Mathproblems
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PROBLEMS AND SOLUTIONS

Proposals and solutions must be legible and should appear on separate sheets, each indicating the name of the sender. Drawings must be suitable for reproduction. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editors have supplied a solution. The editors encourage undergraduate and pre-college students to submit solutions. Teachers can help by assisting their students in submitting solutions. Student solutions should include the class and school name. Solutions will be evaluated for publication by a committee of professors according to a combination of criteria. Questions concerning proposals and/or solutions can be sent by e-mail to: *mathproblems-ks@hotmail.com*

Solutions to the problems stated in this issue should arrive before 2 June 2012

Problems

36. *Proposed by Anastasios Kotronis, Athens, Greece.* Evaluate the sum

$$\sum_{n=1}^{+\infty} n\left(2^{-1/2} - 1 + \binom{1/2}{1}\frac{1}{2} - \binom{1/2}{2}\frac{1}{4} + \cdots + (-1)^{n+1}\binom{1/2}{n}\frac{1}{2^n}\right)$$

37. *Proposed by Mihály Bencze, Braşov, Romania.* If $A, B \in M_2(\mathbb{R})$ then prove that

$$2\left(\det A\right)^2 + \det(AB + BA) + 2\left(\det B\right)^2 \geq \det\left(A^2 - B^2\right) + 4\det AB$$

38. *Proposed by Florin Stanescu, School Cioculescu Serban, Gaesti, jud. Dambovita, Romania.* Determine all functions $f : [0,1] \to \mathbb{R}$ that have the following properties

a): f is three times differentiable with $f'''(x) \geq 0, \forall x \in [0,1]$;
b): f' is increasing and strictly positive;
c): $f'(1)\Big(2\left(f(1) - f(0)\right) - f'(1)\Big) \int_0^1 \frac{dx}{(f'(x))^2} = 1$

39. *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.* If A, B and C are the angles of a triangle, then prove that:

$$\frac{\sin^k A + \sin^k B + \sin^k C}{A + B + C} \leq \frac{1}{3}\left(\frac{\sin^k A}{A} + \frac{\sin^k B}{B} + \frac{\sin^k C}{C}\right),$$

where $k \in (0, 1]$.

40. *Proposed by Enkel Hysnelaj, University of Technology, Sydney, Australia.* Find all distinct positive integers x, y, z, t all greater than 2, such that

$$\frac{x^3}{x-1} + \frac{y^3}{y-1} + \frac{z^3}{z-1} + \frac{t^3}{t-1}$$

is an integer.

41. *Proposed by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Ital.* Evaluate

$$\int_0^{\pi/2} 4(\cos x)^2 \left(\ln(\cos x)\right)^2 dx$$

42. *Proposed by Cristinel Mortici, Valahia University of Târgovişte, Romania.* Let $f : \mathbb{R} \to \mathbb{R}$ be monotone such that $f + f \circ f \circ f$ is continuous. Prove that f is continuous.

Solutions

No problem is ever permanently closed. We will be very pleased considering for publication new solutions or comments on the past problems.

29. *Proposed by Enkel Hysnelaj, University of Technology, Sydney, Australia.*
1) Let $f : (0, \infty) \to \mathbb{R}$ be a function that satisfies the property

$$\frac{f(x^2) + f(y^2)}{2} = f(xy)$$

for any $(x, y) \in (0, \infty)$. Show that

$$\frac{f(x^3) + f(y^3) + f(z^3)}{3} = f(xyz)$$

for any $(x, y, z) \in (0, \infty)$.
2) Generalize the above statement, so show that if

$$\frac{f(x_1^2) + f(x_2^2)}{2} = f(x_1 x_2)$$

for any $(x_1, x_2) \in (0, \infty)$, then

$$\frac{f(x_1^n) + f(x_2^n) + \ldots + f(x_n^n)}{n} = f(x_1 x_2 \ldots x_n)$$

for any $(x_1, x_2, \ldots, x_n) \in (0, \infty)$ and n a positive integer greater than 1.

Solution 1 by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia. 1) On account of the condition in the statement, we have

$$\begin{aligned}
&\frac{f(x^3) + f(y^3) + f(z^3) + f(xyz)}{4} \\
&= \frac{1}{2}\left(\frac{f\left((x^{\frac{3}{2}})^2\right) + f\left((y^{\frac{3}{2}})^2\right)}{2} + \frac{f\left((z^{\frac{3}{2}}\right)^2) + f\left(((xyz)^{\frac{3}{2}})^2\right)}{2}\right) \\
&= \frac{1}{2}\left(f\left((xy)^{\frac{3}{2}}\right) + f\left(z^{\frac{3}{2}}(xyz)^{\frac{1}{2}}\right)\right) \\
&= \frac{1}{2}\left(f\left(((xy)^{\frac{3}{4}})^2\right) + f\left((z^{\frac{3}{4}}(xyz)^{\frac{1}{4}})^2\right)\right) \\
&= f\left((xy)^{\frac{3}{4}} \cdot z^{\frac{3}{4}} \cdot (xyz)^{\frac{1}{4}}\right) = f(xyz)
\end{aligned}$$

Hence,

$$\frac{f(x^3) + f(y^3) + f(z^3)}{3} = f(xyz)$$

2) First, we prove the statement for $n = 2^k$ arguing by mathematical induction. Indeed, when $n = 2$ it trivially holds. Now suppose that for $n = 2^k$ is true and

then we prove it for $n = 2^{k+1}$. By the given condition, we have

$$\begin{aligned}\frac{1}{2^{k+1}} \sum_{i=1}^{2^{k+1}} f(x_i^{2^{k+1}}) &= \frac{1}{2^k} \sum_{i=1}^{2^k} \frac{1}{2} \left(f(x_i^{2^{k+1}}) + f(x_{2^k+i}^{2^{k+1}}) \right) \\ &= \frac{1}{2^k} \sum_{i=1}^{2^k} f\left((x_i \cdot x_{2^k+i})^{2^k} \right) \\ &\quad \text{(by the hypothesis)} \\ &= f(x_1 x_2 \cdots x_{2^{k+1}})\end{aligned}$$

