1}^{n}\left(1+|z_k|^2\right)^2$$

Traian Ianculescu

PP. 11225. We have the relation

$$|z_1\overline{z_2}+1|^2 + |z_1-z_2|^2 = \left(1+|z_1|^2\right)\left(1+|z_2|^2\right)$$

and

$$|\overline{z_1}z_2+1|^2 + |z_1-z_2|^2 = \left(1+|z_1|^2\right)\left(1+|z_2|^2\right)$$

After addition we get

$$|z_1\overline{z_2}+1|^2 + |\overline{z_1}z_2+1|^2 + 2\,|z_1-z_2|^2 = 2\left(1+|z_1|^2\right)\left(1+|z_2|^2\right)$$

Traian Ianculescu

PP. 11226. We have the relation

$$|z_1\overline{z_2}-1|^2-|z_1-z_2|^2=\left(1-|z_1|^2\right)\left(1-|z_2|^2\right)$$

and

$$|\overline{z_1}z_2-1|^2-|z_1-z_2|^2=\left(1-|z_1|^2\right)\left(1-|z_2|^2\right)$$

After addition we get

$$|z_1\overline{z_2}-1|^2+|\overline{z_1}z_2-1|^2-2\,|z_1-z_2|^2=2\left(1-|z_1|^2\right)\left(1-|z_2|^2\right)$$

Traian Ianculescu

PP. 11227. We have

$$\prod_{cyclic}\left(|z_1\overline{z_2}-1|^2-|z_1-z_2|^2\right)=\prod_{cyclic}\left(1-|z_1|^2\right)\left(1-|z_2|^2\right)=\prod_{k=1}^{n}\left(1-|z_k|^2\right)^2$$

Traian Ianculescu

PP. 11325. If $P_n=\prod\limits_{k=1}^{n}\left(1+\frac{1}{k^3}\right)$, then by induction holds

$$2\le P_n\le 3-\frac{1}{n}<3$$

therefore $[P_n]=2$ for all $n\in N^*$.

Traian Ianculescu

PP. 11338. We have $1\le\sum\limits_{k=1}^{n}\frac{1}{k^2}\le 2-\frac{1}{n}<2$, therefore $\left[\sum\limits_{k=1}^{n}\frac{1}{k^2}\right]=1$ for all $n\in N^*$.

Traian Ianculescu

PP. 11449. In $\frac{9}{x+2}\le 2s\sum\frac{1}{2s+(x-1)a}<\frac{x+5}{x+1}$, for $x=2$ we get

$$\frac{9}{4}\le 2s\sum\frac{1}{2a+b+c}<\frac{7}{3}\text{ or}$$

$$\frac{9}{4\sum a}\le\sum\frac{1}{2a+b+c}<\frac{7}{3\sum a}$$

If $a \to m_a, b \to m_b, c \to m_c$, then

$$\frac{9}{4\sum m_a} \le \sum \frac{1}{2m_a + m_b + m_c} < \frac{7}{3\sum m_a}$$

Traian Ianculescu

PP. 11456. If $f(x) = x + e^{-\alpha x}$ then the equation $f'(x) = 1 - \alpha e^{-\alpha x} = 0$ have the solution $x = \frac{\ln\alpha}{\alpha}$ which is global minium point, so $f(x) \ge f\left(\frac{\ln\alpha}{\alpha}\right)$ or $x + e^{-\alpha x} \ge \frac{1+\ln\alpha}{\alpha}$ for all $x \in R$ and $\alpha > 0$. If $x = \binom{n}{k}$, then

$$\sum_{k=0}^{n} e^{-\alpha\binom{n}{k}} \ge \frac{(n+1)(1+\ln\alpha)}{\alpha} - \sum_{k=0}^{n}\binom{n}{k} = \frac{(n+1)(1+\ln\alpha)}{\alpha} - 2^n$$

Traian Ianculescu

PP. 11457. Using the previous inequality we get

$e^{-\alpha\sin^2 x} + e^{-\alpha\cos^2 x} \ge \left(\frac{1+\ln\alpha}{\alpha} - \sin^2 x\right) + \left(\frac{1+\ln\alpha}{\alpha} - \cos^2 x\right) = \frac{2(1+\ln\alpha)}{\alpha} - 1$ but $e^{-\alpha} \ge \frac{1+\ln\alpha}{\alpha} - 1$ so

$$e^{-\alpha\sin^2 x} + e^{-\alpha\cos^2 x} \ge \frac{3}{\alpha}(1+\ln\alpha) - 2 - e^{-\alpha}$$

Traian Ianculescu

PP. 11459. We have

$$\sum_{k=1}^{n} e^{-k\alpha} \ge \frac{n(1+\ln\alpha)}{\alpha} - \sum_{k=1}^{n} k = n\left(\frac{1+\ln\alpha}{\alpha} - \frac{n+1}{2}\right)$$

Traian Ianculescu

PP. 11461. We have

$$\sum\left(A + e^{-\alpha A}\right) \ge \sum \frac{1+\ln\alpha}{\alpha} \text{ or } \pi + \sum \frac{1}{e^{\alpha A}} \ge \frac{3(1+\ln\alpha)}{\alpha}$$

Traian Ianculescu

PP. 11469. If in $x + e^{-\alpha x} \ge \frac{1+\ln\alpha}{\alpha}$ we take $x = \ln k$ then we obtain

$$\sum_{k=1}^{n}\left(\ln k + \left(\frac{1}{k}\right)^{\alpha}\right) \ge \sum_{k=1}^{n}\frac{1+\ln\alpha}{\alpha} \text{ or}$$

$$\sum_{k=1}^{n}\left(\frac{1}{k}\right)^{\alpha} \geq \frac{n}{\alpha}+\ln\left(\frac{\alpha^{\frac{n}{\alpha}}}{n!}\right)$$

Traian Ianculescu

PP. 11470. We have

$$\left(\ln x+\frac{1}{x^{\alpha}}\right)\left(\ln y+\frac{1}{y^{\alpha}}\right)+\left(\ln xy+\frac{1}{(xy)^{\alpha}}\right) \geq \frac{3(1+\ln\alpha)}{\alpha} \text{ or}$$

$$2\ln xy+\frac{x^{\alpha}+1+y^{\alpha}}{(xy)^{\alpha}} \geq \frac{3(1+\ln\alpha)}{\alpha}$$

Traian Ianculescu

PP. 11471. If we take $x=1$ and $\alpha=\pi$ and $x=\pi$, $\alpha=e$ then we obtain $1+e^{-\pi} \geq \frac{1+\ln\pi}{\pi}$ and $\ln\pi+\pi^{-\alpha} \geq \frac{2}{e}$, therefore

$$\left(1+e^{-\pi}\right)\left(\ln\pi+\pi^{-e}\right) \geq \frac{2(1+\ln\pi)}{\pi e}$$

Traian Ianculescu

PP. 11639. The authors J. Sándor and M. Bencze in "On a Problem of William Lowell Putman Competition" (Octogon Mathematical Magazine, Vol. 14, No. 1, April) have proved the inequalities

$$\frac{2a+2}{2a+1} < e\left(1+\frac{1}{a}\right)^{-a} < \sqrt{1+\frac{1}{a}} < \frac{2a+1}{2a}$$

for all $a>0$.
If $a=k$ then we get $e\sqrt{\frac{k}{k+1}} < \left(\frac{k+1}{k}\right)^{k} < e$ for all $k \in \{1,2,...,n\}$, therefore

$$\prod_{k=1}^{n} e\sqrt{\frac{k}{k+1}} = \frac{e^{n}}{\sqrt{n+1}} < \prod_{k=1}^{n}\left(\frac{k+1}{k}\right)^{k} = \frac{(n+1)^{n}}{n!} < \prod_{k=1}^{n} e = e^{n}$$

and finally

$$1 \leq \left(\frac{e}{n+1}\right)^{n} n! \leq \sqrt{n+1}$$

Traian Ianculescu

PP. 11658. If $f(x)=\ln\frac{1}{1-x^2}, g(x)=x^3$, then from Cauchy theorem we obtain that exist $c\in(a,b)$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(c)}{g'(c)}=\frac{2}{3c(1-c^2)}\text{ but}$$

$c\left(1-c^2\right)\leq\frac{2}{3\sqrt{3}}$ so $\frac{f(b)-f(a)}{g(b)-g(a)}\geq\sqrt{3}$ or $\ln\left(\frac{1-a^2}{1-b^2}\right)\geq\sqrt{3}\left(b^3-a^3\right)$ and if $a=\frac{1}{\sqrt[3]{k(k+1)(k+2)}}$ and $b=\frac{1}{\sqrt[3]{k(k+1)}}$, then we get:

$$\sum_{k=1}^{n}\ln\left(\frac{\sqrt[3]{k^2(k+1)^2(k+2)^2}-1}{\sqrt[3]{k^2(k+1)^2(k+2)^2}-\sqrt[3]{(k+2)^2}}\right)=\sum_{k=1}^{n}\ln\left(\frac{1-a^2}{1-b^2}\right)\geq$$

$$\geq\sqrt{3}\sum_{k=1}^{n}\left(b^3-a^3\right)=\sqrt{3}\sum_{k=1}^{n}\frac{1}{k(k+2)}=\frac{n(3n+5)\sqrt{3}}{4(n+1)(n+2)}$$

Traian Ianculescu

PP. 11665. If $f(x)=\ln\frac{1}{1-x^2}, g(x)=x^5;\ a,b\in(0,1),\ b\geq a$, then from Cauchy theorem we get $\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(c)}{g'(c)}$, where $c\in(a,b)$ and finally

$$\ln\left(\frac{1-a^4}{1-b^4}\right)\geq\sqrt[4]{5}\left(b^5-a^5\right)$$

Traian Ianculescu

PP. 11666. If $a=\frac{1}{\sqrt[5]{k(k+1)(k+2)}}$, $b=\frac{1}{\sqrt[5]{k(k+1)}}$, then

$$\sum_{k=1}^{n}\ln\left(\frac{\sqrt[5]{k^4(k+1)^4(k+2)^4}-1}{\sqrt[5]{k^4(k+1)^4(k+2)^4}-\sqrt[5]{(k+2)^4}}\right)=\sum_{k=1}^{n}\ln\left(\frac{1-a^4}{1-b^4}\right)\geq$$

$$\geq\sqrt[4]{5}\sum_{k=1}^{n}\left(b^5-a^5\right)=\sqrt[4]{5}\sum_{k=1}^{n}\frac{1}{k(k+2)}=\frac{n(3n+5)\sqrt[4]{5}}{4(n+1)(n+2)}$$

Traian Ianculescu

PP. 11667. The function $f(x)=x\left(1-x^4\right), x\in(0,1)$ have a maximum point in $x=\frac{1}{\sqrt[4]{5}}$, therefore $\frac{1}{x(1-x^4)}\geq\frac{5\sqrt[4]{5}}{4}$ for all $x\in(0,1)$.
If $a,b,c\in(0,1)$ and $\sum a^{4n+k}=1$, then

$$\sum \frac{a^k}{\left(1-a^4\right)^{4n}} = \sum \frac{a^{4n+k}}{\left(a\left(1-a^4\right)\right)^{4n}} \geq \left(\frac{5\sqrt[4]{5}}{4}\right)^{4n} \sum a^{4n+k} = \left(\frac{3125}{256}\right)^n$$

Traian Ianculescu

PP. 11668. If $a, b, c \in (0,1)$, then

$$\sum \left(\frac{bc}{1-a^4}\right)^{4n} = \sum \left(\frac{1}{a\left(1-a^4\right)}\right)^{4n} (abc)^{4n} \geq 3\,(abc)^{4n} \left(\frac{5\sqrt[4]{5}}{4}\right)^{4n} =$$

$$= 3\left(\frac{3125}{256}\right)^n (abc)^{4n}$$

Traian Ianculescu

PP. 11734. The function $f(x) = x\left(1-x^2\right), x \in (0,1)$ have a maximum point in $x = \frac{1}{\sqrt{3}}$, therefore $\frac{1}{x(1-x^2)} \geq \frac{3\sqrt{3}}{2}$ for all $x \in (0,1)$, therefore

$$\sum \frac{\sin A}{x\left(1-x^2\right)} \geq \frac{3\sqrt{3}}{2} \sum \sin A = \frac{3\sqrt{3}s}{2R}$$

Traian Ianculescu

PP. 11765. Using the inequality $\sin x \geq \frac{x}{\sqrt{1+x^2}}$ for all $x \in \left[0, \frac{\pi}{2}\right]$ we get for $x = \frac{1}{\sqrt{k^2(k+1)^2-1}}$ $(k = 1, 2, ..., n)$ the following:

$$\sum_{k=1}^{n} \sin \frac{1}{\sqrt{k^2\left(k+1\right)^2-1}} \geq \sum_{k=1}^{n} \frac{1}{k\left(k+1\right)} = \frac{n}{n+1}.$$

Traian Ianculescu

PP. 11771. Because $e^x \geq 1 + x^2 \geq \left(\frac{1+x}{2}\right)^2$ for $x = 2k-1$ we get $e^{2k-1} \geq 2k^2 \geq k\left(k+1\right)$ and $e^{1-2k} \leq \frac{1}{k(k+1)}$ $(k = 1, 2, ..., n)$, therefore

$$\sum_{k=1}^{n} e^{1-2k} \leq \frac{n}{n+1}$$

Traian Ianculescu

PP. 11772. From inequality $\ln\left(1+x^2\right)\leq xarctgx$ we get

$$x\ln\left(1+x^2\right)\leq arctg\frac{1}{x}\text{ or }tg\left(x\ln\left(1+\frac{1}{x^2}\right)\right)\leq\frac{1}{x.}$$

If $x=k\left(k+1\right)\left(k=1,2,...,n\right)$, then

$$\sum_{k=1}^{n}tg\left(k\left(k+1\right)\ln\left(1+\frac{1}{k^2\left(k+1\right)^2}\right)\right)\leq\sum_{k=1}^{n}\frac{1}{k\left(k+1\right)}=\frac{n}{n+1}$$

Traian Ianculescu

PP. 11783. If $x=\sqrt{e^x-1}$, then from inequality $\ln\left(1+x^2\right)\leq xarctgx$ we get

$$\sqrt{e^x-1}arctg\sqrt{e^x-1}\geq x$$

and for $x=\frac{1}{k(k+1)}\left(k=1,2,...,n\right)$ we obtain:

$$\sum_{k=1}^{n}\sqrt{e^{\frac{1}{k(k+1)}}-1}\cdot arctg\sqrt{e^{\frac{1}{k(k+1)}}-1}\geq\sum_{k=1}^{n}\frac{1}{k\left(k+1\right)}=\frac{n}{n+1}$$

Traian Ianculescu

PP. 11791 and PP. 11792. We have the inequality

$$\sum_{k=1}^{n}\ln\left(\frac{1+\sqrt{k^2\left(k+1\right)^2+1}}{k\left(k+1\right)}\right)\leq\frac{n}{n+1}\leq\sum_{k=1}^{n}\ln\left(\frac{k^2+k}{k^2+k-1}\right)$$

(Problem 257, Gazeta matematica seria A, Nr. 1/2008, author M. Bencze)
We starting from $\ln\left(x+\sqrt{1+x^2}\right)\leq x\leq\ln\frac{1}{1-x}$ in which we take $x=\frac{1}{k(k+1)}$ $\left(k=1,2,...,n\right)$ therefore

$$\sum_{k=1}^{n}\ln\left(\frac{1+\sqrt{k^2\left(k+1\right)^2+1}}{k\left(k+1\right)}\right)\leq\sum_{k=1}^{n}\frac{1}{k\left(k+1\right)}=\frac{n}{n+1}\leq\sum_{k=1}^{n}\ln\left(\frac{k^2+k}{k^2+k-1}\right)$$

Traian Ianculescu

PP. 11871. By induction we get

$$(k+1)\left((k+1)!\right)^{\frac{1}{k+1}} - k\left(k!\right)^{\frac{1}{k}} \leq k+1,$$

therefore

$$\sum_{k=1}^{n} (k+1)\left((k+1)!\right)^{\frac{1}{k+1}} - k\left(k!\right)^{\frac{1}{k}} \leq \sum_{k=1}^{n} (k+1) = \frac{n\left(n+3\right)}{2}$$

Traian Ianculescu

PP. 11892. By induction we get

$$\sin\frac{1}{(k+1)(k+2)} \geq (k+1)\sin\frac{1}{k+2} - k\sin\frac{1}{k+1} \text{ or}$$

$$\arcsin\left((k+1)\sin\frac{1}{k+2} - k\sin\frac{1}{k+1}\right) \leq \frac{1}{(k+1)(k+2)}$$

$(k = 1, 2, ..., n)$ therefore

$$\sum_{k=1}^{n} \arcsin\left((k+1)\sin\frac{1}{k+2} - k\sin\frac{1}{k+1}\right) \leq \sum_{k=1}^{n} \frac{1}{(k+1)(k+2)} = \frac{n}{2(n+2)}$$

Traian Ianculescu

PP. 12337. If $x \in (0,1)$ then $\frac{1}{x(1-x^2)} \geq \frac{3\sqrt{3}}{2}$, and if $\sum x^{2n+k} = 1$, then

$$\sum \frac{x^k}{(1-x^2)^{2n}} = \sum \frac{x^{2n+k}}{(x(1-x^2))^{2n}} \geq \left(\frac{3\sqrt{3}}{2}\right)^{2n} \sum x^{2n+k} = \left(\frac{27}{4}\right)^n$$

Traian Ianculescu

PP. 12341. We have

$$\sum \left(\frac{yz}{1-x^2}\right)^{2n} = \sum \frac{(xyz)^{2n}}{(x(1-x^2))^{2n}} \geq \frac{3^{3n+1}}{4^n}(xyz)^{2n}$$

Traian Ianculescu

PP. 12342. If in PP. 12337 we take $k=0$ then we obtain

$$\sum \frac{1}{\left(1-x^2\right)^{2n}} \geq \left(\frac{27}{4}\right)^n$$

Traian Ianculescu

PP. 12395.

1). If $x=\frac{r}{h_a}, x=\frac{r}{h_b}, z=\frac{r}{h_c}, t=\frac{r}{h_d}$, then $\sum \frac{r}{h_a}=1$ and

$$\sum \frac{h_b h_c}{h_a\left(h_b h_c+2h_a h_c+3h_a h_b\right)}=\frac{1}{r}\sum \frac{x^2}{x+2y+3z} \geq$$

$$\geq \frac{1}{r}\frac{\left(\sum x\right)^2}{\sum (x+2y+3z)}=\frac{1}{6r}\sum x=\frac{1}{6r}$$

2). If $x=\frac{r}{r_a}, x=\frac{r}{r_b}, z=\frac{r}{r_c}, t=\frac{r}{r_d}$, $\sum x=2$ and

$$\sum \frac{r_b r_c}{r_a\left(r_b r_c+2r_a r_c+3r_a r_b\right)}=\frac{1}{r}\sum \frac{x^2}{x+2y+3z} \geq \frac{1}{6r}\sum x=\frac{1}{3r}$$

Traian Ianculescu

PP. 12409. If $x=\frac{r}{h_a}, x=\frac{r}{h_b}, z=\frac{r}{h_c}, t=\frac{r}{h_d}$, then

$$\sum \frac{h_a^2}{h_a^2-r^2}=\sum \frac{1}{1-x^2} \geq \frac{3\sqrt{2}}{2}\sum x=\frac{3\sqrt{3}}{2}$$

If $x=\frac{r}{2r_a}, x=\frac{r}{2r_b}, z=\frac{r}{2r_c}, t=\frac{r}{2r_d}$, then

$$\sum \frac{r_a^2}{4r_a^2-r^2}=\sum \frac{1}{1-x^2} \geq \frac{3\sqrt{2}}{2}\sum x=\frac{3\sqrt{3}}{8}$$

Traian Ianculescu

PP. 12410. If $x=\frac{r}{h_a}, x=\frac{r}{h_b}, z=\frac{r}{h_c}, t=\frac{r}{h_d}$, then

$$\sum \frac{h_a^4}{h_a^4-r^4}=\sum \frac{1}{1-x^4} \geq \frac{5\sqrt[4]{5}}{4}\sum x=\frac{5\sqrt[4]{5}}{4}$$

If $x=\frac{r}{2r_a}, x=\frac{r}{2r_b}, z=\frac{r}{2r_c}, t=\frac{r}{2r_d}$, then

$$\sum \frac{r_a^4}{16r_a^4 - r^4} = \sum \frac{1}{1-x^4} \geq \frac{5\sqrt[4]{5}}{4} \sum x = \frac{5\sqrt[4]{5}}{64}$$

Traian Ianculescu

PP. 12411. If $x \in (0,1)$, then

$$\frac{1}{1-x^2} \geq \frac{3\sqrt{3}}{2}x \text{ and } \frac{1}{1-x^4} \geq \frac{5\sqrt[4]{5}}{4}x$$

therefore

$$\frac{1}{\left(1-x^2\right)^2\left(1+x^2\right)} \geq \frac{15\sqrt[4]{45}}{8}x^2$$

If $\sum x^2 = 1$, then

$$\sum \frac{1}{\left(1-x^2\right)^2\left(1+x^2\right)} \geq \frac{15\sqrt[4]{45}}{8} \sum x^2 = \frac{15\sqrt[4]{45}}{8}$$

Traian Ianculescu

PP. 12413. The function $f:(0,1) \to R$, where $f(x) = \frac{x}{2-x}$ is convex,

therefore from Jensen‘s inequality we get for $x = \frac{r}{h_a}, x = \frac{r}{h_b}, z = \frac{r}{h_c}, t = \frac{r}{h_d}$, the following

$$\sum \frac{1}{2h_a - r} = \frac{1}{r} \sum \frac{x}{2-x} \geq \frac{4}{r} f\left(\frac{1}{4}\sum x\right) = \frac{4}{7r}$$

If $x = \frac{r}{2r_a}, x = \frac{r}{2r_b}, z = \frac{r}{2r_c}, t = \frac{r}{2r_d}$, then in same way we get

$$\sum \frac{1}{4r_a - r} \geq \frac{4}{7r}$$

Traian Ianculescu

PP. 12442. If $x = \frac{r}{h_a}, x = \frac{r}{h_b}, z = \frac{r}{h_c}$ and after then $x = \frac{r}{r_a}, x = \frac{r}{r_b}, z = \frac{r}{r_c}$, then we get

$$\sum \frac{h_b}{h_a\left(h_a + h_b\right)} \geq \frac{1}{2r} \geq \sum \frac{h_a}{h_a^2 + h_b^2}$$

and

$$\sum \frac{r_a}{r_a\left(r_a+r_b\right)} \geq \frac{1}{2r} \geq \sum \frac{r_a}{r_a^2+r_b^2}$$

Traian Ianculescu

PP. 12443. If $x=\frac{r}{h_a}, x=\frac{r}{h_b}, z=\frac{r}{h_c}, t=\frac{r}{h_d}$, then

$$\sum \frac{h_b}{h_a\left(h_a+h_b\right)}=\frac{1}{r}\sum \frac{x^2}{x+y} \geq \frac{\left(\sum x\right)^2}{r\sum (x+y)}=\frac{1}{2r}$$

and

$$\sum \frac{h_a}{h_a^2+h_b^2}=\frac{1}{r}\sum \frac{xy^2}{x^2+y^2} \leq \frac{1}{2r}\sum y=\frac{1}{2r}$$

In same manner we obtain

$$\sum \frac{r_a}{r_a\left(r_a+r_b\right)} \geq \frac{1}{r} \geq \sum \frac{r_a}{r_a^2+r_b^2}$$

Traian Ianculescu

PP. 12445. If $x=\frac{r}{h_a}, x=\frac{r}{h_b}, z=\frac{r}{h_c}$ and after then $x=\frac{r}{r_a}, x=\frac{r}{r_b}, z=\frac{r}{r_c}$, then $\sum x=1$ and

$$\sum \frac{h_b h_c}{h_a\left(h_b h_c+2h_a h_c+3h_a h_b\right)}=\frac{1}{r}\sum \frac{x^2}{x+2y+3z} \geq \frac{1}{6r}$$

and

$$\sum \frac{r_b r_c}{r_a\left(r_b r_c+2r_a r_c+3r_a r_b\right)}=\frac{1}{r}\sum \frac{x^2}{x+2y+3z} \geq \frac{1}{6r}$$

Traian Ianculescu

PP. 12577. From $\left(1+\frac{1}{k}\right)^k<e<\left(1+\frac{1}{k}\right)^{k+1}$ we get $\frac{(k+1)^k}{k!}<e^k<\frac{(k+1)^{k+1}}{k!}$ $(k=1,2,...,n)$, therefore

$$1<e^{\frac{n(n+1)}{2}}\prod_{k=1}^{n}\frac{k!}{(k+1)^k}<(n+1)!$$

Traian Ianculescu

PP. 12611. We have $\frac{a_1}{a_1^2+a_2^2} \leq \frac{1}{2a_2}$, therefore

$$\sum \frac{a_1}{a_1^2+a_2^2} \leq \frac{1}{2} \sum \frac{1}{a_2} = \frac{1}{2} \sum_{k=1}^{n} \frac{1}{a_k}$$

If $a_k = \frac{1}{b_k}$ $(k = 1, 2, ..., n)$, then

$$\sum \frac{a_2}{a_1\left(a_1+a_2\right)} = \sum \frac{b_1^2}{b_1+b_2} \geq \frac{\left(\sum b_1\right)^2}{\sum\left(b_1+b_2\right)} = \frac{1}{2} \sum b_1 = \frac{1}{2} \sum_{k=1}^{n} \frac{1}{a_k}$$

Traian Ianculescu

Open questions

OQ. 3154. If $x_k \in [0,1]$ $(k = 1, 2, ..., n)$, $y_k \in [-1, 1]$ $(k = 1, 2, ..., n)$ such that $\sum\limits_{k=1}^{n} \left(1 + y_k \prod\limits_{i=1}^{n} x_i\right)^m \geq 1$.

Mihály Bencze

OQ. 3155. If $\alpha, x_k > 0$ $(k = 1, 2, ..., n)$ then

$\prod\limits_{k=1}^{n} \left(x_k^2 + \alpha\right) \geq \frac{(\alpha+1)^n}{n} \left(\sum\limits_{cyclic} x_1 x_2\right)$.

Mihály Bencze

OQ. 3156. If $x_k, a_k > 0$ $(k = 1, 2, ..., n)$ then $\frac{\sqrt{a_1x_1x_2+a_2x_2x_3+...+a_nx_nx_1}}{x_2+...+x_n}+$

$$+\frac{\sqrt{a_1x_2x_3+a_2x_3x_4+...+a_nx_1x_2}}{x_3+...+x_1}+...+\frac{\sqrt{a_1x_nx_1+a_2x_1x_2+...+a_nx_{n-1}x_n}}{x_1+...+x_{n-1}} \geq \frac{n\sqrt{a_1+a_2+...+a_n}}{n-1}.$$

Mihály Bencze

OQ. 3157. If $x_k > 0$ $(k = 1, 2, ..., n)$, then

$$\sqrt{\left(\sum_{cyclic} x_1^{n-1} x_2\right)\left(\sum_{cyclic} x_1 x_2^{n-1}\right)} \geq (n-2)\prod_{k=1}^{n} x_k + \sqrt[n]{\prod_{cyclic}\left(x_1^n + x_1 x_2 ... x_n\right)}.$$

Mihály Bencze and Zhao Changjian

OQ. 3158. If $x_i > 0$ $(i = 1, 2, ..., n)$, then

$\sum\limits_{cyclic} \frac{x_1 x_2 ... x_k}{x_{k+1}(x_1+x_{k+1})(x_2+x_{k+1})...(x_{k-1}+x_{k+1})} \geq \sum\limits_{cyclic} \left(\frac{x_1}{x_2+x_3}\right)^{k-1}$, for all $k \in \{2, 3, ..., n-1\}$.

Mihály Bencze and Shanhe Wu

OQ. 3159. If $x_i > 0$ $(i = 1, 2, ..., n)$, then

$\sum\limits_{cyclic} \sqrt[k]{x_1^k + \frac{1}{x_2+...+x_{k+1}}} \geq \sqrt[k]{1 + \frac{n^{k+1}}{k}}$.

Mihály Bencze and Yu-Dong Wu

OQ. 3160. If $x_i > 0$ $(i = 1, 2, ..., n)$, then determine all $\alpha > 0$ such that the inequality $\sum\limits_{1 \le i_1 < ... < i_k \le n} x_{i_1} x_{i_2} ... x_{i_k} \le \alpha + \left(\binom{n}{k} - \alpha n^k\right) \prod\limits_{k=1}^{n} x_k$ is the best possible.

Mihály Bencze

OQ. 3161. 1). If $x, y > 0$ then determine all $a, b > 0$ such that

$\frac{1}{(x+y)^a} + \frac{1}{(x+2y)^a} + ... + \frac{1}{(x+ny)^a} \le \frac{n}{(x(x+ny))^b}$.

2). If $x > 0$, then determine all $a, b, c > 0$ such that
$\sum\limits_{k=1}^{n} \frac{1}{(1+kx)^a} \le n\,(1 + bnx)^{-c}$.

Mihály Bencze

OQ. 3162. In all triangle ABC denote N the Nagel point, H is orthocentre, O is circumcentre, I is incentre. Determine all $x, y > 0$ such that
$\max\left\{(NI)^x (HI)^y ; (NI)^y (HI)^x\right\} \le (x + y)(OI)^{x+y}$.

Mihály Bencze

OQ. 3163. If $x_k \in (-1, 1)$ $(k = 1, 2, ..., n)$, then
1). $\frac{1}{\prod\limits_{k=1}^{n}(1-x_k)} + \frac{1}{\prod\limits_{k=1}^{n}(1+x_k)} \ge 2$
2). If $y_k, z_k \in [-1, 1]$ $(k = 1, 2, ..., n)$ such that $\sum\limits_{k=1}^{n} y_k = \sum\limits_{k=1}^{n} z_k = 0$, then
$\frac{1}{\prod\limits_{k=1}^{n}(1+x_k y_k)} + \frac{1}{\prod\limits_{k=1}^{n}(1+x_k z_k)} \ge 2$.

Mihály Bencze

OQ. 3164. If $x_i > 0$ $(i = 1, 2, ..., n)$ and $S_k = \sum\limits_{1 \le i_1 < ... < i_k \le n} x_{i_1} x_{i_2} ... x_{i_k}$, then
$\frac{S_1}{\binom{n}{1}} \ge \left(\frac{S_2}{\binom{n}{2}}\right)^{\frac{1}{2}} \ge \left(\frac{S_3}{\binom{n}{3}}\right)^{\frac{1}{3}} \ge ... \ge \left(\frac{S_n}{\binom{n}{n}}\right)^{\frac{1}{n}}$.

Mihály Bencze

OQ. 3165. In all triangle ABC hold $\left(\frac{9R}{2}\right)^2 \le \left(\sum \frac{a^2b^2}{c^2}\right)\left(\sum \frac{r_a^2}{a^2}\right) \le \left(\frac{9R^2}{4r}\right)^2$.

Mihály Bencze

OQ. 3166. If $x_k > 0$ $(k = 1, 2, ..., n)$, then

$$\left(\sum_{k=1}^{n} x_k\right)^2 \left(\sum_{cyclic} x_1x_2\right)^{n-1} \leq \frac{n^{n+1}}{3^n} \prod_{cyclic} \left(x_1^2 + x_1x_2 + x_2^2\right).$$

Mihály Bencze

OQ. 3167. If $x_k > 0$ $(k = 1, 2, ..., n)$ and $S = \sum\limits_{k=1}^{n} x_k$, then

1). $\sum\limits_{k=1}^{n} \frac{x_k}{S-x_k} \leq \frac{n\sum\limits_{k=1}^{n} x_k^2}{(n-1)\sum\limits_{cyclic} x_1x_2}$ 2). $\sum\limits_{k=1}^{n} \left(\frac{x_k}{S-x_k}\right)^2 \geq \frac{n\sum\limits_{k=1}^{n} x_k^2}{(n-1)^2\sum\limits_{cyclic} x_1x_2}$

Mihály Bencze

OQ. 3168. If $a_, x_k > 0$ $(k = 1, 2, ..., n)$ and

$A_1 = \frac{a_1x_1+a_2x_2+...+a_nx_n}{a_nx_1+a_1x_2+...+a_{n-1}x_n}$,
$A_2 = \frac{a_2x_1+a_3x_2+...+a_1x_n}{a_1x_1+a_2x_2+...+a_nx_n}, ..., A_n = \frac{a_nx_1+a_1x_2+...+a_{n-1}x_n}{a_{n-1}x_1+a_nx_2+...+a_{n-2}x_n}$, then
$\max\{A_1, A_2, ..., A_n\} \geq 1 \geq \min\{A_1, A_2, ..., A_n\}$.

Mihály Bencze

OQ. 3169. If $x_k \in R$ $(k = 1, 2, ..., n)$ and $A \subseteq \{1, 2, ..., n\}$, then

$$\left(\sum_{i\in A} x_i\right)^k \leq \sum_{1\leq i_1<...<i_k\leq n} (x_{i_1} + ... + x_{i_2}) ... (x_{i_1} + ... + x_{i_k}).$$

Mihály Bencze

OQ. 3170. If $x_k > 0$ $(k = 1, 2, ..., n)$ then

$$\sum_{cyclic} x_1 (x_1 - x_2) ... (x_1 - x_n) \geq \frac{(n+1)\left|(x_1-x_2)^{n-1}(x_2-x_3)^{n-1}...(x_n-x_1)^{n-1}\right|}{\prod\limits_{cyclic}(x_1+x_2)\left(1-\left|\prod\limits_{cyclic}\frac{x_1-x_2}{x_1+x_2}\right|^{n-1}\right)}.$$

Mihály Bencze

OQ. 3171. If $x_k, p_k > 0$ $(k = 1, 2, ..., n)$ and
$f(p_{i_1}, ..., p_{i_k}; x_{i_1}, ..., x_{i_k}) = \frac{p_{i_1}x_{i_1}+...+p_{i_k}x_{i_k}}{p_{i_1}+...+p_{i_k}}$, then

$$\sum_{k=1}^{n} (-1)^{k-1} \sum_{1\leq i_1<...<i_k\leq n} \ln f(p_{i_1}, ..., p_{i_k}; x_{i_1}, ..., x_{i_k}) \leq 0.$$

Mihály Bencze

OQ. 3172. If $x_k > 0$ $(k = 1, 2, ..., n)$ and $\prod\limits_{k=1}^{n} x_k = 1$, then

$\sum\limits_{k=1}^{n} \sqrt{1 + (a^2 - 1)\, x_k^2} \leq a \sum\limits_{k=1}^{n} x_k$ for all $a \geq 1$.

Mihály Bencze

OQ. 3173. If $x_k > 0$ $(k = 1, 2, ..., n)$ then
$2 \sum\limits_{cyclic} x_1 x_2^{n-1} \geq n \prod\limits_{k=1}^{n} x_k + \sum\limits_{cyclic} x_1^{n-1} x_2.$

Mihály Bencze

OQ. 3174. Let $f : I \to R$ $(I \subseteq R)$ be a convex function, and $x_k \in I$ $(k = 1, 2, ..., n)$. Denote $S_k = \binom{n}{k}^{-1} \sum\limits_{1 \leq i_1 < ... < i_k \leq n} f\left(\frac{x_{i_1} + ... + x_{i_k}}{k}\right)$, then determine all $a_i \in R$ $(i = a, 1, ..., m)$ such that $\sum\limits_{i=0}^{m} (-1)^i\, a_i S_{i+1} \geq 0$.

Mihály Bencze

OQ. 3175. If $x_k > 0$ $(k = 1, 2, ..., n)$ then $\sum\limits_{cyclic} \frac{a x_1^2 + x_2 x_3}{(x_1 + x_2)^2} \geq \frac{n(a+1)}{4}$ for all $a > 0$.

Mihály Bencze

OQ. 3176. If $x_k > 0$ $(k = 1, 2, ..., n)$ and $\sum\limits_{k=1}^{n} x_k = 1$ then determine all $a_k > 0$ $(k = 1, 2, ..., n)$ such that, the inequalities
$\sum\limits_{k=1}^{n} x_k^2 \geq \sum\limits_{cyclic} x_1 \left(a_1 + a_2 x_2 + ... + a_n x_2^{n-1}\right) \geq x_1 x_2 + x_2 x_3 + ... + x_n x_1$ are the best possible.

Mihály Bencze

OQ. 3177. Determine all $y_k > 0$ $(k = 1, 2, ..., n)$ for which
$\sum\limits_{j=1}^{n} \frac{1}{\sum\limits_{i=1}^{n} x_j^{y_i}} \geq \frac{n}{\sum\limits_{i=1}^{n} (\sqrt[n]{x_1 x_2 ... x_n})^{y_i}}$ for all $x_k > 0$ $(k = 1, 2, ..., n)$.