Thus our claim is proved. Now assume that $2^k < n < 2^{k+1} = N$ and we choose $x_{n+1}, x_{n+2}, \ldots, x_N$ such that $x_{n+1} = x_{n+2} = \cdots = x_N = (x_1 x_2 \cdots x_n)^{\frac{1}{n}}$. Then

$$\begin{aligned}&\frac{f(x_1^n) + \cdots + f(x_n^n) + f(x_{n+1}^n) + \cdots + f(x_N^n)}{N} \\ &= \frac{f((x_1^{\frac{n}{N}})^N) + \cdots + f((x_n^{\frac{n}{N}})^N) + f((x_{n+1}^{\frac{n}{N}})^N) + \cdots + f((x_N^{\frac{n}{N}})^N)}{N} \\ &= f\left((x_1 \cdots x_n)^{\frac{n}{N}} \cdot (x_{n+1} \cdots x_N)^{\frac{n}{N}} \right) \\ &= f\left((x_1 \cdots x_n)^{\frac{n}{N}} \cdot (x_1 \cdots x_n)^{\frac{N-n}{n} \cdot \frac{n}{N}} \right) \\ &= f(x_1 \cdots x_n)\end{aligned}$$

Hence,

$$\frac{f(x_1^n) + \cdots + f(x_n^n)}{n} = f(x_1 \cdots x_n)$$

as we wanted to prove.

Solution 2 by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria. We prove directly generalization 2). Indeed, let $f : (0, \infty) \to \mathbb{R}$ be a function that satisfies

$$\forall\, (x, y) \in (0, \infty)^2, \qquad \frac{f(x^2) + f(y^2)}{2} = f(xy) \tag{1}$$

We will consider two cases according to the value of $f(1)$:

(a): $f(1) = 0$. Choosing $y = 1$ in (1) we see that $f(x^2) = 2f(x)$ for every $x > 0$. Using this from (1) we have for all $(x, y) \in (0, \infty)^2$,

$$f(x) + f(y) = f(xy)$$

Now, it is a straightforward task to show by induction on n that for every integer $n > 1$ and for all , $(x_1, \ldots, x_n) \in (0, +\infty)^n$, that

$$f(x_1) + f(x_2) + \cdots + f(x_n) = f(x_1 x_2 \cdots x_n)$$

Also, choosing $x_1 = x_2 = \cdots = x_n = t$ we get $nf(t) = f(t^n)$ for every $t > 0$. So, for all $(x_1, \ldots, x_n) \in (0, +\infty)^n$ is

$$\frac{f(x_1^n) + f(x_2^n) + \ldots + f(x_n^n)}{n} = f(x_1 x_2 \cdots x_n)$$

as desired.

(b): $f(1) \neq 0$. In this case we obtain the conclusion by applying the preceding case to the function $\overline{f} : (0, \infty) \to \mathbb{R}$ defined by $\overline{f}(x) = f(x) - f(1)$.

Also solved by Islam Foniqi, Department of Mathematics, Prishtinë, Republic of Kosova; Adrian Naco, Department of Mathematics, Polytechnic University of Tirana, Albania; Florin Stanescu, Serban Cioculescu School, Gaesti, Dambovita, Romania; and the proposer

30. *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.* Evaluate

$$\lim_{x\to 0}\int_{2011\,x}^{2012\,x}\frac{\sin^n t}{t^m}\,dt,$$

where $n, m \in \mathbb{N}$.

Solution by Anastasios Kotronis, Athens, Greece. More generally, let $0 < a < b$ and $n, m \in \mathbb{N}$. We consider the following cases:

(1) For $n-m \geq -1$ we have

$$\begin{aligned}\int_{ax}^{bx}\frac{\sin^n t}{t^m}\,dt &= \int_{ax}^{bx} t^{n-m}\left(1+\mathcal{O}(t^2)\right)^n\\ &= \int_{ax}^{bx} t^{n-m}\left(1+\mathcal{O}(t^2)\right)\\ &= \int_{ax}^{bx} t^{n-m}+\mathcal{O}\left(t^{n-m+2}\right)\,dt\\ &= \begin{cases}\left.\frac{t^{n-m+1}}{n-m+1}\right|_{ax}^{bx}+\mathcal{O}\left(\left.\frac{t^{n-m+3}}{n-m+3}\right|_{ax}^{bx}\right), & n-m\geq 0\\[2ex] \ln|t|\Big|_{ax}^{bx}+\mathcal{O}\left(t^2\Big|_{ax}^{bx}\right), & n-m=-1\end{cases}\\ &= \begin{cases}\frac{b^{n-m+1}-a^{n-m+1}}{n-m+1}x^{n-m+1}+\mathcal{O}(x^{n-m+3}), & n-m\geq 0\\[2ex] \ln\frac{b}{a}+\mathcal{O}(x^2), & n-m=-1\end{cases}\\ &\xrightarrow{x\to 0}\begin{cases}0, & n-m\geq 0\\ \ln\frac{b}{a}, & n-m=-1\end{cases}.\end{aligned}$$

Carrying out the change of variable $t=-y$, we get

$$\int_{ax}^{bx}\frac{sin^n t}{t^m}\,dt = (-1)^{n-m+1}\int_{a(-x)}^{b(-x)}\frac{\sin^n y}{y^m}\,dy, \tag{1}$$

(2) For $n-m \leq -2$ we distinguish two cases:
- If $n-m$ is odd, then for some $0<\varepsilon<1$ and while $x\to 0^+$ we have

$$(1-\varepsilon)\leq\frac{\sin t}{t}\leq 1 \Rightarrow \frac{(1-\varepsilon)^n}{t^{m-n}}\leq\frac{\sin^n t}{t^m}\leq\frac{1}{t^{m-n}}\ \ \textit{and}$$
$$(1-\varepsilon)^n\frac{b^{n-m+1}-a^{n-m+1}}{(n-m+1)x^{m-n-1}}\leq\int_{ax}^{bx}\frac{\sin^n t}{t^m}\leq\frac{b^{n-m+1}-a^{n-m+1}}{(n-m+1)x^{m-n-1}}$$

Thus $\lim_{x\to 0^+}\int_{ax}^{bx}\frac{\sin^n t}{t^m}\,dt = +\infty$ and from (1)

$$\lim_{x\to 0^-}\int_{ax}^{bx}\frac{\sin^n t}{t^m}\,dt = \lim_{x\to 0^+}\int_{ax}^{bx}\frac{\sin^n t}{t^m}\,dt = +\infty$$

- If $n-m$ is even, then similarly while $x\to 0^+$ we have

$$(1-\varepsilon)^n\frac{b^{n-m+1}-a^{n-m+1}}{(n-m+1)x^{m-n-1}} \le \int_{ax}^{bx}\frac{\sin^n t}{t^m} \le \frac{b^{n-m+1}-a^{n-m+1}}{(n-m+1)x^{m-n-1}}$$

Thus $\lim_{x\to 0^+}\int_{ax}^{bx}\frac{\sin^n t}{t^m}\,dt = +\infty$ and from (1)

$$\lim_{x\to 0^-}\int_{ax}^{bx}\frac{\sin^n t}{t^m}\,dt = \lim_{x\to 0^+} -\int_{ax}^{bx}\frac{\sin^n t}{t^m}\,dt = -\infty$$

and the limit does not exist.