Mihály Bencze

OQ. 3178. If $x_k \in [0,1]$ $(k = 1,2,...,n)$ then determine the minimum and the maximum of the expression $\frac{\prod\limits_{cyclic}(1+x_1+x_1x_2)}{\prod\limits_{cyclic}(1+x_2+x_1x_2)}$.

Mihály Bencze

OQ. 3179. If $x_k > 0$ $(k = 1,2,...,n)$ and $S = \sum\limits_{k=1}^{n} x_k$ then

$$\sum_{k=1}^{n} \frac{(s-x_k)}{x_k(s+x_k)} \geq \frac{(n-1)2^n\left(\sum\limits_{k=1}^{n} x_k\right)^{n-1}}{n(n+2)\prod\limits_{cyclic}(x_1+x_2)}.$$

Mihály Bencze

OQ. 3180. If $x_k > 0$ $(k = 1,2,...,n)$ and $S = \sum\limits_{k=1}^{n} x_k$, then

$$\sum_{k=1}^{n} \frac{x_k}{x_k^n+(s-x_k)^n} \geq \frac{\sum\limits_{k=1}^{n} x_k}{\sum\limits_{k=1}^{n} x_k^3+n!\sum\limits_{1\leq i<j<k\leq n} x_ix_jx_k}.$$

Mihály Bencze

OQ. 3181. If $x_k > 0$ $(k = 1,2,...,n)$ and $\sum\limits_{k=1}^{n} x_k = 1$ then

$$\sum_{cyclic} \frac{x_1}{\sqrt{1-x_2}} \geq \sqrt{2}\sum_{k=1}^{n}\sqrt{x_k} - \frac{n+2}{n+1}.$$

Mihály Bencze

OQ. 3182. If $x_k > 0$ $(k = 1,2,...,n)$ and $\alpha \geq 1$, then $\frac{\sqrt{n\sum\limits_{k=1}^{n} x_k^{2\alpha}}}{\sum\limits_{k=1}^{n} x_k^{\alpha-1}} \geq \frac{\sum\limits_{k=1}^{n} x_k^{\alpha+1}}{\sum\limits_{k=1}^{n} x_k^{\alpha}}$.

Mihály Bencze

OQ. 3183. If $x_k > 0$ $(k = 1,2,...,n)$ then determine all $a,b,c > 0$ such that

$$\left(\sum_{cyclic}\frac{x_1}{(x_1+ax_2)^b}\right)^{\frac{1}{b}} \geq \sum\left(\frac{x_1x_2\left(x_1^c+x_2^c\right)}{2}\right)^{\frac{1}{c+2}}.$$

Mihály Bencze

OQ. 3184. If $0 < \alpha \leq 1$ and $x_k > 0$ $(k = 1, 2, ..., n)$, $S = \sum\limits_{k=1}^{n} x_k$, then $\sum\limits_{k=1}^{n} \left(\frac{x_k}{S - x_k}\right)^{\alpha} \geq \frac{n}{(n-1)^{\alpha}}$.

Mihály Bencze

OQ. 3185. If $x_k > 0$ $(k = 1, 2, ..., n)$ then determine all $\alpha > 0$ such that $\sum\limits_{k=1}^{n} \frac{1}{x_k} \geq \sum\limits_{cyclic} \frac{x_1^2 - \alpha x_2^2}{x_1^3 + \alpha x_2^3}$.

Mihály Bencze

OQ. 3186. Determine all $A_k \in M_m(C)$ $(k = 1, 2, ..., n)$ such that $A_1^{n-1} = A_2 A_3 ... A_n$, $A_2^{n-1} = A_1 A_3 ... A_n$, ..., $A_n^{n-1} = A_1 A_2 ... A_{n-1}$ for all $n, m \geq 3$.

If n is odd then $A_1 = A, A_2 = \varepsilon A, A_3 = \varepsilon^2 A, ..., A_n = \varepsilon^{n-1} A$, where $\varepsilon = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}$ is a solution.

Mihály Bencze

OQ. 3187. Determine all $A, B \in M_n(C)$ for which $rang(AB) - rang(BA) = \left[\frac{n}{2}\right] - 1$, where $[\cdot]$ denote the integer part.

Mihály Bencze

OQ. 3188. Determine all $A, B \in M_n(R)$ for which $\det\left(A^{2k} + A^{2k-1}B + ... + AB^{2k-1} + B^{2k}\right) \geq 0$ for all $k \geq 1$. We have the following result, if $AB = BA$ then the affirmation is true.

Mihály Bencze

OQ. 3189. If $x_k > 0$ $(k = 1, 2, ..., n)$ then $\sum\limits_{cyclic} \frac{x_1}{x_2} + \alpha \frac{\sum\limits_{cyclic} x_1 x_2}{\left(\sum\limits_{k=1}^{n} x_k\right)^2} \geq n + \frac{\alpha}{n}$, where $\alpha > 0$.

Mihály Bencze

OQ. 3190. If $x_k > 0$ $(k = 1, 2, ..., n)$ then

$$\sum_{cyclic} \frac{x_1^{n-1}}{x_2^n + x_3^n + x_1 x_2 ... x_n} \geq \sqrt[n]{\frac{\sum\limits_{k=1}^{n} x_k^n}{\sum\limits_{cyclic} x_1^n x_2^n}}.$$

Mihály Bencze

OQ. 3191. If $x_k > 0$ $(k = 1, 2, ..., n)$ and $\sum\limits_{k=1}^{n} x_k = a$, $\sum\limits_{k=1}^{n} x_k^2 = b > 1$, then

$$1 + \frac{b-1}{\prod\limits_{k=1}^{n} x_k} \geq \frac{1}{a} \sum_{cyclic} \frac{x_1}{x_2}.$$

Mihály Bencze

OQ. 3192. If $x_k > 0$ $(k = 1, 2, ..., n)$ and $\sum\limits_{k=1}^{n} \sqrt{x_k} = 1$, then

$$\sum_{cyclic} \frac{x_1^r + x_2 x_3 ... x_{r+1}}{x_1 \sqrt[r]{x_2 + ... + x_{r+1}}} \geq \frac{2\sqrt[r]{n^2}}{n^{2r-1}\sqrt[r]{r}}, \text{ for all } r \in \{2, 3, ..., n-1\}.$$

Mihály Bencze

OQ. 3193. Determine all $A_k \in M_n(R)$ $(k = 1, 2, ..., n)$ such that

$$\det\left(\sum_{k=1}^{m} A_k^t A_k\right) = 0.$$

Mihály Bencze

OQ. 3194. If $x_k > 0$ $(k = 1, 2, ..., n)$ then

$$\min\left\{\frac{\sum\limits_{k=1}^{n} x_k^3}{\sum\limits_{cyclic} x_1^2 x_2}; \frac{\sum\limits_{k=1}^{n} x_k^3}{\sum\limits_{cyclic} x_1 x_2^2}\right\} + n - 1 \geq \sum_{cyclic} \frac{x_1 + x_2}{x_2 + x_3}.$$

Mihály Bencze

OQ. 3195. Let $A_1A_2...A_{2n}A_{2n+1}$ be a convex polygon and $B_1, B_2, ..., B_{2n+1}$ are on the sides $A_{n+1}A_{n+2}, A_{n+2}A_{n+3}, ..., A_{2n}A_{2n+1}, ..., A_nA_{n+1}$ such that $A_1B_1, A_2B_2, ..., A_{2n+1}B_{2n+1}$ are ceviens of rank p in triangles $A_{n+1}A_1A_{n+2}, A_{n+2}A_2A_{n+3}, ..., A_{2n+1}A_nA_1, ..., A_nA_{2n+1}A_{n+1}$.
Prove that if $A_1B_1, A_2B_2, ..., A_{2n}B_{2n}$ are concurent in point M, then $M \in A_{2n+1}B_{2n+1}$.

Mihály Bencze

OQ. 3196. If $-1 \leq x_k \leq 1$ $(k=1,2,...,n)$ then determine the best constants $m, M > 0$ such that $m \leq \sum_{cyclic} |f(x_1)+f(x_2)-2f(x_3)| \leq M$, when $f(x)=4x^3-3x+1$.

Mihály Bencze

OQ. 3197. Denote R_1, R_2, R_3 the distances from an arbitrary point M to the vertices A, B, C of the triangle ABC.

Prove $\left(aR_1^2+bR_2^2+cR_3^2\right)\left(bR_1^2+cR_2^2+aR_3^2\right)\left(cR_1^2+aR_2^2+bR_3^2\right) \geq$
$\geq \frac{abc\left(a^3b+b^3c+c^3a\right)\left(a^3c+b^3a+c^3b\right)}{(a+b+c)^2}$. Can be strongened this inequality?

Mihály Bencze

OQ. 3198. Determine all $x_k > 0$ $(k=1,2,...,n)$ for which from

$\sum_{k=1}^{n} x_k > \sum_{cyclic} \frac{x_1}{x_2}$ holds $\sum_{k=1}^{n} x_k < \sum_{cyclic} \frac{x_2}{x_1}$.

Mihály Bencze

OQ. 3199. If $x_k \in (0,1) \cup (1,+\infty)$ $(k=1,2,...,n)$, then determine all $a, b > 0$ such that $\sum_{cyclic} \frac{x_1^a}{(x_2-1)^{2b}} \geq 1$.

Mihály Bencze

OQ. 3200. If $x_k > 0$ $(k=1,2,...,n)$ then
$2\sum_{cyclic} \sqrt{x_1^2-x_1x_2+x_2^2} \geq \sum_{cyclic} \sqrt{x_1^2+x_1x_2+x_2^2}$.

Mihály Bencze

OQ. 3201. If $x_k \in R$ $(k=1,2,...,n)$ and $\sum_{k=1}^{n} x_k = a$, $\sum_{k=1}^{n} x_k^2 = b$, then determine the best $m_r, M_r \in R$ $(r=1,2)$ such that $m_1 \leq \sum_{cyclic} x_1^2x_2 \leq M_1$ and $m_2 \leq \sum_{cyclic} x_1x_2^2 \leq M_2$.

Mihály Bencze

OQ. 3202. If $x_k > 0$ $(k = 1, 2, ..., n)$ then determine the best $m, M > 0$ such that $m \leq \sum\limits_{cyclic} \frac{x_1}{\sqrt[r]{x_1^r + (a^r - 1) x_2 x_3 ... x_r x_{r+1}}} \leq M$ where $r \in \{2, 3, ..., n-1\}$ and $a \geq 2$.

Mihály Bencze

OQ. 3203. If $x_k > 0$ $(k = 1, 2, ..., n)$ and $\sum\limits_{k=1}^{n} x_k = 1$, then determine $\max \sum\limits_{cyclic} \frac{(x_1 x_2 ... x_p)^{\alpha}}{1 - (x_2 x_3 ... x_{p+1})^{\beta}}$, where $\alpha, \beta > 0$ and $p \in \{2, 3, ..., n-1\}$.

Mihály Bencze

OQ. 3204. If $x_k > 0$ $(k = 1, 2, ..., n)$ then

$$\left(\sum_{cyclic} x_1 x_2 ... x_p \right) \left(\sum_{cyclic} \frac{x_1}{x_2^p + x_3} \right) \geq \frac{n}{n^{p-1} + 1}, \text{ for all } p \in \{2, 3, ..., n-1\}.$$

Mihály Bencze

OQ. 3205. Let ABC be a triangle, then determine $\max \sum\limits_{cyclic} (\sin A)^b (\sin B)^c (\sin C)^a$.

Mihály Bencze

OQ. 3206. If $x_k > 0$ $(k = 1, 2, ..., n)$, $\sum\limits_{k=1}^{n} x_k = 1$, then determine the maximal constant $\alpha > 0$ such that $\sum\limits_{cyclic} \sqrt{x_1 + \alpha (x_2 - x_3)^2} + 2 \sum\limits_{k=1}^{n} \sqrt{x_k} \leq 3n$.

Mihály Bencze

OQ. 3207. If $x_k > 0$ $(k = 1, 2, ..., n)$ and $\sum\limits_{k=1}^{n} x_k^2 = 1$, then $1 \leq \sum\limits_{cyclic} \frac{x_1}{1 + x_2 x_3 ... x_n} \leq \frac{(\sqrt{n})^n}{(\sqrt{n})^{n-1} + 1}$.

Mihály Bencze

OQ. 3208. If $x_k > 0$ $(k = 1, 2, ..., n)$ and $\sum\limits_{k=1}^{n} x_k = 1$, then
$\prod\limits_{k=1}^{n} x_k + \frac{\alpha}{\sum\limits_{cyclic} x_1 x_2 ... x_p} \geq 1 + \frac{\alpha}{n}$ for all $\alpha > 0$ and all $p \in \{2, 3, ..., n-1\}$.

Mihály Bencze

OQ. 3209. If $x_k > 0$ $(k = 1, 2, ..., n)$ and $S = \sum\limits_{k=1}^{n} x_k$, then $\sum\limits_{k=1}^{n} x_k^{S - x_k} \geq 1$.

Mihály Bencze

OQ. 3210. If $x_k > 0$ $(k = 1, 2, ..., n)$ then
$\sum\limits_{cyclic} \frac{x_1}{x_2} \geq \frac{n^2}{n+1} \sqrt[3]{\frac{\sum\limits_{k=1}^{n} x_k^3}{\sum\limits_{cyclic} x_1 x_2 x_3} + \frac{3n^2+3n+1}{n^3}}$.

Mihály Bencze

OQ. 3211. If $f : I \to R$ $(I \subseteq R)$ is a convex function, $x_k \in I$ $(k = 1, 2, ..., n)$ such that $x_1 \leq x_2 \leq ... \leq x_n$, then determine all $y_k \in R$ $(k = 1, 2, ..., n)$
$\sum\limits_{k=1}^{n} y_k = 1$ such that $f\left(\sum\limits_{k=1}^{n} x_k y_k\right) \leq \sum\limits_{k=1}^{n} y_k f(x_k)$.

Mihály Bencze

OQ. 3212. If $x_k > 0$ $(k = 1, 2, ..., n)$ and $\prod\limits_{k=1}^{n} x_k = P^n$, then
$\sum\limits_{cyclic} \frac{1 + x_1 x_2}{1 + x_1} \geq \frac{n(P^n + 1)}{P+1}$.

Mihály Bencze

OQ. 3213. Let ABC be a triangle. Determine all $x, y, z > 0$ and $n \in N$ such that $\prod (xa^n + yb^n + zc^n) \geq ((y^x + z^x) sr)^{n-1}$.

Mihály Bencze

OQ. 3214. If $f : [0, 1] \to (0, +\infty)$ is a concave function, then determine all $s, r \in R$ such that $\left((s+1) \int\limits_0^1 f^s(x)\, dx\right)^r \leq \left((r+1) \int\limits_0^1 f^r(x)\, dx\right)^s$.

Mihály Bencze

OQ. 3215. If $x_k > 0$ $(k = 1, 2, ..., n)$ and $\sum\limits_{cyclic} x_1 x_2 ... x_{n-1} = n$, then
$\sum\limits_{cyclic} \frac{x_1}{n x_1^n + x_2 x_3 ... x_n} \geq \prod\limits_{k=1}^{n} x_k.$

Mihály Bencze

OQ. 3216. If $x_k > 0$ $(k = 1, 2, ..., n)$ then
$\sum\limits_{cyclic} \left(x_1 + \frac{x_2^{n-1}}{x_3} \right)^{n-1} \geq \frac{n(n-1)^2 \sum\limits_{k=1}^{n} x_k^n}{\sum\limits_{k=1}^{n} x_k}.$

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OQ. 3217. If $x_k > 0$ $(k = 1, 2, ..., n)$ and $\sum\limits_{k=1}^{n} x_k^a = 1$, where $a \in R$, then find the minimum value of $\sum\limits_{k=1}^{n} \frac{x_k^b}{1 - x_k^c}$, when $b, c \in R$.

Mihály Bencze

OQ. 3218. If $0 < x_k < 1$ $(k = 1, 2, ..., n)$, then determine all $f : R^n \to R$ for which $\sum\limits_{k=1}^{n} \frac{1}{1 - x_k} \geq \frac{n}{1 - f(x_1, x_2, ..., x_n)} \geq \frac{n}{1 - \frac{1}{n} \sum\limits_{k=1}^{n} x_k}$.

Mihály Bencze

OQ. 3219. If $x_k > 0$ $(k = 1, 2, ..., n)$ then

$$\frac{n-1}{\left(\sum\limits_{k=1}^{n} x_k \right) \left(\sum\limits_{cyclic} x_1 x_2 \right)} \geq \frac{1}{n \sum\limits_{cyclic} x_1^2 x_2} + \frac{1}{n \sum\limits_{cyclic} x_1 x_2^2}.$$

Mihály Bencze

OQ. 3220. If $y, x_k > 0$ $(k = 1, 2, ..., n)$ and $\sum\limits_{k=1}^{n} x_k^\alpha = y^\alpha$, where $\alpha \geq 1$, then

$$\sum_{k=1}^{n} x_k^{\alpha-1} - y^{\alpha-1} \geq n(\alpha - 1) \prod_{k=1}^{n} (y - x_k).$$

Mihály Bencze

OQ. 3221. If $x_k, y_k > 0$ $(k = 1, 2, ..., n)$ and $S = \sum\limits_{k=1}^{n} y_k$, then

$$\sum_{k=1}^{n} (S - y_k) x_k \ge (n-1) \sqrt{\left(\sum_{cyclic} x_1 x_2\right)\left(\sum_{cyclic} y_1 y_2\right)}.$$

Mihály Bencze

OQ. 3222. If $x_k > 0$ $(k = 1, 2, ..., n)$ and $\sum\limits_{k=1}^{n} x_k = 1$ and $\alpha \ge 1$, then $n \le \sum\limits_{cyclic} \frac{x_1^\alpha + 1}{x_2^\alpha + 1} \le n + \frac{1}{n-1}$.

Mihály Bencze

OQ. 3223. If $x_k > 0$ $(k = 1, 2, ..., n)$ and $\sum\limits_{k=1}^{n} x_k = 1$ then find the minimum and the maximum value of the expression $\sum\limits_{cyclic} (x_1 x_2 ... x_{n-1})^\alpha$ when $\alpha \ge 1$ is given.