Finally, collecting yields

$$\lim_{x\to 0}\int_{ax}^{bx}\frac{\sin^n t}{t^m}\,dt \begin{cases} = 0, & n-m\ge 0\\ = \ln\frac{b}{a}, & n-m=-1\\ = +\infty, & n-m\le -2 \quad\text{and}\quad n-m=\text{odd}\\ \text{does not exist}, & n-m\le -2 \quad\text{and}\quad n-m=\text{even}\end{cases}$$

Also solved by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; and the proposer

31. (Correction) *Proposed by Valmir Bucaj, Texas Lutheran University, Seguin, TX.* If the vertices of a polygon, in clockwise order, are:

$$\left(\frac{1}{\sqrt{a_1}},\frac{1}{\sqrt{a_{n+1}}}\right),\left(\frac{2}{\sqrt{a_2}},\frac{1}{\sqrt{a_{2n}}}\right),\left(\frac{3}{\sqrt{a_3}},\frac{1}{\sqrt{a_{2n-1}}}\right),\left(\frac{4}{\sqrt{a_4}},\frac{1}{\sqrt{a_{2n-2}}}\right),\ldots,$$
$$\left(\frac{n-1}{\sqrt{a_{n-1}}},\frac{1}{\sqrt{a_{n+3}}}\right),\left(\frac{n}{\sqrt{a_n}},\frac{1}{\sqrt{a_{n+2}}}\right),$$

where $(a_n)_{n\ge 1}$ is a decreasing geometric progression, show that the area of this polygon is

$$A=\frac{1}{4\sqrt{a_1}}\left(\frac{(n+3)(n-2)}{\sqrt{a_{2n+2}}}-\frac{n(n+1)}{\sqrt{a_{2n}}}+\frac{6}{\sqrt{a_{n+2}}}\right)$$

Solution 1 by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria. Let $P_1=\left(\frac{1}{\sqrt{a_1}},\frac{1}{\sqrt{a_{n+1}}}\right)$ and for $2\le k\le n$ let $P_k=\left(\frac{k}{\sqrt{a_k}},\frac{1}{\sqrt{a_{2n+2-k}}}\right)$. Then the area A of the polygon $(P_1,P_2,\ldots,P_n)$ is given by

$$\begin{aligned} A &= \frac{1}{2}\sum_{k=2}^{n-1}\det(\overrightarrow{P_1P_{k+1}},\overrightarrow{P_1P_k})\\ &= \frac{1}{2}\sum_{k=2}^{n-1}\begin{vmatrix} \frac{k+1}{\sqrt{a_{k+1}}}-\frac{1}{\sqrt{a_1}} & \frac{k}{\sqrt{a_k}}-\frac{1}{\sqrt{a_1}}\\ \frac{1}{\sqrt{a_{2n+1-k}}}-\frac{1}{\sqrt{a_{n+1}}} & \frac{1}{\sqrt{a_{2n+2-k}}}-\frac{1}{\sqrt{a_{n+1}}}\end{vmatrix}\end{aligned}$$

Noting that $a_k a_{2n+1-k} = a_1 a_{2n}$ and $a_{k+1} a_{2n+2-k} = a_1 a_{2n+2}$ we conclude that

$$\begin{aligned} A &= \sum_{k=2}^{n-1} \left(\frac{k+1}{\sqrt{a_1 a_{2n+2}}} - \frac{k}{\sqrt{a_1 a_{2n}}} - \frac{1}{\sqrt{a_{n+1}}} \left(\frac{k+1}{\sqrt{a_{k+1}}} - \frac{k}{\sqrt{a_k}} \right) \right. \\ &\quad \left. - \frac{1}{\sqrt{a_1}} \left(\frac{1}{\sqrt{a_{2n+2-k}}} - \frac{1}{\sqrt{a_{2n+1-k}}} \right) \right) \\ &= \frac{n(n+1)-6}{2\sqrt{a_1 a_{2n+2}}} - \frac{n(n-1)-2}{2\sqrt{a_1 a_{2n}}} - \frac{1}{\sqrt{a_{n+1}}} \left(\frac{n}{\sqrt{a_n}} - \frac{2}{\sqrt{a_2}} \right) \\ &\quad - \frac{1}{\sqrt{a_1}} \left(\frac{1}{\sqrt{a_{2n}}} - \frac{1}{\sqrt{a_{n+2}}} \right) \end{aligned}$$

But $a_{n+1} a_n = a_1 a_{2n}$ and $a_2 a_{n+1} = a_1 a_{n+2}$. So, the above formula simplifies to

$$A = \frac{1}{4\sqrt{a_1}} \left(\frac{(n+3)(n-2)}{\sqrt{a_{2n+2}}} - \frac{n(n+1)}{\sqrt{a_{2n}}} + \frac{6}{\sqrt{a_{n+2}}} \right)$$

as claimed.

Also solved by the proposer.

32. *Proposed by Mihály Bencze, Braşov, Romania.* Let $f : [a, b] \to \mathbb{R}$ be a two times differentiable function such that f'' and f' are continuous. If $m = \min\limits_{x\in[a,b]} f''(x)$ and $M = \max\limits_{x\in[a,b]} f''(x)$, then prove that

$$\frac{m\,(b^2-a^2)}{2} \le bf'(b) - af'(a) - f(b) + f(a) \le \frac{M\,(b^2-a^2)}{2}$$

Solution by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy. We have

$$f(b) - f(a) = \int_a^b f'(x)dx = xf'(x)\Big|_a^b - \int_a^b xf''(x)\,dx$$

from which follows

$$bf'(b) - af'(a) - f(b) + f(a) = \int_a^b xf''(x)\,dx$$

Moreover,

$$\min_{x\in[a,b]} f''(x) \int_a^b xdx \le \int_a^b xf''(x)dx \le \max_{x\in[a,b]} f''(x) \int_a^b xdx$$

and then

$$m\frac{b^2-a^2}{2} \le \int_a^b xf''(x)dx \le M\frac{b^2-a^2}{2}$$

concluding the proof.