Mihály Bencze

OQ. 3224. Let $A_1 A_2 ... A_n$ be a convex polygon inscribed in the unit circle. If $M \in Int\left(A_1 A_2 ... A_n\right)$, then $\prod\limits_{k=1}^{n} MA_k \le 1 + \frac{1}{n^2} + \frac{2}{n^n}$.

Mihály Bencze

OQ. 3225. If $P_0(x) = 0$, $P_{n+1}(x) = P_n(x) + \frac{x - P_n^k(x)}{k}$ for all $n \in N^*$, where $k \in N, k \ge 2$ is given. Prove that, for all $n \in N$ holds the inequalities $0 \le \sqrt[k]{x} - P_n(x) \le \frac{k}{n+1}$, when $x \in [0, 1]$.

Mihály Bencze

OQ. 3226. If $x_k > 0$ $(k = 1, 2, ..., n)$ and $P = \prod\limits_{k=1}^{n} x_k$, $\sum\limits_{k=1}^{n} x_k = 1$ then $\sum\limits_{k=1}^{n} \frac{x_k}{x_k^2 + P + 1} \le \frac{n^n}{n^n + n^{n-2} + 1}$.

Mihály Bencze

OQ. 3227. If $x_k > 0$ $(k = 1, 2, ..., n)$ and $\sum\limits_{k=1}^{n} x_k = n$ then determine all $p, q \in N^*$ such that $\sum\limits_{k=1}^{n} \frac{1}{x_k^p} \geq \sum\limits_{k=1}^{n} x_k^q$.

Mihály Bencze

OQ. 3228. If $x_k, y_k > 0$ $(k = 1, 2, ..., n)$ then $\frac{1}{\sum\limits_{k=1}^{n} \frac{1}{x_k}} + \frac{1}{\sum\limits_{k=1}^{n} \frac{1}{y_k}} \leq \frac{1}{\sum\limits_{k=1}^{n} \frac{1}{x_k} + \sum\limits_{k=1}^{n} \frac{1}{y_k}}$ and more general, if $x_{ij} > 0$ $(i = 1, 2, ..., n; j = 1, 2, ..., m)$, then $\sum\limits_{j=1}^{m} \frac{1}{\sum\limits_{i=1}^{n} \frac{1}{x_{ij}}} \leq \frac{1}{\sum\limits_{j=1}^{m} \frac{1}{\sum\limits_{i=1}^{n} x_{ij}}}$.

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OQ. 3229. Let be $a_1 = 1$, $16a_{n+1} = 1 + 4a_n + \sqrt{1 + 24a_n}$ for all $n \geq 1$. Determine all $n \in N^*$ for which a_n is prime.

Mihály Bencze

OQ. 3230. The equation $x^2 + 4xy + y^2 = 1$ have infinitely many solution in Z, because the sequences $x_{n+1} = x_n^2 - y_n^2$, $y_{n+1} = 2x_n y_n + 4y_n^2$, $x_1 = 1, y_1 = -4$ offer infinitely many solution in Z.

1). Determine all solution in Z 2). Determine all solutution in Q

Mihály Bencze

OQ. 3231. Let be $f(x) = \prod\limits_{k=1}^{n} \frac{\ln(a_k x + b_k)}{\ln(c_k x + d_k)}$, where $x > 0$

1). Determine all $a_k, b_k, c_k, d_k > 0$ $(k = 1, 2, ..., n)$ for which f is increasing (decreasing)

2). Determine all $a_k, b_k, c_k, d_k > 0$ $(k = 1, 2, ..., n)$ for which f is convex (concav)

Mihály Bencze

OQ. 3232. Let be $f(x)=\prod\limits_{k=1}^{n}\sin(a_kx+b_k)$
1). Determine all $a_k, b_k \in R$ $(k=1,2,...,n)$ and $x\in R$ for which f is increasing (decreasing)
2). Determine all $a_k, b_k \in R$ $(k=1,2,...,n)$ and $x\in R$ for which f is convex (concav)

Mihály Bencze

OQ. 3233. 1). If $H_n=1+\frac{1}{2}+...+\frac{1}{n}$, then $1^{H_n}+2^{H_n}+...+k^{H_n}\le\binom{n+k}{n+1}$ for all $k\in N^*$
2). Determine the best constants $0<a<b\le 1$ such that
$a\binom{n+k}{n+1}\le 1^{H_n}+2^{H_n}+...+k^{H_n}\le b\binom{n+k}{n+1}$
3). Determine the assymptotical expansion of the sum $1^{H_n}+2^{H_n}+...+k^{H_n}$

Mihály Bencze

OQ. 3234. Let ABC be a triangle. Determine all $x,y,z>0$ such that
$\sin\frac{A}{x}\sin\frac{B}{y}\sin\frac{C}{z}\le\left(\frac{3}{x+y+z}\right)^{\frac{x+y+z}{2}}$.

Mihály Bencze

OQ. 3235. Let ABC be a triangle. Determine all $x_k, y_k, z_k>0$ $(k=1,2,3)$ such that $x_1\sin\frac{A}{y_1}+x_2\sin\frac{B}{y_2}+x_3\sin\frac{C}{y_3}\le\frac{3\left(x_1y_1^2+x_2y_2^2+x_3y_3^2\right)}{4(y_1+y_2+y_3)}$.

Mihály Bencze

OQ. 3236. 1). Prove that
$\frac{1}{2}\ln 3\,(n+2)\ln\frac{n+2}{3}<\sum\limits_{k=3}^{n+1}\frac{\ln k}{k}<\frac{1}{2}\ln 3\,(n+1)\ln\frac{n+1}{3}+\frac{1}{3}\ln 3$
2). Determine the best constants $a,b,c,d>0$ such that
$\sum\limits_{k=1}^{n}\frac{\ln x}{k}=a\ln bn\ln cn+d+O(n)$.

Mihály Bencze

OQ. 3237. 1). If $x\ge y>0$ then $(x+1)^{x-\frac{1}{x}}y^y\ge(y+1)^{y-\frac{1}{y}}x^x$
2). Determine all $a_k, b_k, c_k>0$ and $d_k\in R$ $(k=1,2)$ for which
$(a_1x+b_1)^{c_1x+\frac{d_1}{x}}y^y\ge(a_2y+b_2)^{c_2y+\frac{d_2}{y}}x^x$ for all $x\ge y>0$.

Mihály Bencze

OQ. 3238. 1). If $x \geq y > 0$, then $\left(1+\frac{1}{x}\right)^x + (1+y)^{\frac{1}{y}} \geq \left(1+\frac{1}{y}\right)^y + (1+x)^{\frac{1}{x}}$
2). Determine all $a_k, b_k, c_k, d_k > 0$ $(k = 1, 2)$ such that
$\left(a_1+\frac{b_1}{x}\right)^x + (c_1+d_1y)^{\frac{1}{y}} \geq \left(a_2+\frac{b_2}{y}\right)^y + (c_2+d_2x)^{\frac{1}{x}}$ for all $x \geq y > 0$.

Mihály Bencze

OQ. 3239. Suppose that $A_1, A_2, ..., A_n$ are the vertices of a simplex S. On the faces opposite to $A_1, A_2, ..., A_{n-1}$, construct simplex outside S with apexes $B_1, B_2, ..., B_{n-1}$ and volumes $V_1, V_2, ..., V_{n-1}$, respectively.
Let B_n be the point such that $\overline{A_1B_n} = \overline{BA_n}$, where B is the point of intersection of the planes through B_i parallel to the respective bases $(i = 1, 2, ..., n-1)$. Let V_n be the volume of the simplex $A_1A_2...A_{n-1}B_n$.
Prove that $V_n = V_1 + V_2 + ... + V_{n-1}$.

Mihály Bencze

OQ. 3240. If M, N, K are the mid-points of sides BC, CA, AB in triangle ABC, then $1 \geq \prod \cos\frac{A-B}{2} \geq \sin(AMB)\sin(BNC)\sin(CKA) \geq \left(\frac{2r}{R}\right)^3$, a refinement of Euler's inequality.
Determine all $M \in BC, N \in CA, K \in AB$ such that
$1 \geq \prod \cos\frac{A-B}{2} \geq \sin(AMB)\sin(BNC)\sin(CKA) \geq \left(\frac{2r}{R}\right)^3$.

Mihály Bencze

OQ. 3241. Let ABC be a triangle, $A_1 \in (BC), B_1 \in (CA), C_1 \in (AB)$ such that $AA_1 \cap BB_1 \cap CC_1 = \{M\}$

1). Determine all points M for which $\sum \frac{1}{\sqrt{MA}+\sqrt{MB}-\sqrt{MC}} \geq \frac{3\sum\sqrt{MA}}{\sum MA}$.
I have obtained $M \equiv G$.

2). Determine all points M for which $\sum \frac{1}{MA^\lambda+MB^\lambda-MC^\lambda} \geq \frac{9}{a^\lambda+b^\lambda+c^\lambda}$, where $\lambda \in [0, 1]$.

Mihály Bencze

OQ. 3242. Let ABC be a triangle and $M \in Int(ABC)$, such that $MAB\measuredangle + MBC\measuredangle + MCA\measuredangle = 90^\circ$. Determine all M for which the triangle is isoscele.

Mihály Bencze

OQ. 3243. Determine all $a, b, c, d, e \in Z$ such that
$\sum\limits_{j=0}^{n} \sum\limits_{k=1}^{n} (-1)^{j+k} \binom{an}{bj}\binom{cn}{dk+e} = 0$

Mihály Bencze

OQ. 3244. Let ABC be a triangle. Determine all $x_k, y_k, z_k > 0$ $(k = 1, 2, 3)$ such that $x_1 \cos \frac{A}{y_1} + x_2 \cos \frac{A}{y_2} + x_3 \cos \frac{A}{y_3} \leq \frac{1}{2} + \frac{x_1+x_2+x_3}{x_1y_1+x_2y_2+x_3y_3}$.

Mihály Bencze

OQ. 3245. Let ABC be a triangle. Determine all $x_k, y_k, z_k > 0$ $(k = 1, 2, 3)$ such that $x_1 \cos \frac{A}{y_1} + x_2 \cos \frac{A}{y_2} + x_3 \cos \frac{A}{y_3} \leq \frac{3}{y_1+y_2+y_3} \left(\frac{x_1y_1+x_2y_2+x_3y_3}{2}\right)^{\frac{3}{2}}$.

Mihály Bencze

OQ. 3246. Solve in Z the following equation

$$\left(x_1 + x_2^2 + x_3^3 + ... + x_n^n\right)\left(x_2 + x_3^2 + x_4^3 + ... + x_1^n\right) ... \left(x_n + x_1^2 + x_2^3 + ... + x_{n-1}^n\right) = \left(x_1 + x_2 + ... + x_n\right)^{\frac{n(n+1)}{2}}.$$

Mihály Bencze

OQ. 3247. Let ABC be a triangle, and denote A the areea of the triangle. Determine the best constants $1 \leq x < y \leq 3$, such that

$$x \sum (a-b)^2 \leq \sum a^2 - 4A\sqrt{3} \leq y \sum (a-b)^2.$$

Mihály Bencze

OQ. 3248. Determine all $n \in N$ for which $\Phi(n)$ divides $\sigma(n) + \Psi(n)$.

Mihály Bencze

OQ. 3249. Solve in Z the equation $\sum\limits_{k=1}^{n} \begin{vmatrix} a_k & b_k & c_k & d_k \\ -b_k & a_k & -d_k & c_k \\ -c_k & d_k & a_k & -b_k \\ -d_k & -c_k & b_k & a_k \end{vmatrix} = x^n,$

where $n \in N, n \geq 2$.

Mihály Bencze

OQ. 3250. Let $A_1A_2...A_{n+1}$ be a concyclic $(n+1)$ −gon, denote Ω_k the anticentres of $A_1A_2...A_{k-1}A_{k+1}...A_{n+1}$ $(k = 1, 2, ..., n+1)$. Prove that $\Omega_1\Omega_2...\Omega_{n+1}$ is concyclic.

Mihály Bencze

OQ. 3251. Determine all functions $f : R \to R$ for which from $\sum\limits_{k=1}^{n} f(\alpha_k x) \geq \sum\limits_{k=1}^{n} f(\beta_k x)$ when $\alpha_k, \beta_k \in R$ $(k = 1, 2, ..., n)$ for $x \in (-\varepsilon, \varepsilon)$, $\varepsilon > 0$, implies $\sum\limits_{k=1}^{n} \alpha_k \geq \sum\limits_{k=1}^{n} \beta_k$.

Mihály Bencze

OQ. 3252. Denote Q_3 the set of important points of a triangle $(H, G, O, I$ etc)

1). Let $ABCD$ be a concyclic quadrilateral and denote $M_A, M_B, M_C, M_D \in Q_3$ the important points of triangles BCD, CDA, DAB, ABC. Determine all $M_A, M_B, M_C, M_D \in Q_3$ for which the quadrilaterals $M_AM_BM_CM_D$ are concyclic

2). Prove that $cardQ_3 \geq 2$. We can show that $H, G \in Q_3$ satisfys the point 1). From Sylvester's theorem we have $\overline{OH_A} = \overline{OB} + \overline{OC} + \overline{OD}$ and his permutations. From others we have $\overline{OA} + \overline{OB} + \overline{OC} + \overline{OD} = \overline{OH_A} + \overline{OA} =$ $= \overline{OH_B} + \overline{OB} + \overline{OH_C} + \overline{OC} = \overline{OH_D} + \overline{OD} = \overline{QT}$. From $\overline{OH_A} + \overline{OA} = \overline{OT}$ we have $\overline{TH_A} = \overline{AO}$, O is the circumcenter of $ABCD$, etc.

We have $TH_A = OA = TH_B = OB = TH_C = OC = TH_D = OD$, which means that $H_AH_BH_CH_D$ is concyclic. From $\overline{H_AG_A} = 2\overline{G_AO}$ holds that $G_AG_BG_CG_D$ is concyclic. Finally $cardQ_3 \geq 2$.

3). Denote Q_n the set of important points of the convex n-gon.

Let $A_1A_2...A_{n+1}$ be a concyclic convex $(n+1)$ −gon. Denote $M_k \in Q_n$ $(k = 1, 2, ..., n+1)$ the important points of $A_1A_2...A_{k-1}A_{k+1}...A_{n+1}$ $(k = 1, 2, ..., n+1)$. Determine all $M_k \in Q_n$ $(k = 1, 2, ..., n)$ for which $A_1A_2...A_{k-1}A_{k+1}...A_{n+1}$ $(k = 1, 2, ..., n)$ are concyclic.

Mihály Bencze

OQ. 3253. Denote A_{2k} the denominator of Bernoullli's number B_{2k}.

1). Compute $\sum\limits_{k=1}^{\infty} \frac{1}{A_{2k}}$

2). Compute $\sum\limits_{k=1}^{\infty} \frac{1}{A_{2k}^2}$

3). More general determine $\sum\limits_{k=1}^{\infty} \frac{1}{A_{2k}^{\alpha}}$
4). How many prime exist between A_{2k} and A_{2k+2}?
5). Determine all $k \in N$ for which A_{2k} is prime
6). Determine all $n \in N$ for which $\sum\limits_{k=1}^{n} A_{2k}$ is prime

Mihály Bencze

OQ. 3254. Let be $A_1A_2...A_n$ a convex polygon with sides a_k $(k = 1, 2, ..., n)$. Prov that
1). $\frac{a_1}{\min(a_2,a_3,...,a_n)} + \frac{a_2}{\min(a_1,a_3,...,a_n)} + ... + \frac{a_n}{\min(a_1,a_2,...,a_{n-1})} \geq n$
2). $\frac{a_1}{\max(a_2,a_3,...,a_n)} + \frac{a_2}{\max(a_1,a_3,...,a_n)} + ... + \frac{a_n}{\max(a_1,a_2,...,a_{n-1})} \geq n$

Mihály Bencze

OQ. 3255. Denote B_n the $n-$th Bernoulli's number. Determine all n for which $k + \sum\limits_{i=1}^{k} n_i B_{n_i - 1} \equiv 0 \,(\text{mod } n)$ if and only if n is prime, when $n = \sum\limits_{i=1}^{k} n_i$.

Mihály Bencze

OQ. 3256. Determine all n for which $n!B_1B_2...B_{n-1} + (-1)^n \equiv 0 \,(\text{mod } n)$ if and only if n is prime.

Mihály Bencze

OQ. 3257. If $x, y, z > 0$ and $\lambda \in [1, 2]$, then $\sum\limits_{cyclic} \frac{(y+z)^2}{x^2+\lambda yz} \leq \frac{12}{\lambda+1}$.

Mihály Bencze

OQ. 3258. Solve in Z the following equation
$x_1a^3 + x_2b^3 + x_3c^3 - \left(xa^2 + yb^2 + zc^2\right)(xa + yb + zc) =$
$= y_1 (a + b) (a - b)^2 + y_2 (b + c) (b - c)^2 + y_3 (c + a) (c - a)^2$.

Mihály Bencze

OQ. 3259. Compute the following sums:
1). $F = \frac{1}{F_1} + \frac{1}{F_1F_2} + ... + \frac{1}{F_1F_2...F_n} + ...$, when F_k denote the k-th Fibonacci number
2). $P = \frac{1}{p_1} + \frac{1}{p_1p_2} + ... + \frac{1}{p_1p_2...p_n} + ...$, when p_k denote the k-th prime number
3). If $e_n = \left(1 + \frac{1}{n}\right)^n$, then compute $E = \frac{1}{e_1} + \frac{1}{e_1e_2} + ... + \frac{1}{e_1e_2...e_n} + ...$

4). $D = \frac{1}{d(1)} + \frac{1}{d(1)d(2)} + ... + \frac{1}{d(1)d(2)...d(n)} + ...$
5). $\Phi = \frac{1}{\Phi(1)} + \frac{1}{\Phi(1)\Phi(2)} + ... + \frac{1}{\Phi(1)\Phi(2)...\Phi(n)} + ...$
6). $\Psi = \frac{1}{\Psi(1)} + \frac{1}{\Psi(1)\Psi(2)} + ... + \frac{1}{\Psi(1)\Psi(2)...\Psi(n)} + ...$
7). $\sigma = \frac{1}{\sigma(1)} + \frac{1}{\sigma(1)\sigma(2)} + ... + \frac{1}{\sigma(1)\sigma(2)...\sigma(n)} + ...$
8). If $c_k = 1 + \frac{1}{2} + ... + \frac{1}{k} - \ln k$, then compute $C = \frac{1}{c_1} + \frac{1}{c_1 c_2} + ... + \frac{1}{c_1 c_2 ... c_n} + ...$
9). $N = \frac{1}{1!} + \frac{1}{1!2!} + ... + \frac{1}{1!2!...n!} + ...$
10). $R = \frac{1}{1} + \frac{1}{1 \cdot 3} + ... + \frac{1}{1 \cdot 3 ... (2^n - 1)} + ...$

Mihály Bencze

OQ. 3260. Determine all $n \in N$ for which $2^{\frac{n(n+1)}{2}} - 1$ and $2^{\frac{n(n+1)}{2}} + 1$ are primes.

Mihály Bencze

OQ. 3261. Prove that the sequence $(2^n - 1, 2^n + 1)$ contain infinitely many twin primes.

Mihály Bencze

OQ. 3262. If p_k denote the $k-$th prime $(p_1 = 2, p_2 = 3, ...)$, then prove that exist infinitely many twin primes which have the following forms
$p_1 p_2 ... p_n - 1$ and $p_1 p_2 ... p_n + 1$.

Mihály Bencze

OQ. 3263. Determine all prime p_k, q for which $\prod_{k=1}^{n} p_k - q$ and $\prod_{k=1}^{n} p_k + q$ are prime.

Mihály Bencze

OQ. 3264. 1). Prove that exist infinitely many numbers of the form
$n^k + n^{k-1} + ... + n^2 + n + 1$ which can be expressed as a product of at most k primes.
2). Prove that exist infinitely many prime numbers of the form
$n^k + n^{k-1} + ... + n^2 + n + 1$.
3). Prove that exist infinitely many prime p and k, and infinitely many
$n, k \in N$ such that $n^k + n^{k-1} + ... + n^2 + n + 1 = p^2 + q^2$.

Mihály Bencze

OQ. 3265. Prove that exist infinitely many numbers of the form n^k+1 which can be expressed as a poroduct of at most k primes.

Mihály Bencze

OQ. 3266. Denote F_k the $k-$th Fibonacci number. Prove that the sequence $(F_1F_2...F_n-1, F_1F_2...F_n+1)$ contain infinitely many twin primes.

Mihály Bencze

OQ. 3267. Exist infinitely many prime p, q and infinitely many $n \in N$ such that $(n-1)\left(n^2-1\right)...(n^p-1) \equiv 0 \left(\operatorname{mod} q^{\frac{p(p++1)}{2}}\right)$.

Mihály Bencze

OQ. 3268. Determine all $n \in N$ for which
1). $\sum\limits_{k=1}^{n} (d(k), \sigma(k))$ is prime 2). $\sum\limits_{k=1}^{n} (\Phi(k), \Psi(k))$ is prime

Mihály Bencze

OQ. 3269. Determine all $a, n_k \in N$ $(k=1,2,...,m)$ such that
$\prod\limits_{k=1}^{m} \left(a^{n_k-1}-1\right) \equiv 0 \left(\operatorname{mod} \prod\limits_{k=1}^{m} n_k\right)$.

Mihály Bencze

OQ. 3270. Let be F_k the $k-$ th Fibonacci number. Determine all $n \in N$. For which $\sum\limits_{k=1}^{n} F_k^{n-1} + 1 \equiv 0 \,(\operatorname{mod} n)$ if and only if n is prime.

Mihály Bencze

OQ. 3271. Determine all $n \in N$ for which $\sum\limits_{k=1}^{n} (d(k))^{n-1} + 1 \equiv 0 \,(\operatorname{mod} n)$ if and only if n is prime.

Mihály Bencze

OQ. 3272. Determine all $n \in N$ for which $\sum\limits_{k=1}^{n} (F(k))^{n-1} + 1 \equiv 0 \,(\operatorname{mod} n)$ if and only if n is prime, where $F \in \{\sigma, \Phi, \Psi, P, S, ...\}$.

Mihály Bencze

OQ. 3273. Determine all $n \in N$ for which $\sum\limits_{k=1}^{n} \left(\binom{n}{k}\right)^{n-1} + 1 \equiv 0 \,(\bmod\, n)$ if and only if n is Carmichael number.

Mihály Bencze

OQ. 3274. Determine all $n \in N$ for which $\sum\limits_{k=1}^{n} (k!)^{n-1} + 1 \equiv 0 \,(\bmod\, n)$ if and only if n is prime.

Mihály Bencze

OQ. 3275. Denote B_n the n-th Bernoulli's number. Determine all $n, k, p \in N$ such that $k + n^k B_{n-k} \equiv 0 \,(\bmod\, nk)$.

Mihály Bencze

OQ. 3276. Determine all function $F : N \to N$ for which $\sum\limits_{k=0}^{n-1} F(k) \not\equiv (\bmod\, n)$ if and only if n is prime.

Mihály Bencze

OQ. 3277. Let be S the set of numbers n which have the following property: $\sum\limits_{k=1}^{n} k^{n-1} + 1 \equiv 0 \,(\bmod\, n)$ if and only if n is pseudoprime.

1). Prove that $cardS = +\infty$.

2). Compute $\sum\limits_{t \in S} \frac{1}{F(t)}$, where $F \in \{d, \sigma, \Phi, \Psi, ...\}$.

Mihály Bencze

OQ. 3278. Determine all primes p and all composites $n \in N$ such that $p | \left(\left(\sum\limits_{i=1}^{k} n_i \right) \left(\sum\limits_{i=1}^{k} \frac{1}{p_i} \right) - 1 \right)$ and $p_i | n_i$ $(i = 1, 2, ..., k)$ and $p = \sum\limits_{i=1}^{k} p_i$, $n = \sum\limits_{i=1}^{k} n_i$.

Mihály Bencze

OQ. 3279. Let be $G = \left\{ n \text{ composite for which } p | \left(\frac{n}{p} - 1 \right) \text{ for each prime } p | n \right\}$ the set of Giuga's numbers. Compute

1). $\sum\limits_{k\in G}\frac{1}{k}, \sum\limits_{k\in G}\frac{1}{k^2}, ..., \sum\limits_{k\in G}\frac{1}{k^\alpha},$

2). Prove that $cardG = +\infty$

3). Determine all Giuga's number n for which $n = \prod\limits_{i=1}^{k} n_i$, where n_i $(i = 1, 2, ..., k)$ are Giuga's numbers.

Mihály Bencze

OQ. 3280. Any large composite n satisfies

$$\left(\sum_{p_1\le \sqrt[k]{n_1}} p_1\right)\left(\sum_{p_2\le \sqrt[k]{n_2}} p_2\right)...\left(\sum_{p_k\le \sqrt[k]{n_k}} p_r\right) \ge \sum_{p_s\le \sqrt[k]{n}} p_s, \text{ where } n = n_1n_2...n_k.$$

Mihály Bencze

OQ. 3281. Determine all $n \in N$ and all prime p and q such that $\sum\limits_{k=1}^{n}\frac{1}{k} = \frac{1}{p} + \frac{1}{q}$.

Mihály Bencze

OQ. 3282. Let ABC be a triangle. Determine the best constants $x, y > 0$ such that $\frac{18sr}{s^2+r^2+4Rr} + x\left((a-b)^2 + (b-c)^2 + (c-a)^2\right) \le \sum\cos\frac{A}{2} \le$ $\le \frac{3\sqrt{3}}{2} - y\left((a-b)^2 + (b-c)^2 + (c-a)^2\right)$, which give a refinement of V.E. Olov's inequality.

Mihály Bencze

OQ. 3283. 1). Determine all $k \in N$ such that
$\binom{kp+m}{kp+n} + \binom{m}{n} \equiv k\binom{p+m}{p+n} \left(\bmod\ p^k\right)$, where p is a prime and
$m, n \in \{0, 1, ..., p-1\}$
2). Determine all $k \in N$ such that
$\binom{kp+m}{kp+n} + \binom{m_1}{n_1} + \binom{m_2}{n_2} \equiv \binom{p+m_1}{p+n_1}\binom{p+m_2}{p+n_2} \left(\bmod\ p^k\right)$, where p is a prime and
$m, n, m_1, n_1, m_2, n_2 \in \{0, 1, ..., p-1\}$, $m = m_1 + m_2$, $n = n_1 + n_2$
3). Determine all $m, n, m_i, n_i \in N$ such that
$\binom{kp+m}{kp+n} + \sum\limits_{i=1}^{k}\binom{m_i}{n_i} \equiv \prod\limits_{i=1}^{k}\binom{p+m_i}{p+n_i} \left(\bmod\ p^k\right)$, where $m = \sum\limits_{i=1}^{k} m_i$, $n = \sum\limits_{i=1}^{k} n_i$ and p is a prime.

Mihály Bencze

OQ. 3284. Let ABC be a triangle. Determine all function

$f:(0,+\infty)\times(0,+\infty)\to(0,+\infty)$ such that $\frac{aw_a}{f(b,c)}+\frac{bw_c}{f(c,a)}+\frac{cw_a}{f(a,b)}\geq s$

Mihály Bencze

OQ. 3285. Solve in Z the equation $\sum\limits_{k=1}^{n}\frac{1}{x_k^2+1}+\frac{n-1}{\sqrt{\prod\limits_{k=1}^{n}\left(x_k^2+1\right)}}=1.$

Mihály Bencze

OQ. 3286. 1). The equation $1+x\left(x_1-1\right)\left(x_2-1\right)...\left(x_n-1\right)=x_1x_2...x_n$ have infinitely many solution in N. A solution is $x=2$, $x_k=2^{2^{k-1}}+1$ $(k=1,2,...,n)$

2). Determine all solutions in N
3). Determine all solutions in Z
4). Determine all solutions in Q
5). Solve in N the equation $y+x\left(x_1-y\right)\left(x_2-y\right)...\left(x_n-y\right)=x_1x_2...x_n$
6). Solve in Z 7). Solve in Q

Mihály Bencze

OQ. 3287. If $a_{n,k}=\sum\limits_{m=1}^{\infty}\frac{m^{\frac{1}{k}}}{n^{\frac{1}{k}}(m+n)}$, when $n\in N^*$ and $k\geq 2$.

1). Determine the best constants $b,c>0$ such that $b(k)\leq a_{n,k}\leq c(k)$
2). Compute $\sum\limits_{n=1}^{\infty}\frac{1}{a_{n,k}^{\alpha}}$ when $\alpha\geq 2$
3). Compute $\sum\limits_{k=1}^{\infty}\frac{1}{a_{n,k}^{\beta}}$, when $\beta\geq 2$.

Mihály Bencze

OQ. 3288. If $n\in\{1,4\}$, then $a^n(b-c)+b^n(c-a)+c^n(a-b)$ is divisible by $a^2+b^2+c^2+ab+bc+ca$ for all $a,b,c\in Z$. Determine all $n,k\in N$ for which $a_1^n(a_2-a_3)+a_2^n(a_3-a_4)+...+a_k^n(a_1-a_2)$ is divisible by $a_1^2+a_2^2+...+a_n^2+\sum\limits_{1\leq i<j\leq k}a_ia_j$ for all $a_i\in Z$ $(i=1,2,...,k)$.

Mihály Bencze

OQ. 3289. 1). Solve in Z the equations $\prod\limits_{k=1}^{n}(x_k+y_k)=x^2\pm y^2$

2). Solve in Z the equations $\prod\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{m}x_{ij}\right)=y_1^2\pm y_2^2\pm ...\pm y_m^2$

Mihály Bencze

OQ. 3290. Determine all $x_k\in Z$ and all $p,r\in Z$ such that

$\left(\sum\limits_{k=1}^{n}x_k\right)^p\left(\prod\limits_{k=1}^{n}x_k\right)^r\in Z$. If $x_k\in\{-1,1\}$ and $r\in Z$, $p\in N$ then we have a solution. If $p<0$, $p\in Z$, and exist $x_i=0$ then $x_j\neq 0$ $(j\neq 1)$ is a solution.

Mihály Bencze

OQ. 3291. If $P(n)x_{n+1}=Q(n)x_n-R(n)x_{n-1}$ for all $n\in N^*$, then determine all $P,Q,R\in Z[x]$ and all $x_1,x_2\in Z$ for which $x_n\in Z$ if and only if n is prime.
A solution is $P(x)=x^2-x-2$, $Q(x)=x^3-x^2-x$, $R(x)=(x-1)^3$, $x_1=x_2=1$.

Mihály Bencze

OQ. 3292. Determine all $a_i,b_i\in Z$ $(i=0,1,...,k)$, $n_0\in N$ such that $\Phi\left(a_0n^k+a_1n^{k-1}+...+a_k\right)\leq\Phi\left(b_0n^k+b_1n^{k-1}+...+b_k\right)$ for all $n\geq n_0$, $n\in N$.

Mihály Bencze

OQ. 3293. 1). Prove that
$n-\frac{1}{2}(1+\ln n)<\sum\limits_{k=1}^{n}k\ln\left(1+\frac{1}{k}\right)<n+2-\frac{1}{n}-\frac{1}{2}\ln(n+1)$
2). Determine $\alpha,\beta\in R$ such that $\sum\limits_{k=1}^{n}k\ln\left(1+\frac{1}{k}\right)=n+\alpha+\beta\ln n+O(n)$

Mihály Bencze

OQ. 3294. Determine the best constants $a,b>0$ such that
$\frac{ax}{1+x^2}\leq\min\left\{arctgx;xe^{-xarctgx}\right\}$ for all $x\geq 0$ and
$\max\left\{arctgx;xe^{-xarctgx}\right\}\leq\frac{bx}{1+x^2}$ for all $x\in R$.

Mihály Bencze

OQ. 3295. Determine all $a_k \in \{0, 1, ..., 9\}$ such that $\left\{ \begin{array}{l} \overline{a_1 a_2 ... a_k} = n! \\ a_1 + ... + a_k = k! \end{array} \right.$.

Mihály Bencze

OQ. 3296. 1). Solve in Z the equation
$10\left(x^2+y^2\right)^3 = 8\left(x^6+y^6\right) + (x+y)^6 + (x-y)^6$

2). Solve in Z the equation $28\left(x^2+y^2+z^2\right)^3 =$
$8\left(x^6+y^6+z^6\right) + (x+y+z)^6 + (-x+y+z)^6 + (x-y+z)^6 + (x+y-z)^6$
3). Solve in Z the equation
$\left(n^4 - 5n^3 + 15n^2 - 20n + 15\right)\left(\sum\limits_{k=1}^{n} x_k^2\right)^3 = (6n-1)\sum\limits_{k=1}^{n} x_k^6 + \left(\sum\limits_{k=1}^{n} x_k\right)^6 +$
$+\left(-x_1 + x_2 + ... + x_n\right)^6 + \left(x_1 - x_2 + x_3 + ... + x_n\right)^6 + ...$
$+\left(x_1 + x_2 + ... + x_{n-1} - x_n\right)^6$.

Mihály Bencze

OQ. 3297. 1). If $x \geq 0$ then $x \geq \max\left\{arctg\left(x + \frac{x^3}{3}\right); arctgx + \frac{1}{3}arctg^3 x\right\}$
2). Determine the best constants $a_1, a_2, b_1, b_2 > 0$ such that the inequality
$x \geq \max\left\{arctg\left(a_1 + b_1 x^3\right); a_2 arctgx + b_2 arctg^3 x\right\}$ are the best possible
3). Determine all polynomial $P \in R[x]$ such that the inequality
$x \geq \max\left\{arctgP(x); P(arctgx)\right\}$ are the best possible.

Mihály Bencze

OQ. 3298. Determine the best constants $a, b > 0$ such that
$\pi a \zeta(k\alpha) \leq \sum\limits_{i_1, i_2, ..., i_k = 1}^{\infty} \frac{1}{(i_1 + ... + i_k)(i_1 i_2 ... i_k)^\alpha} \leq \pi b \zeta(k\alpha)$, where $\alpha > 1$, where ζ denote the Riemann zeta function.

Mihály Bencze

OQ. 3299. Determine $\sum\limits_{k=1}^{\infty}\left(1 + \frac{1}{2} + ... + \frac{1}{k} - \ln\left(k + \frac{1}{2}\right)\right)$ in function of π, e, γ. Determine the best constants $\frac{1}{24}\left(\frac{\pi^2}{6} - 1\right) \leq a < b \leq \frac{\pi^2}{144}$ and $a, b \in Q$ such that $a \leq \sum\limits_{k=1}^{n}\left(1 + \frac{1}{2} + ... + \frac{1}{k} - \ln\left(k + \frac{1}{2}\right)\right) \leq b$.

Mihály Bencze

OQ. 3300. If $x > 1$ and $a \in \left[\frac{1}{3}, \frac{2}{3}\right]$, then determine the best constants $b, c > 0$ such that $\frac{b}{e}x^{\frac{x}{x-1}} < a\sqrt{x} + (1-a)\frac{x+1}{2} < \frac{c}{e}x^{\frac{x}{x-1}}$.

Mihály Bencze

OQ. 3301. Determine the best constant $c > 0$ such that $\sum\limits_{i_1,i_2,...,i_k}^{\infty} \frac{1}{(i_1+...+i_k)i_1^{a_1}...i_k^{a_k}} \leq c\sqrt[k]{\prod\limits_{j=1}^{k}\zeta(ka_j)}$, where $a_j > 1$ $(j = 1, 2, ..., k)$ and ζ denote the Riemann zeta function.

Mihály Bencze

OQ. 3302. If $x_n = \sum\limits_{k=1}^{n}\frac{1}{k} - \ln\left(n + \frac{1}{2}\right)$, then $\frac{1}{24(n+1)^2} < x_n < \frac{1}{24n^2}$. Denote $y_n = n^3\sum\limits_{k=1}^{n}\frac{1}{x_k}$. Prove that $(y_n)_{n\geq 1}$ is convergent and compute its limit. Compute $\sum\limits_{n=1}^{\infty}\frac{1}{y_n}$.

Mihály Bencze

OQ. 3303. 1). Compute $\lim\limits_{n\to\infty}\frac{(n!)^2}{24^n}\prod\limits_{k=1}^{n}\left(1 + \frac{1}{2} + ... + \frac{1}{k} - \ln\left(k + \frac{1}{2}\right)\right)$

2). Compute $\lim\limits_{n\to\infty}\frac{(n!)^\alpha}{a^n}\prod\limits_{k=1}^{n}\left(1 + \frac{1}{2} + ... + \frac{1}{k} - \ln\left(k + \frac{1}{\alpha}\right)\right)$, where $\alpha \geq 1$ and $a \geq 2$.