Also solved by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria; Angel Plaza, Department of Mathematics, Universidad de Las Palmas de Gran Canaria, Spain; Florin Stanescu, Serban Cioculescu School, Gaesti, Dambovita, Romania; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; and the proposer.

33. *Proposed by Ovidiu Furdui, Cluj, Romania.* Find the value of

$$\lim_{n\to\infty}\int_0^{\pi/2}\sqrt[n]{\sin^n x+\cos^n x}\,dx$$

Solution by Moubinool Omarjee, Paris France. Let $f_n(x)=\sqrt[n]{\sin^n x+\cos^n x}=f_n\left(\frac{\pi}{2}-x\right)$. Then,

$$\int_0^{\frac{\pi}{2}} f_n(x)dx = 2\int_0^{\frac{\pi}{4}} f_n(x)\,dx$$

Let us denote by

$$y_n=\int_0^{\frac{\pi}{4}} f_n(x)dx=\int_0^{\frac{\pi}{4}}\cos x\left(1+(\tan x)^n\right)^{\frac{1}{n}}\,dx$$

The functions $g_n(x)=\cos x\left(1+(\tan x)^n\right)^{\frac{1}{n}}$ are continuous on $\left[0;\frac{\pi}{4}\right]$ and the sequence (g_n) converges to the function $x\mapsto\cos x$ on $\left[0;\frac{\pi}{4}\right]$ as can be easily checked. Furthermore,

$$|g_n(x)|\le\cos x\left(1+\left(\tan\frac{\pi}{4}\right)^n\right)^{\frac{1}{n}}\le\cos x\cdot 2^{\frac{1}{n}}\le\cos x\cdot 2\le 2$$

So, by the Dominated Convergence theorem, we have

$$\lim_{n\to\infty} y_n=\int_0^{\frac{\pi}{4}}\cos x\,dx=\frac{\sqrt{2}}{2}$$

Finally,

$$\lim_{n\to\infty}\int_0^{\frac{\pi}{2}}\sqrt[n]{\sin^n x+\cos^n x}\,dx=2\cdot\frac{\sqrt{2}}{2}=\sqrt{2}$$

Also solved by Albert Stadler, Switzerland; Moubinnol Omarjee, Paris, France; Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria; Anastasios Kotronis, Athens, Greece; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; and the proposer

34. *Proposed by Mihály Bencze, Braşov, Romania.* Solve the following equation

$$\underbrace{\sqrt{2+\sqrt{2+\ldots+\sqrt{2+x}}}}_{\text{n-times}}+\underbrace{\sqrt{2-\sqrt{2+\ldots+\sqrt{2+x}}}}_{\text{n-times}}=x\sqrt{2},$$

where $n\ge 3$.

Solution by Islam Foniqi, Department of Mathematics, Prishtinë, Republic of Kosova. We see that x has to be in $(0,2)$, so we can take $x=2\cos y$ where $0<y<\frac{\pi}{2}$. Using the formula $\sqrt{2(1+\cos 2a)}=2\cos a$, we have

$$\underbrace{\sqrt{2+\sqrt{2+\cdots+\sqrt{2+x}}}}_{(n-1)-times}=2\cos\frac{y}{2^{n-1}}$$

and the given equation becomes

$$\sqrt{2(1+\cos\frac{y}{2^{n-1}})}+\sqrt{2(1-\cos\frac{y}{2^{n-1}})}=2\sqrt{2}\cos y$$

or

$$\cos\frac{y}{2^n}+\sin\frac{y}{2^n}=\sqrt{2}\cos y$$

which can be written as

$$\cos(\frac{y}{2^n}-\frac{\pi}{4})=\cos y$$

Now $\frac{y}{2^n}-\frac{\pi}{4}=y$ or $\frac{y}{2^n}-\frac{\pi}{4}=-y$ which is equivalent to $y(1-\frac{1}{2^n})=-\frac{\pi}{4}$ or $y(1+\frac{1}{2^n})=\frac{\pi}{4}$, with $0<y<\frac{\pi}{2}$. The equation $y(1-\frac{1}{2^n})=-\frac{\pi}{4}$ does not have solution because $n\geq 3$ and $0<y<\frac{\pi}{2}$, but the equation $y(1+\frac{1}{2^n})=\frac{\pi}{4}$ has the solution $y=\frac{2^n\pi}{4(2^n+1)}$ which is clearly between 0 and $\frac{\pi}{2}$. Therefore, $x=2\cos y=2\cos\frac{2^n\pi}{4(2^n+1)}$ when $n\geq 3$.

Also solved by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; and the proposer

35. *Proposed by Florin Stănescu, School Cioculescu Serban, Găeşti, jud. Dambovita, Romania.* Let $P(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0$ be a polynomial with positive real coefficients of degree $n\geq 3$, such that P' has only real zeros. If $0\leq a<b$ show that

$$\frac{\int_a^b\frac{1}{P'(x)}dx}{\int_a^b\frac{1}{P''(x)}dx}\geq\frac{1}{b-a}\ln\left(\frac{P'(b)}{P'(a)}\right)\geq\frac{P'(b)-P'(a)}{P(b)-P(a)}$$

Solution by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy. The roots of $P'(x)=0$ are evidently all negative and

$$P'(x)=na_n\prod_{k=1}^{n-1}(x-\xi_k),\quad \xi_k<0,\ 1\leq k\leq n-1$$

Thus we have

$$\frac{P''(x)}{P'(x)}=\sum_{k=1}^{n-1}\frac{1}{x-\xi_k}$$

that it is a decreasing function as well as $\frac{1}{P'(x)}$ and $\frac{1}{P''(x)}$. Now

$$\ln\left(\frac{P'(b)}{P'(a)}\right)=\int_a^b\frac{P''(x)}{P'(x)}dx$$

$$\int_a^b\frac{P''(x)}{P'(x)}dx\cdot\int_a^b\frac{1}{P''(x)}dx\leq(b-a)\int_a^b\frac{1}{P'(x)}dx$$

on account of Chebyshev's inequality for integrals applied to decreasing (increasing) functions. This proves the LHS inequality.