Mihály Bencze

OQ. 3304. If $x_k > 0$ $(k = 1, 2, ..., n)$, then

$$\left(\sum_{k=1}^{n}x_k\right)\left(\sum_{k=1}^{n}\frac{1}{x_k}\right) \geq 2n - 1 + \frac{(n-1)^3\sum\limits_{k=1}^{n}x_k^2}{2\sum\limits_{1\leq i<j\leq n}x_ix_j}.$$

Mihály Bencze

OQ. 3305. If $x_k \geq \alpha > 0$ $(k = 1, 2, ..., n)$, then

$$\sum_{k=1}^{n}x_k^3 + 3\sum_{1\leq i<j<k\leq n}x_ix_jx_k \geq$$

$$\geq \sum_{cyclic}x_1x_2(x_1 + x_2) + \alpha\left(\sum_{k=1}^{n}x_k^2 - \frac{2}{n-1}\sum_{1\leq i<j\leq n}x_ix_j\right).$$

Mihály Bencze

OQ. 3306. If $x_k > 0$ $(k = 1, 2, ..., n)$ and $a, b, c > 0$ then
$$\sum_{cyclic} \frac{\sqrt{x_1^2+ax_2x_3}}{(x_2+bx_3)(x_1+cx_3)} \geq \frac{n^3\sqrt{a+1}}{(b+1)(c+1)\sum\limits_{k=1}^{n} x_k}.$$

Mihály Bencze

OQ. 3307. If $x_k > 0$ $(k = 1, 2, ..., n)$ and $\alpha \geq 1$, then
$$\sum_{cyclic} \frac{x_1^3}{x_2^2+\alpha x_3^2} \geq \frac{1}{\alpha+1} \sum_{k=1}^{n} x_k.$$

Mihály Bencze

OQ. 3308. If $\alpha, \beta, x_k > 0$ $(k = 1, 2, ..., n)$, then
$$\sum_{cyclic} \left(\frac{x_1}{x_2+x_3}\right)^{\alpha} + \frac{\beta \prod\limits_{k=1}^{n} x_k}{\prod\limits_{cyclic} (x_1+x_2)} \geq n \cdot 2^{-\alpha} + \beta \cdot 2^{-n}.$$

Mihály Bencze

OQ. 3309. If $x_k > 0$, then determine all $y_k > 0$ $(k = 1, 2, ..., n)$ such that
$$\sum_{cyclic} \frac{x_1^2+x_2^2}{x_1+x_2} \geq \sqrt{n \sum_{k=1}^{n} x_k^2 + \frac{\sum\limits_{cyclic} y_1(x_1-x_2)^2}{(n-1)\left(\sum\limits_{k=1}^{n} x_k\right)^2}}.$$

Mihály Bencze

OQ. 3310. If $x_i, y_i > 0$ $(i = 1, 2, ..., n)$, then

$$\sum_{i=1}^{n} y_i x_i^{k-1} \max\{x_1, x_2, ..., x_n\} \geq \binom{n}{k}^{-1} \sum_{i=1}^{n} y_i \sum_{1 \leq i_1 < ... < i_k \leq n} x_{i_1} x_{i_2} ... x_{i_k}$$ for $n \geq 2$ and $k \in \{2, 3, ..., n-1\}$.

Mihály Bencze

OQ. 3311. If $\alpha x_k > 0$ $(k = 1, 2, ..., n)$ and $\sum\limits_{k=1}^{n} x_k = n$, then
$$\sum_{cyclic} \left(\frac{x_1+x_2}{x_1^2+x_1x_2+x_2^2}\right)^{\alpha} \geq n \left(\frac{2}{3}\right)^{\alpha}.$$

Mihály Bencze

OQ. 3312. If $x_k > 0$ $(k = 1, 2, ..., n)$ and $\prod\limits_{k=1}^{n} x_k = 1$, then

$\prod\limits_{cyclic} \left(x_1 + \frac{1}{x_2} - 1 \right) \leq 1.$

Mihály Bencze

OQ. 3313. If $x_k > 0$ $(k = 1, 2, ..., n)$, then determine all $a, b, c > 0$ such that

$\max \left\{ \sum\limits_{cyclic} \frac{x_1}{ax_1^2 + bx_2^2 + cx_3^2}; \sum\limits_{cyclic} \frac{x_1}{ax_1^2 + bx_2^2 + cx_1x_3} \right\} \leq \frac{n^2}{(a+b+c) \sum\limits_{k=1}^{n} x_k}.$

Mihály Bencze

OQ. 3314. If $x_k > 0$ $(k = 1, 2, ..., n)$ and $\sum\limits_{k=1}^{n} x_k = n$, then
$\sum\limits_{cyclic} \frac{x_1}{ax_1^2 + bx_1x_2x_3 + c} \leq \frac{n}{a+b+c}$, where $a, b, c > 0$.

Mihály Bencze

OQ. 3315. If $x_k > 0$ $(k = 1, 2, ..., n)$ and $\prod\limits_{k=1}^{n} x_k = 1$, then
$\max \left\{ \sum\limits_{cyclic} \frac{x_2 + ... + x_{p+1}}{x_1^p + p - 1}; \sum\limits_{cyclic} \frac{x_2 + ... + x_{p+1}}{x_1^{p+1} + (p-1) x_2 ... x_{p+1}} \right\} \leq \sum\limits_{k=1}^{n} \frac{1}{x_k^p}$, for all
$p \in \{2, 3, ..., n-1\}$.

Mihály Bencze

OQ. 3316. If $x_k > 0$ $(k = 1, 2, ..., n)$ and $\prod\limits_{k=1}^{n} x_k = 1$, then
$\sum\limits_{k=1}^{n} x_k^2 + \sum\limits_{1 \leq i < j \leq n} x_i x_j \geq \frac{n(n-1)}{2} + \sum\limits_{k=1}^{n} x_k.$

Mihály Bencze

OQ. 3317. If $x_k > 0$ $(k = 1, 2, ..., n)$ then determine all $a, b > 0$ such that
$\sum\limits_{cyclic} \frac{x_1}{\sqrt{x_1^2 + ax_2^2 + bx_3^2}} \geq 1.$

Mihály Bencze

OQ. 3318. If $x_k > 0$ $(k = 1, 2, ..., n)$ then

$\sum\limits_{cyclic} \frac{1}{x_1^{p-1}(x_2+...+x_{p+1})} + \frac{1}{\sum\limits_{k=1}^{n} x_k^p} \geq \sum\limits_{cyclic} \frac{1}{(p-1)x_1^p+x_2...x_{p+1}} + \frac{1}{\sum\limits_{cyclic} x_1x_2...x_p}$, where $p \in \{2, 3, ..., n-1\}$.

Mihály Bencze

OQ. 3319. If $x_k > 0$ $(k = 1, 2, ..., n)$ and $\alpha \geq n - 1$, then

$$\sum_{cyclic} \frac{x_1+\alpha x_2}{(x_2+\alpha x_3)^2} \geq \frac{\sum\limits_{k=1}^{n} x_k}{\sum\limits_{k=1}^{n} x_k^2+(\alpha-n+1) \sum\limits_{cyclic} x_1x_2}.$$

Mihály Bencze

OQ. 3320. If $x_k, y_k > 0$ $(k = 1, 2, ..., n)$, $c > 0$, $\alpha \leq 1$, then

$$\sum_{k=1}^{n} (x_k + y_k + c)^{\alpha} \geq (n-2)\, c^{\alpha} + \left(c + \sum_{k=1}^{n} x_k\right)^{\alpha} + \left(c + \sum_{k=1}^{n} y_k\right)^{\alpha}.$$

Mihály Bencze

OQ. 3321. If $x_k > 0$ $(k = 1, 2, ..., n)$ and $\alpha > 0$, then

$\sum\limits_{cyclic} \left(\frac{x_1}{x_2+...+x_{p+1}}\right)^{p+1} + \alpha \sum\limits_{cyclic} \frac{x_1x_2...x_p}{(x_1+x_2...+x_p)^p} \geq \frac{n(\alpha p+1)}{p^{p+1}}$, where $p \in \{2, 3, ..., n-1\}$.

Mihály Bencze

OQ. 3322. If $x_k > 0$ $(k = 1, 2, ..., n)$ and $\sum\limits_{k=1}^{n} x_k = 1$,

$\sum\limits_{cyclic} \sqrt[p]{1 - px_1x_2...x_p} \geq \sqrt[p]{n^p - p}$, where $p \in \{2, 3, ..., n-1\}$.

Mihály Bencze

OQ. 3323. If $x_k > 0$ $(k = 1, 2, ..., n)$ and $\sum\limits_{k=1}^{n} x_k = 1$, then

$\prod\limits_{cyclic} (1 - x_1x_2...x_p) \geq \left(1 - \frac{1}{n^p}\right)^n$, where $p \in \{2, 3, ..., n-1\}$.

Mihály Bencze

OQ. 3324. If $x_k > 0$ $(k = 1, 2, ..., n)$ and $\sum\limits_{k=1}^{n} x_k = 1$, then

$\sum\limits_{cyclic} \frac{1}{1-x_1x_2...x_p} \leq \frac{n^{p+1}}{n^p-1}$, where $p \in \{2, 3, ..., n-1\}$.

Mihály Bencze

OQ. 3325. If $x_k > 0$ $(k = 1, 2, ..., n)$ then
$\left(\sum\limits_{cyclic} \frac{x_1^p+...+x_p^p}{x_1+...+x_p}\right)^p \geq (p+1)\sum\limits_{k=1}^{n} x_k^p + p\prod\limits_{k=1}^{n} x_k \sum\limits_{cyclic} \frac{x_1+...+x_p-px_{p+1}}{x_1(x_2+...+x_{p+1})}$.

Mihály Bencze

OQ. 3326. If $x_k > 0$ $(k = 1, 2, ..., n)$ and $\alpha > 0$, then
$\sum\limits_{cyclic} \frac{x_1^p+x_2x_3...x_{p+1}}{x_1(x_2+x_3+...+x_{p+1})} + \alpha \frac{\sum\limits_{cyclic} x_1x_2...x_p}{\sum\limits_{k=1}^{n} x_k^p} \geq \frac{2n}{p} + \alpha$, where $p \in \{2, 3, ..., n-1\}$.

Mihály Bencze

OQ. 3327. If $\alpha, x_k \in R$ $(k = 1, 2, ..., n)$ and $\sum\limits_{k=1}^{n} x_k = 0$, then
$\sum\limits_{k=1}^{n} |\sin(\alpha + x_k)| \geq (n-1)\sin 1$.

Mihály Bencze

OQ. 3328. If $x_k > 0$ $(k = 1, 2, ..., n)$ then
$\sum\limits_{cyclic} \frac{x_1}{x_2+x_3} \geq \frac{\sum\limits_{k=1}^{n} x_k^2}{\sum\limits_{cyclic} x_1x_2} + \left(n2^{n-1} - 2^n\right) \frac{\prod\limits_{k=1}^{n} x_k}{\prod\limits_{cyclic}(x_1+x_2)}$.

Mihály Bencze

OQ. 3329. If $x_k > 0$ $(k = 1, 2, ..., n)$ and $S = \sum\limits_{k=1}^{n} x_k$, then
$\sum\limits_{k=1}^{n} \frac{x_k}{\sqrt[r]{S-x_1-x_2-...-x_{n-r}}} \geq \frac{1}{\sqrt[r]{r}} \sum\limits_{k=1}^{n} x_k$, when $r \in \{2, 3, ..., n-1\}$.

Mihály Bencze

OQ. 3330. If $x_k > 0$ $(k = 1, 2, ..., n)$ then

$$\sum_{cyclic} \left(\frac{x_1}{x_2+x_3}\right)^n + \frac{n^2 \prod\limits_{k=1}^{n} x_k}{\prod\limits_{cyclic} (x_1+x_2)} \geq \frac{n+1}{2^{n-1}} \sum_{cyclic} \frac{x_1}{x_2+x_3}.$$

Mihály Bencze

OQ. 3331. Determine all $a > 0$, $\alpha > 0$ for which if $a - \frac{1}{n} < x < a + \frac{1}{n}$, where $n \in N^*$, then $a - \frac{1}{n+k} < \left(a + \left(\underbrace{a + ... + (a+x)^{\alpha}}_{k-time}\right)^{\alpha}\right)^{\alpha} < a + \frac{1}{n+k}$ for all $k \in N^*$.

Mihály Bencze

OQ. 3332. Solve the equation

$$\sigma(n-3)+\sigma(n-2)+\sigma(n-1)+\sigma(n+1)+\sigma(n+2)+\sigma(n+3) = 6\sigma(n+m).$$

Mihály Bencze

OQ. 3333. Determine all functions $f : R \to (0, +\infty)$ such that

$$\sum f(\pm x_1 \pm x_2 \pm ... \pm x_n) \leq 2^n f^n\left(\sqrt[n]{\prod_{k=1}^{n} x_k}\right) \text{ for all } x_k > 0 \ (k = 1, 2, ..., n).$$

Mihály Bencze

OQ. 3334. Let ABC be a triangle. Determine all $n \in N^*$ such that $\cos\frac{A}{n} + \cos\frac{B}{n} + \cos\frac{\pi+C}{n} \geq 1 + \frac{2}{n}$.

Mihály Bencze

OQ. 3335. Solve in N the equation

$$\left[\sum_{d|n} d^r \sum_{k|d} \frac{\left|\Phi(k)-\sigma\left(\frac{d}{k}\right)\right|}{k}\right] = \left[\sum_{d|m} d^p \sum_{k|d} \frac{\left|\sigma(k)-\Phi\left(\frac{d}{k}\right)\right|}{k}\right],$$

where $[\cdot]$ denotethe integer part.

Mihály Bencze

OQ. 3336. Solve in N the equation

$$\sum_{d|n}\sum_{k|d} \Phi^d(k)\,\sigma^k\left(\frac{d}{k}\right) = \sum_{k|m}\sum_{k|d} \sigma^d(k)\,\Phi^k\left(\frac{d}{k}\right).$$

Mihály Bencze

OQ. 3337. Determine all $n, m \in N^*$ for which
$\sum\limits_{d|n} d \sum\limits_{k|d} \frac{\Phi(k)\sigma\left(\frac{d}{k}\right)}{k} + \sum\limits_{d|m} d \sum\limits_{k|d} \frac{\sigma(k)\Phi\left(\frac{d}{k}\right)}{k}$ is a perfect cube.

Mihály Bencze

OQ. 3338. Solve in N the equation
$\left[\sum\limits_{d|n} d \sum\limits_{k|d} \frac{\Phi(k)\sigma\left(\frac{d}{k}\right)}{d(k)}\right] = \left[\sum\limits_{d|m} d \sum\limits_{k|d} \frac{\sigma(k)\Phi\left(\frac{d}{k}\right)}{d(k)}\right]$, where $[\cdot]$ denotethe integer part.

Mihály Bencze

OQ. 3339. We have the following equation $y^n = \frac{n(n+1)}{2} + \sum\limits_{k=1}^{n} x^k$
1). Solve in N 2). Solve in Z 3). Solve in Q

Mihály Bencze

OQ. 3340. Determine all $a_k, b_k, c_k \in R$ $(k = 1, 2, 3)$ such that
$\left(a_1x^2 + b_1x + c_1\right) chx + \left(a_2x^2 + b_2x + c_2\right) shx \geq a_3x^2 + b_3x + c_3$ for all $x \in R$.
I have obtained $a_1 = 1, b_1 = 0, c_1 = 6$, $a_2 = 0, b_2 = -4, c_2 = 0$, $a_3 = b_3 = 0$, $c_3 = 6$.

Mihály Bencze

OQ. 3341. Let ABC be a triangle. Determine the best constants
$x, y, z, t > 0$ for which $x\,(ab + bc + ca) - y\left(a^2 + b^2 + c^2\right) \leq$
$\leq 4\sqrt{3}Area\,[ABC] \leq z\,(ab + bc + ca) - t\left(a^2 + b^2 + c^2\right)$.
I have obtained $x = 6, y = 5, z = 2, t = 1$.

Mihály Bencze

OQ. 3342. Determine all $\alpha > 0$ for which
$n^{\alpha+1} \leq \sum\limits_{k=1}^{n} (2k-1)^{\alpha} \leq n^{\alpha+1}\left(\left(\alpha + \frac{1}{3}\right) n^2 - \left(\alpha - \frac{1}{3}\right)\right)^{\alpha-1}$.
I have obtained $\alpha \in \left\{1, \frac{3}{2}\right\}$.

Mihály Bencze

OQ. 3343. The prime p is called $(n, q)-$ prime if $2^n p + q$ is also prime, where q is a prime.
1). Prove that exist infinitely many $(n, q)-$ prime

2). If $n=1$ and $q=1$, then we obtain the Sophie Germain's prime: $2,3,5,11,23,41,53,83,84,...$
3). Let be $B_\alpha\left(n,q\right)=\frac{1}{p_1^\alpha}+\frac{1}{p_2^\alpha}+...$ where $p_1,p_2,...$ are $(n,q)-$ primes. Compute $B_\alpha\left(n,q\right)$, when $\alpha\geq 1$
4). Prove that $B_\alpha\left(n,q\right)$ is irrational and transcendental.

Mihály Bencze

OQ. 3344. Let be $(3,5),(5,7),(11,13),(17,19),...$ the sequence of twin primes, and $B\left(\alpha\right)=\left(\frac{1}{3^\alpha}+\frac{1}{5^\alpha}\right)+\left(\frac{1}{5^\alpha}+\frac{1}{7^\alpha}\right)+\left(\frac{1}{11^\alpha}+\frac{1}{13^\alpha}\right)+...$

1). Compute $B\left(\alpha\right)$ and prove that is irrational and transcendential
2). If $\alpha=1$, then we obtain the Brun's constant $B\left(1\right)=1,90216054...$
3). Let be $(3,3+2^n),(5,5+2^n),(7,7+2^n),...$ the sequence of 2^n- twin primes, and $B_n\left(\alpha\right)=\left(\frac{1}{3^\alpha}+\frac{1}{3^\alpha+2^n}\right)+\left(\frac{1}{5^\alpha}+\frac{1}{5^\alpha+2^n}\right)+...$
4). Compute $B_n\left(\alpha\right)$ and prove that are irrational and transcendental.

Mihály Bencze

OQ. 3345. Let be $y^n=x^m+k$ the $(n,m)-$ Bachet equation, when $n,m,x,y\in N$ and $k\in Z$. For $n=2$ and $m=3$ we obtain the classical Bachet equation.