The RHS inequality is actually

$$\int_a^b\frac{P''(x)}{P'(x)}\,dx\geq(b-a)\frac{\int_a^bP''(x)\,dx}{\int_a^bP'(x)\,dx}$$

That is,

$$\int_a^b P'(x)dx \cdot \int_a^b \frac{P''(x)}{P'(x)}dx \geq (b-a) \int_a^b P''(x)\, dx$$

and this follows also by applying Chebyshev's result again with $P'(x)$ increasing while $\frac{P''(x)}{P'(x)}$ is decreasing.

Comment by the Editor. This problem has appeared as part of the following paper by the same author: *Aplicaţi ale inegalităţii lui Cebişev in formă integrală, Gazeta Matematică, Seria B, Anul CXVII, nr. 3 (2012) 113–121.*

Also solved by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; and the proposer

MATHCONTEST SECTION

This section of the Journal offers readers an opportunity to solve interesting and elegant mathematical problems mainly appeared in Math Contest around the world and most appropriate for training Math Olympiads. Proposals are always welcomed. The source of the proposals will appear when the solutions be published.

Proposals

26. Let $f : [0,1] \to [0,1]$ be a differentiable function such that $|f'(x)| \neq 1$ for all $x \in [0,1]$. Show that there exist two unique points $\alpha, \beta \in [0,1]$ such that $f(\alpha) = \alpha$ and $f(\beta) = 1 - \beta$.

27. Prove that the equation

$$(x + y\sqrt{3})^4 + (z + t\sqrt{3})^4 = 7 + 6\sqrt{3}$$

does not have rational solutions.

28. Find all polynomials $p(x)$ with real coefficients such that

$$p(a + b - 2c) + p(b + c - 2a) + p(c + a - 2b) = 3p(a - b) + 3p(b - c) + 3p(c - a)$$

for all $a, b, c \in \mathbb{R}$.

29. Equation $x^3 - 2x^2 - x + 1 = 0$ has three real roots $a > b > c$. Find the value of $ab^2 + bc^2 + ca^2$.

30. Let $\{a_n\}_{n\geq 1}$ be a strictly increasing sequence of positive integers such that $\lim\limits_{n\to+\infty} \dfrac{a_{n+1}}{a_1 a_2 \ldots a_n} = +\infty$. Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{a_n}$ is an irrational number.

Solutions

21. *Let $a > -3/4$ be a real number. Show that*

$$\sqrt[3]{\frac{a+1}{2}+\frac{a+3}{6}\sqrt{\frac{4a+3}{3}}}+\sqrt[3]{\frac{a+1}{2}-\frac{a+3}{6}\sqrt{\frac{4a+3}{3}}}$$

is an integer and determine its value.

(XXVI Spanish Math Olympiad 1989-1990)

Solution by José Luis Díaz-Barrero, BARCELONA TECH, Barcelona, Spain. Putting

$$x=\sqrt[3]{\frac{a+1}{2}+\frac{a+3}{6}\sqrt{\frac{4a+3}{3}}}\ \text{ and }\ y=\sqrt[3]{\frac{a+1}{2}-\frac{a+3}{6}\sqrt{\frac{4a+3}{3}}}$$

and adding up and multiplying up the preceding expressions, we get $x+y=a+1$ and $xy=-a^3/27$, respectively. Now, we call $z=\sqrt[3]{x}+\sqrt[3]{y}$ and rising to cube, yields

$$z^3=x+y+3\sqrt[3]{xy}\left(\sqrt[3]{x}+\sqrt[3]{y}\right)=a+1-az$$

or equivalently,

$$z^3+az-(a+1)=0\Leftrightarrow(z-1)(z^2+z+a+1)=0$$

Since the discriminant of $z^2+z+a+1=0$ is $\delta=-(3+4q)<0$, then it does not have real roots. So,

$$\sqrt[3]{\frac{a+1}{2}+\frac{a+3}{6}\sqrt{\frac{4a+3}{3}}}+\sqrt[3]{\frac{a+1}{2}-\frac{a+3}{6}\sqrt{\frac{4a+3}{3}}}=1$$

and we are done. □

Also solved by José Gibergans Báguena, BARCELONA TECH, Barcelona, Spain

22. *Let a_1,a_2,a_3,a_4 be nonzero real numbers defined by $a_k=\dfrac{\sin(k\beta+\alpha)}{\sin k\beta}$, $(1\le k\le 4),\alpha,\beta\in\mathbb{R}$. Calculate*

$$\begin{vmatrix} 1+a_1^2+a_2^2 & a_1+a_2+1/a_4 & 1+a_2(a_1+a_3) \\ a_1+a_2+1/a_4 & 2+1/a_4^2 & a_2+a_3+1/a_4 \\ 1+a_2(a_1+a_3) & a_2+a_3+1/a_4 & 1+a_2^2+a_3^2 \end{vmatrix}$$

(József Wildt Mathematics Competition 2005)

Solution by José Gibergans Báguena, BARCELONA TECH, Barcelona, Spain. First, we evaluate

$$\begin{aligned}\Delta = \begin{vmatrix} a_1 & 1 & a_2 \\ 1 & 1/a_4 & 1 \\ a_2 & 1 & a_3 \end{vmatrix} &= \frac{1}{a_4}\begin{vmatrix} a_1 & 1 & a_2 \\ a_4 & 1 & a_4 \\ a_2 & 1 & a_3 \end{vmatrix} \\ &= \frac{1}{a_4}[a_1(a_3-a_4)+a_2(a_4-a_2)+a_4(a_2-a_3)]\end{aligned}$$

Taking into account that

$$\begin{aligned} a_k - a_h &= \frac{\sin(k\beta+\alpha)}{\sin k\beta} - \frac{\sin(h\beta+\alpha)}{\sin h\beta} \\ &= \frac{\sin(k\beta+\alpha)\sinh\beta - \sin(h\beta+\alpha)\sin k\beta}{\sin k\beta \sin h\beta} \\ &= \frac{1}{2}\frac{\cos[(k-h)\beta+\alpha]-\cos[(k-h)\beta-\alpha]}{\sin k\beta\sin h\beta} \\ &= -\frac{\sin(k-h)\beta\sin\alpha}{\sin k\beta\sin h\beta}\end{aligned}$$

we have

$$\begin{aligned}\Delta &= \frac{-\sin\alpha}{\sin(4\beta+\alpha)}\left[\frac{\sin 3\beta\sin(2\beta+\alpha)-\sin 2\beta\sin(\beta+\alpha)}{\sin 2\beta\sin 3\beta} - \frac{\sin\beta\sin(4\beta+\alpha)}{\sin 2\beta\sin 3\beta}\right] \\ &= \frac{-\sin\alpha}{\sin(4\beta+\alpha)}\left[\frac{\sin(4\beta+\alpha)\sin\beta}{\sin 2\beta\sin 3\beta} - \frac{\sin(4\beta+\alpha)\sin\beta}{\sin 2\beta\sin 3\beta}\right] = 0.\end{aligned}$$

Now, it is easy to see by direct calculations that

$$\begin{aligned}&\begin{vmatrix} 1+a_1^2+a_2^2 & a_1+a_2+1/a_4 & 1+a_2(a_1+a_3) \\ a_1+a_2+1/a_4 & 2+1/a_4^2 & a_2+a_3+1/a_4 \\ 1+a_2(a_1+a_3) & a_2+a_3+1/a_4 & 1+a_2^2+a_3^2 \end{vmatrix} \\ &= \begin{vmatrix} a_1 & 1 & a_2 \\ 1 & 1/a_4 & 1 \\ a_2 & 1 & a_3 \end{vmatrix}^2 = 0\end{aligned}$$

and we are done. □

Also solved by José Luis Díaz-Barrero, BARCELONA TECH, Barcelona, Spain

23. *Let A be a set of positive integers. If the prime divisors of elements in A are among the prime numbers $p_1, p_2, \ldots, p_n$ and $|A| > 3 \cdot 2^n + 1$, then show that it contains one subset of four distinct elements whose product is the fourth power of an integer.*

(Training Sessions of Catalonia Team for OME 2012)

Solution by José Luis Díaz-Barrero, BARCELONA TECH, Barcelona, Spain. To each element a in A we associate an n-tuple $(x_1, x_2, \ldots, x_n)$, where x_i is 0 if the exponent of p_i in the prime factorization of a is even, and 1 otherwise.

These n-tuples are the "pigeons". The "holes" are the 2^n possible choices of $0'$s and $1'$s. Hence, by the PHP, every subset of 2^n+1 elements of A contains two distinct elements say a_{11}, a_{12} with the same associated n-tuple, and the product of these two elements is then a square, as all exponents are even. Thus, $\sqrt{a_{11}a_{12}}$ is an integer. Likewise, from the remaining numbers in A we can choose a_{21} and a_{22} such that $\sqrt{a_{21}a_{22}}$ is an integer. Similarly, from the set A, which has at least $3\cdot 2^n+1$ elements, we can select 2^n+1 such pairs or more. Consider the 2^n+1 integer numbers that are the square roots of products of the two elements of each pair. That is,

$$\sqrt{a_{11}a_{12}},\ \sqrt{a_{21}a_{22}},\ \sqrt{a_{31}a_{32}},\ \ldots,\ \sqrt{a_{2^n+1,1}a_{2^n+1,2}}$$

Since all the previous numbers have the same divisors, the previous arguments give us two of them $\sqrt{a_{i1}a_{i2}}$, $\sqrt{a_{j1}a_{j2}}$ with the same $0-1$ n-tuple. The product of them is a perfect square

$$\sqrt{a_{i1}a_{i2}}\cdot\sqrt{a_{j1}a_{j2}}=x^2,$$

where x is an integer. So, $a_{i1}a_{i2}a_{j1}a_{j2}=x^4$ and we are done.

□

Also solved by José Gibergans Báguena, BARCELONA TECH, Barcelona, Spain

24. *Find all triples (x,y,z) of real numbers such that*

$$\left.\begin{aligned}12x-4z^2&=25,\\24y-36x^2&=1,\\20z-16y^2&=9.\end{aligned}\right\}$$

(Training Sessions for COM-2011)

Solution by José Luis Díaz-Barrero and José Gibergans Báguena, BARCELONA TECH, Barcelona, Spain. First, we write the given system in the most convenient form

$$\left.\begin{aligned}12x&=25+4z^2,\\24y&=1+36x^2,\\20z&=9+16y^2.\end{aligned}\right\}$$

Subtracting $20z$ to both members of the first equation, $12x$ to the second, and $24y$ to the third, we obtain

$$\left.\begin{aligned}12x-20z&=25+4z^2-20z,\\24y-12x&=1+36x^2-12x,\\20z-24y&=9+16y^2-24y.\end{aligned}\right\}\Leftrightarrow\left.\begin{aligned}12x-20z&=(5-2z)^2,\\24y-12x&=(1-6x)^2,\\20z-24y&=(3-4y)^2.\end{aligned}\right\}$$

Adding up the last three equations, yields

$$(1-6x)^2+(3-4y)^2+(5-2z)^2=0$$

which is possible only when

$$\left.\begin{aligned}1-6x&=0,\\3-4y&=0,\\5-2z&=0.\end{aligned}\right\}$$

But, triple $(1/6, 3/4, 5/2)$ does not satisfy the system and therefore it does not have solution.

□

Also solved by Iván Geffner Fuenmayor, BARCELONA TECH, Barcelona, Spain.

25. *Let a, b, c be the lengths of the sides of a triangle ABC with inradius r and cicumradius R. Prove that*

$$\frac{r}{2r+R} \le \sqrt[3]{\left(\frac{a}{b+c+2a}\right)\left(\frac{b}{c+a+2b}\right)\left(\frac{c}{a+b+2c}\right)} \le \frac{1}{4}$$

(Training Sessions of Spanish Math Team for IMO 2011)

Solution by José Luis Díaz-Barrero, BARCELONA TECH, Barcelona, Spain. To prove the LHS inequality we can assume that $a+b+c=1$ on account of the homogeneity. Consider the function $f:(0,+\infty) \to \mathbb{R}$ define by $f(t) = \frac{t}{1+t}$: Then, we have $f'(t) = (1+t)^{-2}$ and $f''(t) = -2(1+t)^{-3} < 0$ for all $t > 0$. So, f is concave and applying Jensen's inequality, we have

$$f\left(\frac{a+b+c}{3}\right) \ge \frac{1}{3}\left(f(a)+f(b)+f(c)\right)$$

That is,

$$\begin{aligned}\frac{1}{4} = \frac{(a+b+c)/3}{1+(a+b+c)/3} &\ge \frac{1}{3}\left(\frac{a}{1+a}+\frac{b}{1+b}+\frac{c}{1+c}\right)\\ &\ge \sqrt[3]{\left(\frac{a}{1+a}\right)\left(\frac{b}{1+b}\right)\left(\frac{c}{1+c}\right)}\\ &= \sqrt[3]{\left(\frac{a}{b+c+2a}\right)\left(\frac{b}{c+a+2b}\right)\left(\frac{c}{a+b+2c}\right)}\end{aligned}$$

on account of AM-GM inequality. On the other hand, we have that

$$\frac{b}{c}+\frac{c}{b} \le \frac{R}{r} \quad \text{(cyclic)}$$

In fact, using the duality principle, there exist three positive numbers x, y, z such that $a = y+z, b = z+x$ and $c = x+y$ for which

$$R = \frac{(x+y)(y+z)(z+x)}{4\sqrt{(x+y+z)xyz}} \quad \text{and} \quad r = \sqrt{\frac{xyz}{x+y+z}}$$