1). Determine all arithmetical progression $a_1,a_2,...,a_k\in Z$ for which the equations $y^n=x^m+a_p$ $(p=1,2,...,k)$ have no solutions
2). Prove that exist infinitely many prime p, for which the equation $y^n=x^m+p$ have no solution.
3). Prove that exist infinitely many prime q, for which the equation $y^n=x^m+q$ have solution.

Mihály Bencze

OQ. 3346. Let $y^2=x^3+k$ the Bachet equation.

1). Determine all $k\in Z$ for which the equations $y^2=x^3+k$ and $y^2=x^3-k$ have no solutions in Z.
2). Determine all $a_k\in Z$ $(k=1,2,...,n)$ which are in arithmetical progression and for which the equations $y^2=x^3+a_k$ $(k=1,2,...,n)$ have no solutions. Same question for geometrical progression.
3). Prove that exist infinitely many prime p for which the equation $y^2=x^3+p$ have no solution.

4). Determine all prime q for which the equation $y^2 = x^3 + q$ have solution in Z. Determine all solutions.
5). Exist infinitely many $k \in Z$ for which the equation $y^2 = x^3 + k$ have no solution.

Mihály Bencze

OQ. 3347. 1). The equation $x^2 + y^2 = u^2 + v^2$ have infinitely many solutions in Z. Determine all solutions:
a). in Z b). in N c). in Q
2). The equation $x^3 + y^3 = u^3 + v^3$ have infinitely many solutions in Z, by example: $x = t\left(1 - (a - 3b)\left(a^2 + 3b^2\right)\right)$, $y = t\left((a + 3b)\left(a^2 + 3b^2\right) - 1\right)$,
$u = t\left((a + 3b) - \left(a^2 + 3b\right)^2\right)^2$, $v = t\left(a^2 + 3b^2\right)^2 - (a - 3b)$, where $a, b, t \in Z$. Determine all solutions
a). in Z b). in N c). in Q
3). The equation $x^4 + y^4 = u^4 + v^4$ have infinitely many solutions in Z, by example: $x = t\left(a^7 + a^5b^2 - 2a^3b^4 + 3a^2b^5 + ab^6\right)$,
$y = t\left(a^6b - 3a^5b^2 - 2a^4b^3 + a^2b^5 + b^7\right)$,
$u = t\left(a^7 + a^5b^2 - 2a^3b^4 - 3a^2b^5 + ab^6\right)$,
$v = t\left(a^6b + 3a^5b^2 - 2a^4b^3 + a^2b^5 + b^7\right)$, where $a, b, t \in Z$. Determine all solutions
a). in Z b). in N c). in Q
4). Let be $x^n + y^n = u^n + v^n$, when $n \in Z$. Solve
a). in Z b). in N c). in Q
5). Let be $\sum\limits_{k=1}^{n} x_k^a = \sum\limits_{k=1}^{m} y_k^b$, when $a, b \in Z$. Solve
a). in Z b). in N c). in Q

Mihály Bencze

OQ. 3348. Let be $(c, n) = (d, n) = (e, n) = (f, n) = (cf - ed, n) = 1$, $A = (a_{ij}) \in M_{n \times n}(N)$, $a_{ij} = k$. Determine all functions $g, h : N \times N \to N$ such that $\begin{cases} i \equiv g(n, k) \pmod{n} \\ j \equiv h(n, k) \pmod{n} \end{cases}$ and for which A is a magic square.
A solution is $g(n, k) = ck + e\left[\frac{k}{n}\right]$, $h(n, k) = dk + f\left[\frac{k}{n}\right]$, when $[\cdot]$ denote the integer part, and $c = 1$, $d = e = f = 2$, $n = 3$.

Mihály Bencze

OQ. 3349. Determine all $a_k, b_k, c_k, d_k, e \in R$ $(k = 1, 2, ..., n)$ such that

$\sum\limits_{k=1}^{n} \frac{1}{\Gamma^2(a_kx+b_k)\Gamma^2(c_kx+d_k)} = e\pi^2$. A solution is
$\frac{1}{\Gamma^2(x)\Gamma^2(1-x)} + \frac{1}{\Gamma^2\left(\frac{1}{2}+x\right)\Gamma^2\left(\frac{1}{2}-x\right)} = \pi^2$.

Mihály Bencze

OQ. 3350. 1). Determine all functions $f, g : R \to R$ such that

$\left(\int\limits_0^\infty t^{x-1} f(t)\, dt\right)^2 + \left(\int\limits_0^\infty t^{x-1} g(t)\, dt\right)^2 = \Gamma^2(x)$, where $0 < \mathrm{Re}(x) < 1$ and Γ denote the Euler's gamma function. A solution is $f(x) = \sin x$, $g(x) = \cos x$
2). If $f, g : R \to R$ are solutions of the given equation, then
$f^2(x) + g^2(x) = 1$?

Mihály Bencze

OQ. 3351. Determine all functions $f, g : R \times R \to R$ such that

$\sum\limits_{k=0}^{\infty} \int\limits_a^b f(x,k)\, g(x,\alpha)\, dx = \zeta(\alpha + c)\,\Gamma(\alpha + c)$, when ζ denote the Riemann zeta function and Γ the Euler's gamma function. Two solutions are:
$\sum\limits_{k=0}^{\infty} \int\limits_0^\infty x^{\alpha-1} e^{-(k+1)x} dx = \zeta(\alpha)\,\Gamma(\alpha)$ and
$\sum\limits_{k=0}^{\infty} \int\limits_0^1 x^k \left(\ln \frac{1}{x}\right)^\alpha dx = \zeta(\alpha+1)\,\Gamma(\alpha+1)$.

Mihály Bencze

OQ. 3352. 1). Determine all $a, b, c, d, e \in R$ such that $\int\limits_0^\infty \frac{x^a+x^b+x^c+x^d}{x+1} dx = e$.
A solution is $a = -\frac{1}{2}$, $b = -\frac{2}{3}$, $c = -\frac{5}{6}$, $d = -\frac{3}{4}$, $e = 3 + \sqrt{2} + \frac{2\sqrt{3}}{3}$

2). Determine all $a_k \in R$ $(k = 1, 2, ..., n + 1)$ such that $\int\limits_0^\infty \frac{1}{x+1} \sum\limits_{k=1}^{n} x^{a_k} = a_{n+1}$.

3). Exist $a_1, a_2, ..., a_{n+1}$ in arithmetical progression?

Mihály Bencze

OQ. 3353. Let be $\pi F(n) = \int\limits_0^\infty \left(\frac{\sin x}{x}\right)^n dx$, where $n \in Z$.

1). Prove that $F(3) = \frac{3}{8}$ and $F(4) = \frac{1}{3}$
2). Deterine $F(n)$ in function of $n \in N$
3). Compute $\sum\limits_{n=1}^{\infty} F^{\alpha}(n)$, when $\alpha \geq 1$
4). Compute $F(-n)$ in function of $n \in N$.

Mihály Bencze

OQ. 3354. Determine all functions $f, g, h : R \to R$ for which $\int\limits_a^b \ln f(x) \ln g(x)\, dx + \int\limits_c^d \ln f(x) \ln h(x)\, dx = e + f\pi^2$. A solution is $f(x) = x$, $g(x) = 1 + x$, $h(x) = 1 - x$, $a = c = 0$, $b = d = 1$, $e = 4 - 2\ln 2$, $f = -\frac{1}{4}$.

Mihály Bencze

OQ. 3355. Determine all functions $f, g : R \to R$ for which $\int\limits_a^b \left(\frac{\ln f(x)}{g(x)} + \frac{\ln g(x)}{f(x)}\right) dx = 0$. Two solutions are:

1). $f(x) = x, g(x) = x + 1, a = 0, b = 1$
2).. $f(x) = x, g(x) = 1 - x, a = 0, b = 1$

Mihály Bencze

Solution of the OQ. 2283

Róbert Szász and Aurel Pál Kupán[41]

ABSTRACT. We determine the set of α, for which the weighted Hölder mean $H_\alpha(a,b)$ is between the $L(a,b)$ Pólya-Szegõ logarithmic and the $I(a,b)$ exponential mean. Other results concerning this means are proved.

1. INTRODUCTION

Let $0 < a < b$. The generalized Hölder mean is

$$Q_\alpha(a,b) = \left(\frac{a^{\frac{1}{\alpha}} + b^{\frac{1}{\alpha}}}{2}\right)^\alpha.$$

The Pólya &Szegő logarithmic mean and the exponential mean are defined by

$$L(a,b) = \frac{b-a}{\ln b - \ln a}, \quad I(a,b) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}.$$

In [4] the authors proved that, the inequality holds:

$$L(a,b) < I(a,b), a, b \in (0,\infty).$$

Other results concerning these means were deduced in [1]. An exhaustive treatment of the topic can be found in [3].

The author of [2] proposed the following open question: determine the values $\alpha \in [2,\infty)$ for which the inequalities hold

$$L(a,b) \le Q_\alpha(a,b) \le I(a,b) \quad \text{for all} \quad a, b \in (0,\infty), \ a < b.$$

The aim of this paper is to determine the desired set of α and to deduce some other inequalities concerning these means.

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2. PRELIMINARIES

We will need in our work the following results.

Lemma 1. Let $\alpha \in (0,2]$ be a fixed number.If $s \in (\frac{1}{2},1)$ then the inequality

$$2^\alpha(s^\alpha(1-s)+(1-s)^\alpha s)<1$$

holds.
Proof. Let $g_1 : [\frac{1}{2},1) \to \mathbb{R}$ be the function defined by the equality:

$$g_1(s)=2^\alpha(s^\alpha(1-s)+(1-s)^\alpha s).$$

By differentiation,

$$g_1'(s)=2^\alpha[\alpha s(1-s)(s^{\alpha-2}-(1-s)^{\alpha-2})+(1-s)^\alpha-s^\alpha].$$

Since $s^{\alpha-2}-(1-s)^{\alpha-2}<0$ and $(1-s)^\alpha-s^\alpha<0$ for $s \in (\frac{1}{2},1)$, shows $g_1'(s)<0$ for all $s \in (\frac{1}{2},1)$.
Thus the function g_1 is strictly decreasing and the inequality

$$g_1(s)<g_1(\frac{1}{2})=1,\ s\in(\frac{1}{2},1)$$

follows.
Lemma 2. If $s \in (\frac{1}{2},1)$ is a fixed number, then the function $g_2 : [2,3] \to R$ defined by

$$g_2(\alpha)=2^\alpha(s^\alpha(1-s)+(1-s)^\alpha s)$$

is strictly increasing.
Proof. The expression of g_2 can be rewritten as follows:

$$g_2(\alpha)=2s(1-s)[(2s)^{\alpha-1}+\big(2(1-s)\big)^{\alpha-1}].$$

We get by differentiation:

$$g_2'(\alpha)=2s(1-s)[(2s)^{\alpha-1}\ln 2s+\big(2(1-s)\big)^{\alpha-1}\ln\big(2(1-s)\big)].$$

Since $2s \in (1,2)$, $2(1-s) \in (0,1)$ and $\alpha-1 \in [1,2]$, follows that

$$g_2'(\alpha)\geq 2s(1-s)[2s\ln 2s+2(1-s)\ln\big(2(1-s)\big)].$$

On the other hand for the derivative of the function
$g_3 : [\frac{1}{2},1) \to \mathbb{R},\ g_3(s)=2s\ln 2s+2(1-s)\ln\big(2(1-s)\big)$ the following inequality holds:

$$g_3'(s)=\ln\frac{s}{1-s}>0,\ \text{for}\quad \text{all}\ \ s\in(\frac{1}{2},1).$$

Therefore g_3 is strictly increasing and $g_3(s) > g_3(\frac{1}{2}) = 0$ for all $s \in (\frac{1}{2}, 1)$. Thus $g_2'(\alpha) > 0$, for all $\alpha \in (2,3)$ and the assertion holds.

Lemma 3. [5]. II.2. Let n be a natural number $n \geq 2$, and $a_k \in (0,\infty)$, $k = \overline{1,n}$ be real numbers, not all equal. The function $h : (0,\infty) \to R$ defined by the equality:

$$h(\alpha) = \left(\frac{1}{n}\sum_{k=1}^{n} a_k^{\frac{1}{\alpha}}\right)^{\alpha}$$

is strictly decreasing.

3. THE MAIN RESULT

Theorem 1. Let $\alpha \in (0,3]$ be a fixed number. For all $a, b \in (0,\infty)$, $a < b$ the following inequality holds:

$$L(a,b) < Q_\alpha(a,b). \tag{1}$$

Proof. Let $x = \frac{b}{a}$, obviously $x \in (1,\infty)$. The inequality (1) is equivalent to

$$\frac{x-1}{\ln x} < \left(\frac{1+x^{\frac{1}{\alpha}}}{2}\right)^{\alpha}, x \in (1,\infty).$$

We let $s = \frac{x^{\frac{1}{\alpha}}}{1+x^{\frac{1}{\alpha}}}$ and this leads to the next equivalent form of (1):

$$\alpha \ln\frac{s}{1-s} - [(2s)^\alpha - \big(2(1-s)\big)^\alpha] > 0, \quad s \in (\frac{1}{2}, 1). \tag{2}$$

Consequently we have to study the function

$$f_1 : [\frac{1}{2}, 1) \to \mathbb{R},\ f_1(s) = \alpha \ln\frac{s}{1-s} - [(2s)^\alpha - \big(2(1-s)\big)^\alpha].$$

We mark out two cases.
We assume first $\alpha \in [0,2]$. Since

$$f_1'(s) = \frac{\alpha}{s(1-s)}[1 - 2^\alpha\big(s^\alpha(1-s) + s(1-s)^\alpha\big)],$$

Lemma 1 implies $f_1'(s) > 0, s \in (\frac{1}{2}, 1)$. Thus f_1 is strictly increasing, and the desired inequality follows: $f_1(s) > f_1(\frac{1}{2}) = 0,\ s \in [\frac{1}{2}, 1)$.
The second case is $\alpha \in [2,3]$.

The equality $f_1'(s) = \frac{\alpha}{s(1-s)}[1 - g_1(s)],\ s \in (\frac{1}{2}, 1)$ and Lemma 3 imply, that it is sufficient to prove

$$f_1'(s) > 0, \text{ for all } s \in (\frac{1}{2}, 1)$$

in case if $\alpha = 3$, and then the inequality follows for every $\alpha \in [2, 3]$.
In case if $\alpha = 3$, we have

$$f_1'(s) = \frac{3}{s(1-s)}[1 - 2^3\big(s^3(1-s) + s(1-s)^3\big)]$$

and

$$g_1(s) = 2^3\big(s^3(1-s) + s(1-s)^3\big).$$

Since $g_1'(s) = 8(1-2s)^3 < 0,\ s \in (\frac{1}{2}, 1)$, it follows that g_1 is a decreasing mapping on $(\frac{1}{2}, 1)$. Thus $g_1(s) < g_1(\frac{1}{2}) = 1$ for all $s \in (\frac{1}{2}, 1)$. Hence

$$f_1'(s) > 0, \text{ for all } s \in (\frac{1}{2}, 1) \text{ and } \alpha \in [2, 3].$$

Consequently $f_1(s) > f_1(\frac{1}{2})$ for $s \in (\frac{1}{2}, 1)$, and (2) holds in this case too.

Remark 1. *If* $\alpha > 3$ *then the inequality*

$$L(a, b) \le Q_\alpha(a, b) \tag{1}$$

does not hold for every $0 < a < b$.
Proof. We have to prove that (2) does not hold provided $\alpha > 3$. Let g_1 be the function defined in the proof of Lemma 1. Since $g_1''(\frac{1}{2}) = \alpha(\alpha - 3) > 0$ the continuity of g_1'' implies the existence of a real number $\varepsilon > 0$ so that $g_1''(s) > 0$ for all $s \in [\frac{1}{2}, \frac{1}{2} + \varepsilon)$. Hence g_1' is strictly increasing on $[\frac{1}{2}, \frac{1}{2} + \varepsilon)$. Thus $g_1'(s) > g_1'(\frac{1}{2}) = 0$ for all $s \in (\frac{1}{2}, \frac{1}{2} + \varepsilon)$. Thus we get that g_1 is a strictly increasing mapping on $[\frac{1}{2}, \frac{1}{2} + \varepsilon)$ and $g_1(s) > g_1(\frac{1}{2}) = 1,\ s \in (\frac{1}{2}, \frac{1}{2} + \varepsilon)$. This leads to

$$f_1'(s) < 0,\ s \in (\frac{1}{2}, \frac{1}{2} + \varepsilon) \text{ and } f_1(s) < f_1(\frac{1}{2}) = 0,\ s \in (\frac{1}{2}, \frac{1}{2} + \varepsilon).$$

Hence (2) cannot be true for every $s \in (\frac{1}{2}, 1)$.

Theorem 2. If $\alpha \in [\frac{3}{2}, \infty)$ then the following inequality holds

$$Q_\alpha(a, b) < I(a, b) \text{ for all } a, b \in (0, \infty),\ a < b. \tag{3}$$

Proof. According to Lemma 3 we have to prove (3) only in case if $\alpha = \frac{3}{2}$. Using the notation $x = \frac{b}{a}$ the inequality

$$\left(\frac{a^{\frac{2}{3}} + b^{\frac{2}{3}}}{2}\right)^{\frac{3}{2}} < \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}, \quad a, b \in (0, \infty),\ a < b$$

is equivalent to

$$\frac{x}{x-1}\ln x - \frac{3}{2}\ln\left(\frac{1 + x^{\frac{2}{3}}}{2}\right) - 1 > 0, \quad x \in (1, \infty). \tag{4}$$

Let

$$f_2 : [1, \infty) \to \mathbb{R}, \quad f_2(x) = \frac{x}{x-1}\ln x - \frac{3}{2}\ln\left(\frac{1 + x^{\frac{2}{3}}}{2}\right) - 1.$$

By differentiation

$$f_2'(x) = \frac{1}{(x-1)^2}\left[-\ln x + x - 1 - (x-1)^2\frac{1}{x + x^{\frac{1}{3}}}\right].$$

Let

$$u : [1\infty) \to \mathbb{R}, \quad u(x) = -\ln x + x - 1 - (x-1)^2\frac{1}{x + x^{\frac{1}{3}}}.$$

Since

$$u'(x) = \frac{(x-1)(x^{\frac{1}{3}} - 1)^3}{3x^{\frac{2}{3}}(x + x^{\frac{1}{3}})^2}$$

is positive for every $x \in (1, \infty)$, it shows that u is strictly increasing on $[1, \infty)$. Therefore we have $u(x) > u(1) = 0, \ \ x \in (1, \infty)$. Thus $f_2'(x)$ is positive for every $x \in (1, \infty)$ and so f_2 is strictly increasing on $[1, \infty)$ which means that the inequality

$$f_2(x) > f_2(1) = 0, \quad x \in (1, \infty)$$

holds, and this is equivalent to (4).

Remark 2. *If* $0 < \alpha < \frac{3}{2}$ *then the inequality*

$$Q_\alpha(a, b) \le I(a, b) \tag{5}$$

does not hold for all $a, b \in (0, \infty),\ a < b$.

Proof. Inequality (5) is equivalent to:

$$x\ln x-\alpha(x-1)\ln\left(\frac{1+x^{\frac{1}{\alpha}}}{2}\right)-x+1\geq 0,\quad x\in(1,\infty). \tag{6}$$

Suppose $0<\alpha<\frac{3}{2}$. The derivatives of the function $f_3:[1,\infty)\to\mathbb{R}$ defined by

$$f_3(x)=x\ln x-\alpha(x-1)\ln\left(\frac{1+x^{\frac{1}{\alpha}}}{2}\right)-x+1$$

are:

$$f_3'(x)=\ln x-(x-1)\frac{x^{\frac{1}{\alpha}-1}}{1+x^{\frac{1}{\alpha}}}-\alpha\ln\left(\frac{1+x^{\frac{1}{\alpha}}}{2}\right),$$

$$f_3''(x)=\frac{1-x^{\frac{1}{\alpha}}}{x(1+x^{\frac{1}{\alpha}})}+(1-x)\frac{(\frac{1}{\alpha}-1)x^{\frac{1}{\alpha}-2}-x^{\frac{2}{\alpha}-2}}{(1+x^{\frac{1}{\alpha}})^2}.$$

Since

$$\lim_{x\searrow 1}\frac{f_3''(x)}{1-x}=\frac{3-2\alpha}{4\alpha}>0$$

there exists a positive real number $\varepsilon>0$, so that $f_3''(x)<0,\ x\in(1,1+\varepsilon)$. Thus f_3' is decreasing on $(1,1+\varepsilon)$ and $f_3'(x)<f_3'(1)=0,\ x\in(1,1+\varepsilon)$.

Therefore f_3 is also decreasing on $(1,1+\varepsilon)$ and it follows that $f_3(x)<f_3(1)=0,\ x\in(1,1+\varepsilon)$, and this inequality is in contradiction with (6).

Remark 3. *If we denote* $x=\frac{b}{a}$*, then the inequality*

$$L(a,b)\geq Q_\alpha(a,b).\ \ a,b\in(0,\infty),a<b \tag{7}$$

is equivalent to

$$\frac{1}{\ln x}-\left(\frac{1+x^{\frac{1}{\alpha}}}{2(x-1)^{\frac{1}{\alpha}}}\right)^{\alpha}\geq 0,\quad x\in(1,\infty).$$

But

$$\lim_{x\to\infty}\frac{1}{\ln x}-\left(\frac{1+x^{\frac{1}{\alpha}}}{2(x-1)^{\frac{1}{\alpha}}}\right)^{\alpha}=-1$$

for every $\alpha \in (0, \infty)$ fixed number. This means that inequality (7) cannot be true for any $\alpha \in (0, \infty)$.

Conclusions

1. The inequalities

$$L(a,b) < Q_\alpha(a,b) < I(a,b), \quad \text{hold for all } a,b \in (0,\infty), a < b$$

if and only if $\alpha \in [\frac{3}{2}, 3]$.

2. Remark 1 shows that there is no $\alpha \in (0, \infty)$ so that

$$Q_\alpha(a,b) \leq L(a,b), \quad \text{for all } a,b \in (0,\infty), \ a < b.$$

Remark 4. A more general version of Theorem 1 and Theorem 2 can be found [6].

REFERENCES

[1] Anisiu Valeriu and Anisiu Mira Cristina, *Refinement of Some Inequalities for Means*, Revue D'Analyse et de Theorie de L'Approximation, Tome 35, No.1, 2006, pp.5-10

[2] Bencze Mihály, *Open Question no 2283.*Octogon Mathematical Magazine, Vol.14. No. 2. October 2006.

[3] Bullen P.S., *Handbook of Means and Their Inequalities*, Series:Mathematics and Its Applications, vol.560, 2nd ed., Kluwer Academic Publishers Group, Dordrecht, 2003

[4] Ivan, M. and Raşa, I., *Some Inequalities for Means*, Seminar of Functional Equations, Approximation and Convexity, Cluj-Napoca, May 23-29,2000,pp.99-102

[5] Pólya, Gy.,Szegő, G., *Aufgaben und Lehrsätze aus der Analysis* Springer Verlag. 1924

[6] Edward Neuman, *A generalization of an inequality of JIA and CAU* Journal of Inequalities in Pure and Applied Mathematics,Vol.5,Issue 1,

Article 15, 2004.

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A conjecture on a number theoretical function and the OQ. 1240

Kramer Alpár-Vajk[42]

In a paper of Subramanian and Bencze, see [1] and in the OQ 1240, see [2] a conjecture regarding a number theoretical function is formulated. Let be $f : \mathbf{N} \to \mathbf{N}$ defined as $\forall x \in \mathbf{N}, f(x) :=$ the smallest positive integer k such that $kx^2 + 1$ is a perfect square. For certain properties of f and for the fact that is well defined see [1].

Although there are some confusions in the notations, in both [1] and [2], it is tempting to believe that the same conjecture is meant, namely that if $(p, p+2)$ is a twin prime pair, then

$$f\left(p \cdot (p+2)\right) = \left(\frac{p+1}{2}\right)^2 - 1. \tag{1}$$

For the twin prime pair $(3, 5)$ we have $f(15) = 3 = \left(\frac{3+1}{2}\right)^2 - 1$.

Similarly, for $(5, 7)$ we obtain $f(35) = 8 = \left(\frac{5+1}{2}\right)^2 - 1$.

We want to find, for a given twin prime pair $(p, p+2)$, the smallest positive

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integer k for which exists $a \in \mathbf{N}$ such that

$$k \cdot p^2 \cdot (p+2)^2 + 1 = a^2.$$

The equation is obviously equivalent to

$$k = \frac{(a-1)(a+1)}{p^2 \cdot (p+2)^2}. \tag{2}$$

Now, p and $p+2$ are both prime numbers. It follows that each of them divides $a-1$ or $a+1$. Moreover, since $p > 2$ and consequently $p+2 > 2$, none of them divides both, $a-1$ and $a+1$.

This leads to the following cases :

- p^2 is divisor of $a-1$ and $(p+2)^2$ is divisor of $a+1$;
- p^2 is divisor of $a+1$ and $(p+2)^2$ is divisor of $a-1$;
- $p^2 \cdot (p+2)^2$ is a divisor of $a-1$;
- $p^2 \cdot (p+2)^2$ is a divisor of $a+1$;

Case 1. p^2 is divisor of $a-1$ and $(p+2)^2$ is divisor of $a+1$.
This means that there exist $u, v \in \mathbf{N}$ such that

$$a - 1 = p^2 \cdot u \qquad \text{and} \qquad a + 1 = (p+2)^2 \cdot v.$$

Expliciting in both identities a, we obtain

$$p^2 \cdot u + 2 = (p+2)^2 \cdot v. \tag{3}$$

We consider this equation, with unknown u, in the ring $Z_{(p+2)^2}$,

$$\hat{p^2} \cdot \hat{u} + \hat{2} \equiv \hat{0} \left(\bmod\ (p+2)^2\right). \tag{4}$$

Since $\gcd(p, p+2) = 1, \hat{p}$ is invertible in $Z_{(p+2)^2}$, and since his inverse is unique, equation (4) has a unique solution

$$\hat{u} = -(\hat{2}) \cdot (\hat{p^2})^{-1}$$

in $Z_{(p+2)^2}$.

Assuming $\hat{u}$ as solution of (4) it follows that u is the smallest solution for (3) and the resulting v is the smallest v too satisfying (3).

Now we will show the existence of the solution.

It is subject of a simple calculus that

$$p^2 \cdot \left(\frac{2p^2 + 7p + 5}{2}\right) + 2 = (p+2)^2 \cdot \left(\frac{2p^2 - p + 1}{2}\right).$$

On the other hand

$$\frac{2p^2+7p+5}{2} < p^2+4p+4 = (p+2)^2,$$

thus

$$u = \frac{2p^2+7p+5}{2}.$$

As a consequence,

$$v = \frac{2p^2-p+1}{2} < p^2.$$

Finally, we found $k \in \mathbf{N}$ satisfying the requirements namely

$$k = u \cdot v = \left(\frac{2p^2+7p+5}{2}\right) \cdot \left(\frac{2p^2-p+1}{2}\right).$$

Case 2. p^2 is divisor of $a+1$ and $(p+2)^2$ is divisor of $a-1$.
There exist $u, v \in \mathbf{N}$ such that

$$a+1 = p^2 \cdot u \qquad \text{and} \qquad a-1 = (p+2)^2 \cdot v.$$

Expliciting in both identities a, we obtain

$$p^2 \cdot u - 2 = (p+2)^2 \cdot v.$$

Moving again into the ring $Z_{(p+2)^2}$, we obtain the equation

$$\hat{p^2} \cdot \hat{u} - \hat{2} \equiv \hat{0} \left(\text{mod } (p+2)^2\right),$$

having as unique solution in $Z_{(p+2)^2}$

$$\hat{u} = \hat{2} \cdot (\hat{p^2})^{-1}.$$

Further, the following identity is immediate :

$$p^2 \cdot \left(\frac{p+3}{2}\right) - 2 = (p+2)^2 \cdot \left(\frac{p-1}{2}\right).$$

Since

$$\frac{p-1}{2} < \frac{p+3}{2} < (p+2)^2$$

we obtain

$$u = \frac{p+3}{2} \qquad \text{and} \qquad v = \frac{p-1}{2}.$$

Finally, we found $k \in \mathbf{N}$ which satisfies the requirements, namely

$$k = u \cdot v = \left(\frac{p+3}{2}\right) \cdot \left(\frac{p-1}{2}\right) = \left(\frac{p+1}{2}\right)^2 - 1.$$

Case 3. $p^2 \cdot (p+2)^2$ is a divisor of $a-1$.
Thus there exist $u \in \mathbf{N}$ such that

$$p^2 \cdot (p+2)^2 \cdot u = a - 1.$$

From this relation and from (2) we get that

$$k = u \cdot (a+1) = u \cdot \left(p^2 \cdot (p+2)^2 \cdot u + 2\right).$$

Case 4. $p^2 \cdot (p+2)^2$ is a divisor of $a+1$.
Thus there exist $u \in \mathbf{N}$ such that

$$p^2 \cdot (p+2)^2 \cdot u = a + 1.$$

From this relation and from (2) we get that

$$k = u \cdot (a-1) = u \cdot \left(p^2 \cdot (p+2)^2 \cdot u - 2\right).$$

It is clear that the smallest k among the four cases is the one we look for and since this is

$$\left(\frac{p+1}{2}\right)^2 - 1$$

the conjecture is solved in the affirmative.
In [1] another "function" is defined namely $g : \mathbf{N} \to \mathbf{N}$, where
$\forall x \in \mathbf{N}, g(x) :=$the smallest positive integer k such that $kx^2 - 1$ is a perfect square.

Note that for $x := 3$ we do not have a $k \in \mathbf{N}$ such that $9k - 1$ is a perfect square. Why ? Because if it would be like this it would exist a perfect square, say y^2, such that $y^2 + 1$ is divisible by 9.

This implies that $y^2 + 1$ is divisible by 3 and this is a contradiction.

The same situation is valid for $x := 7$. Thus g cannot be defined on the whole $\mathbf{N}$.

REFERENCES

[1] Subramanian, K. B. and Bencze, M., *On Two Number Theoretic Functions*, Octogon Mathematical Magazine, April 2003.
[2] Subramanian, K. B., *OQ 1240*, Octogon Mathematical Magazine, April 2003.

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The solution of OQ 1156

Kramer Alpár-Vajk[43]

In [1] the following sequence $(a_n)_{n\in N}$ is defined : $a_1 := 3$ and further, for all $n \in N$; $n \geq 2$; $a_n :=$ the smallest number with a_{n-1} divisors.

According to the author of [1] the first six terms are 3; 4; 6; 12; 72; 559872: It is conjectured that $\forall n \in N^*$; $a_n + 1$ is prime.

The first observation is that the fifth term above is false because not 72 is the smallest number with 12 divisors but 60: In light of this, the next term is 5040 and since
$5041 = 71 \cdot 71$ the conjecture is wrong.

REFERENCE

[1] Amarnath Murthy, *OQ 1156*, Octogon Mathematical Magazine, April 2003.

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The solution of OQ 1141

Kramer Alpár-Vajk[44]

The subject of OQ 1141, see [1] is to prove that the sequence $(a_n)_{n\in N}$ is finite.
This sequence is defined in the following way: $a_1 := 1$; and $\forall n \in N$; $a_n :=$ the smallest natural number such that for all $k \in N$; $k < n$; $a_n - a_k$ is a prime or a power of a prime. We have the first six terms : 1,3,5,8,10,12. We will show that there is no other term in this sequence, supposing the opposite and distinguishing two cases.

Case 1. Suppose that x_7 exists and is even. Then

$$\{x_7 - 8, x_7 - 10, x_7 - 12\}$$

are all even and in the same time one of them is divisible by 3, thus divisible by 6 and so neither a prime nor a power of a prime.

Case 2. Suppose that x_7 exists and is odd. Then

$$\{x_7 - 1, x_7 - 3, x_7 - 5\}$$

are all even and in the same time one of them is divisible by 3, thus divisible by 6 and
so neither a prime nor a power of a prime.

REFERENCE

[1] Amarnath Murthy, *OQ 1141*, Octogon Mathematical Magazine, April 2003.

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A logarithmic equation (OQ 19)

Gabriel T. Prăjitura and Tsvetomira Radeva[45]

ABSTRACT. We gave a solution to the Open Question 19.

MAIN RESULT

The Open Question 19 ([1]) asked for all n such that

$$[\log_2 3 + \log_3 4 + ... + \log_n (n+1)] = n+1$$

Equivalently, we are looking for all n such that

$$n+1 \leq \log_2 3 + \log_3 4 + ... + \log_n (n+1) < n+2$$

Let p be a natural number. Notice first that if

$$\log_2 3 + \log_3 4 + ... + \log_{n_0} (n_0+1) < n_o + p$$

then

$$\log_2 3 + \log_3 4 + ... + \log_n (n+1) < n + p$$

for all $n \leq n_0$, while if

$$\log_2 3 + \log_3 4 + ... + \log_{n_0} (n_0+1) \geq n_0 + p$$

then

$$\log_2 3 + \log_3 4 + ... + \log_n (n+1) \geq n + p$$

for all $n \geq n_0$.
This is because

$$\sum_{k=2}^{n+1} \log_k (k+1) - \sum_{k=2}^{n} \log_k (k+1) = \log_{n+1} (n+2) > 1 = (n+1+p) - (n+p)$$

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Therefore, in order to solve the double inequality above we need to find two numbers n_1 and n_2 such that $n_1 < n_2$ and

$$\log_2 3 + \log_3 4 + ... + \log_{n_1} (n_1 + 1) \geq n_1 + 1$$

$$\log_2 3 + \log_3 4 + ... + \log_{n_1 - 1} (n_1) < n_1$$

$$\log_2 3 + \log_3 4 + ... + \log_{n_2} (n_2 + 1) < n_2 + 2$$

$$\log_2 3 + \log_3 4 + ... + \log_{n_2 + 1} (n_2 + 2) \geq n_2 + 3$$

When the two numbers are found, the solution is $n_1 \leq n \leq n_2$.
Next we will show that $n_1 = 70$. We must show that

$$\sum_{k=2}^{69} \log_k (k+1) < 70$$

and

$$\sum_{k=2}^{70} \log_k (k+1) > 71$$

which follows easily from the computation

$$\sum_{k=2}^{69} \log_k (k+1) = 69.998 \text{ and } \sum_{k=2}^{70} \log_k (k+1) = 71.001$$

Now we will show that . We must show that

$$\sum_{k=2}^{105,555} \log_k (k+1) < 105,557$$

and

$$\sum_{k=2}^{105,556} \log_k (k+1) < 105,558$$

which follows easily from the computation

$$\sum_{k=2}^{105,555} \log_k (k+1) < 105,556.99999955755$$

and

$$\sum_{k=2}^{105,556} \log_k (k+1) = 105,558.00000037657$$

Therefore

$$[\log_2 3 + \log_3 4 + ... + \log_n (n+1)] = n+1$$

if and only if $70 \leq n \leq 105,555$.
We will end with some coments about the problem.
The series

$$\sum_{n=2}^{\infty} (\log_n (n+1) - 1)$$

is divergent. To see this notice that

$$\log_n (n+1) - 1 = \frac{\ln (n+1)}{\ln n} - 1 = \frac{\ln (n+1) - \ln n}{\ln n}$$

By the Mean Value Theorem applied to the function ln x on the interval $[n, n+1]$, there is $k_n \in (n, n+1)$ such that

$$\ln (n+1) - \ln n = \frac{\ln (n+1) - \ln n}{(n+1) - n} = \frac{1}{k_n} > \frac{1}{n+1}$$

Therefore

$$\log_n (n+1) - 1 > \frac{1}{(n+1) \ln n}$$

and since

$$\sum_{n=1}^{\infty} \frac{1}{(n+1) \ln n}$$

is a well known divergent series, the Comparison Test implies the divergence of the series we considered above. One of the consequences of this fact is that for every $k \geq 1$ the equation

$$[\log_2 3 + \log_3 4 + ... + \log_n (n+1)] = n + k$$

has only a finite number of solutions. From our computation here it actually follows that there are solutions for every $k \geq 0$. To find the exact number of these solutions turns out to be a very difficult technical problem since, as we showed above, for k = 1 we already need 7 decimals of accuracy.

REFERENCE

[1] Bencze, M., *Open Question 19*, Octogon Mathematical Magazine , Vol 3, nr 1, 1995.

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[1] Rudin, W., *Function Theory in the Unit Ball of* , Springer Verlag, New York.
[2] Kershaw, D., *Some extensions of W. Gautschi's inequality for the gamma function,* Math. Comp. 41(1983), 607-611.
[3] Kečlič, J.D. and Vasić, P.M., *Some inequality for the gamma function ,* Publ. Inst. Math. Beograd N.S. 11(1983), 607-611.

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