Then,

$$\frac{R}{r} = \frac{(x+y)(y+z)(z+x)}{4xyz} \quad \text{and} \quad \frac{b}{c}+\frac{c}{b} = \frac{z+x}{x+y}+\frac{x+y}{z+x}$$

Now, we will see that

$$\frac{(x+y)(y+z)(z+x)}{4xyz} \ge \frac{z+x}{x+y}+\frac{x+y}{z+x}$$

or equivalently,

$$\frac{y+z}{4xyz} \geq \frac{1}{(x+y)^2} + \frac{1}{(z+x)^2}$$

which follows immediately from the fact that $(x+y)^2 \geq 4xy$ and $(z+x)^2 \geq 4zx$.

From the preceding and taking into account GM-HM inequalities, we have

$$\sqrt[3]{\left(\frac{a}{b+c+2a}\right)\left(\frac{b}{c+a+2b}\right)\left(\frac{c}{a+b+2c}\right)}$$

$$\geq 3\left(\frac{b+c+2a}{a} + \frac{c+a+2b}{b} + \frac{a+b+2c}{c}\right)^{-1}$$

$$\geq 3\left(6 + \left(\frac{a}{b}+\frac{b}{a}\right) + \left(\frac{b}{c}+\frac{c}{b}\right) + \left(\frac{c}{a}+\frac{a}{c}\right)\right)^{-1} \geq \frac{r}{2r+R}$$

Equality holds when $a=b=c$ because in this case $R=2r$ and we are done.

□

Also solved by Iván Geffner Fuenmayor, BARCELONA TECH, Barcelona, Spain and José Gibergans Báguena, BARCELONA TECH, Barcelona, Spain.

MATHNOTES SECTION

Note on an Algebraic Inequality

Vandanjav Adiyasuren and Bold Sanchir

Abstract. In this note a constrained inequality is generalized.

1. Introduction

In [1] the following problem was posted: *Let n be a positive integer. Find the largest constant $c_n > 0$ such that, for all positive real numbers $a_1, ..., a_n$,*

$$\frac{1}{a_1^2} + \cdots + \frac{1}{a_n^2} + \frac{1}{(a_1 + \cdots + a_n)^2} \geq c_n \left(\frac{1}{a_1} + \cdots + \frac{1}{a_n} + \frac{1}{a_1 + \cdots + a_n}\right)^2$$

A solution to the preceding proposal and some related results appeared in [2]. Our aim in this short note is to generalize it.

2. Main Results

Theroem 1. *For all positive numbers $a_1, \ldots, a_n$ and for all positive integer $p > 1$, the following inequality holds:*

$$\frac{1}{a_1^p} + \ldots + \frac{1}{a_n^p} + \frac{1}{(a_1 + \ldots + a_n)^p} \geq c_n(p) \left(\frac{1}{a_1} + \ldots + \frac{1}{a_n} + \frac{1}{a_1 + \ldots + a_n}\right)^p, \quad (2)$$

where $c_n(p) = \frac{(n^3+1)^p}{\left(n^{\frac{2p-1}{p-1}}+1\right)^{p-1}(n^2+1)^p}$.

Proof. Denote

$$A = \frac{1}{a_1} + \ldots + \frac{1}{a_n} + \frac{1}{a_1 + \ldots + a_n}, \; B = \frac{1}{a_1} + \ldots + \frac{1}{a_n}$$

By applying Hölder's Inequality, we get

$$\begin{aligned} &\frac{1}{a_1^p} + \cdots + \frac{1}{a_n^p} + \frac{1}{(a_1 + \ldots + a_n)^p} && (3) \\ &\geq \left(\frac{1}{a_1} + \cdots + \frac{1}{a_n} + \frac{1}{n(a_1 + \ldots + a_n)}\right)^p \frac{1}{(1 + \ldots + 1 + \frac{1}{n^{\frac{p}{p-1}}})^{p-1}} \\ &= \frac{n^p}{(n^{\frac{2p-1}{p-1}} + 1)^{p-1}} \left(\frac{1}{n}A + \frac{n(n-1)}{n^2+1}B + \frac{n-1}{n(n^2+1)}B\right)^p \end{aligned}$$

Applying Cauchy-Schwarz Inequality, we have

$$\frac{1}{a_1} + \cdots + \frac{1}{a_n} \geq \frac{n^2}{a_1 + \ldots + a_n} \quad (4)$$

From (3), (4) we get (2). □

Corllary 1. *Let $a_1, \ldots, a_n$ be positive numbers. Then*

$$\frac{1}{a_1^2} + \ldots + \frac{1}{a_n^2} + \frac{1}{(a_1 + \ldots + a_n)^2} \geq \frac{n^3+1}{(n^2+1)^2} \left(\frac{1}{a_1} + \ldots + \frac{1}{a_n} + \frac{1}{a_1 + \ldots + a_n} \right)^2 \tag{5}$$

Proof. Choosing $p = 2$ in (2), we get (5). □

Theroem 2. *Let n, m, k be positive integers. For all positive real numbers $a_1, \ldots, a_n$ and $\alpha, \beta > 0$ with $k\beta - m\alpha > 0$, $k > m$, $p > 1$, we hve*

$$\begin{aligned} &\sum_{1 \leq i_1 < \ldots < i_m \leq n} \frac{1}{(a_{i_1} + \ldots + a_{i_m})^p} + \sum_{1 \leq i_1 < \cdots < i_k \leq n} \frac{1}{(a_{i_1} + \ldots + a_{i_k})^p} \\ &\geq c_{n;m;k}^{[p]}(\alpha, \beta) \left[\sum_{1 \leq i_1 < \ldots < i_m \leq n} \frac{\alpha}{a_{i_1} + \ldots + a_{i_m}} + \sum_{1 \leq i_1 < \ldots < i_k \leq n} \frac{\beta}{a_{i_1} + \ldots + a_{i_k}} \right]^p \end{aligned} \tag{6}$$

where

$$c_{n;m;k}^{[p]}(\alpha, \beta) = \frac{k^p m^p}{(m^{\frac{p}{p-1}} \binom{n}{k} + k^{\frac{p}{p-1}} \binom{n}{m})^{p-1}} \left(\frac{1}{k\beta} + \frac{\binom{k}{m}(k\beta - \alpha m)}{\beta m (\beta m \binom{n-m}{k-m} + \alpha k \binom{k}{m})} \right)^p \tag{7}$$

Proof. Denote

$$\begin{aligned} A &= \sum_{1 \leq i_1 < \ldots < i_m \leq n} \frac{\alpha}{a_{i_1} + \ldots + a_{i_m}} + \sum_{1 \leq i_1 < \ldots < i_k \leq n} \frac{\beta}{a_{i_1} + \ldots + a_{i_k}}, \\ B &= \sum_{1 \leq i_1 < \ldots < i_m \leq n} \frac{1}{a_{i_1} + \ldots + a_{i_m}}. \end{aligned}$$

Using Hölder's Inequality, we get

$$\begin{aligned} L :&= \sum_{1 \leq i_1 < \ldots < i_m \leq n} \frac{1}{(a_{i_1} + \ldots + a_{i_m})^p} + \sum_{1 \leq i_1 < \ldots < i_k \leq n} \frac{1}{(a_{i_1} + \ldots + a_{i_k})^p} \\ &\geq \left(\frac{1}{m} \sum_{1 \leq i_1 < \ldots < i_m < n} \frac{1}{a_{i_1} + \ldots + a_{i_m}} + \frac{1}{k} \sum_{1 \leq i_1 < \ldots < i_k \leq n} \frac{1}{a_{i_1} + \ldots + a_{i_k}} \right)^p \\ &\quad \times \frac{1}{\left(\frac{\binom{n}{m}}{m^{\frac{p}{p-1}}} + \frac{\binom{n}{k}}{k^{\frac{p}{p-1}}} \right)^{p-1}} \\ &= \left(\frac{1}{k\beta} A + \frac{k\beta - m\alpha}{m\beta k} \sum_{1 \leq i_1 < \ldots < i_m \leq n} \frac{1}{a_{i_1} + \ldots + a_{i_m}} \right)^p \\ &\quad \times \frac{k^p m^p}{(m^{\frac{p}{p-1}} \binom{n}{k} + k^{\frac{p}{p-1}} \binom{n}{m})^{p-1}} \\ &= \left(\frac{1}{k\beta} A + \frac{(k\beta - m\alpha)\binom{n-m}{k-m}}{k(\beta m \binom{n-m}{k-m} + \alpha k \binom{k}{m})} B + \frac{\alpha \binom{k}{m}(k\beta - m\alpha)}{m\beta(\beta m \binom{n-m}{k-m} + \alpha k \binom{k}{m})} B \right)^p \\ &\quad \times \frac{k^p m^p}{(m^{\frac{p}{p-1}} \binom{n}{k} + k^{\frac{p}{p-1}} \binom{n}{m})^{p-1}} \end{aligned} \tag{8}$$

Using Cauchy-Schwarz inequality, we get

$$\sum_{1\le i_1<\ldots<i_m\le n}\frac{1}{a_{i_1}+\ldots+a_{i_m}} \tag{9}$$

$$=\frac{1}{\binom{n-m}{k-m}}\sum_{1\le i_1<\ldots<i_k\le n}\Bigg(\underbrace{\frac{1}{a_{i_1}+\ldots+a_{i_m}}+\ldots+\frac{1}{a_{i_{k-m+1}}+\ldots+a_{i_k}}}_{\binom{k}{m}}\Bigg)$$

$$\ge\frac{1}{\binom{n-m}{k-m}}\sum_{1\le i_1<\ldots<i_k\le n}\frac{\binom{k}{m}^2}{\binom{k}{m}\frac{m}{k}(a_{i_1}+\ldots+a_{i_k})}$$

$$=\frac{k\binom{k}{m}}{m\binom{n-m}{k-m}}\sum_{1\le i_1<\ldots<i_k\le n}\frac{1}{a_{i_1}+\cdots+a_{i_k}}$$

Using (8), (9) we get

$$\begin{aligned}L &= \sum_{1\le i_1<\ldots<i_m\le n}\frac{1}{(a_{i_1}+\ldots+a_{i_m})^p}+\sum_{1\le i_1<\ldots<i_k\le n}\frac{1}{(a_{i_1}+\ldots+a_{i_k})^p}\\ &\ge \frac{k^p m^p}{(m^{\frac{p}{p-1}}\binom{n}{k}+k^{\frac{p}{p-1}}\binom{n}{m})^{p-1}}\Bigg(\frac{1}{k\beta}A+\frac{\alpha\binom{k}{m}(k\beta-m\alpha)}{m\beta(\beta m\binom{n-m}{k-m}+\alpha k\binom{k}{m})}B\\ &\quad+\frac{(k\beta-m\alpha)\binom{n-m}{k-m}}{k(\beta m\binom{n-m}{k-m}+\alpha k\binom{k}{m})}\frac{k\binom{k}{m}}{m\binom{n-m}{k-m}}\sum_{1\le i_1<\ldots<i_k\le n}\frac{1}{a_{i_1}+\ldots+a_{i_k}}\Bigg)^p\\ &= c^{[p]}_{n;m;k}(\alpha,\beta)\left[\sum_{1\le i_1<\ldots<i_m\le n}\frac{\alpha}{a_{i_1}+\ldots+a_{i_m}}+\sum_{1\le i_1<\ldots<i_m\le n}\frac{\beta}{a_{i_1}+\ldots+a_{i_k}}\right]^p\end{aligned}$$

Equality occurs when $a_1=\cdots=a_n$. □

References

[1] T. Andreescu and D. Andrica, Problem U193, *Mathematical Reflections.*, **(3)** (2011).

[2] Solution of problem U193, *Mathematical Reflections.*,**(4)** (2011), 13-14. (http://awesomemath.org/wp-content/uploads/reflections/2011_4/MR3solutions.pdf)

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