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PROBLEMS AND SOLUTIONS

Proposals and solutions must be legible and should appear on separate sheets, each indicating the name of the sender. Drawings must be suitable for reproduction. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editors have supplied a solution. The editors encourage undergraduate and pre-college students to submit solutions. Teachers can help by assisting their students in submitting solutions. Student solutions should include the class and school name. Solutions will be evaluated for publication by a committee of professors according to a combination of criteria. Questions concerning proposals and/or solutions can be sent by e-mail to: *mathproblems-ks@hotmail.com*

Solutions to the problems stated in this issue should arrive before 2 April 2012

Problems

29. *Proposed by Enkel Hysnelaj, University of Technology, Sydney, Australia.*
1) Let $f:(0,\infty)\to\mathbb{R}$ be a function that satisfies the property

$$\frac{f(x^2)+f(y^2)}{2}=f(xy)$$

for any $(x,y)\in(0,\infty)$. Show that

$$\frac{f(x^3)+f(y^3)+f(z^3)}{3}=f(xyz)$$

for any $(x,y,z)\in(0,\infty)$.
2) Generalize the above statement, so show that if

$$\frac{f(x_1^2)+f(x_2^2)}{2}=f(x_1x_2)$$

for any $(x_1,x_2)\in(0,\infty)$, then

$$\frac{f(x_1^n)+f(x_2^n)+\ldots+f(x_n^n)}{n}=f(x_1x_2\ldots x_n)$$

for any $(x_1, x_2, \ldots, x_n) \in (0, \infty)$ and n a positive integer greater than 1.

30. *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzau, Romania.* Evaluate:

$$\lim_{x \to 0} \int_{2011\,x}^{2012\,x} \frac{\sin^n t}{t^m}\, dt,$$

where $n, m \in \mathbb{N}$.

31. *Proposed by Valmir Bucaj, Texas Lutheran University, Seguin, TX.* If the vertices of a polygon, in counterclockwise order, are:

$$\left(\frac{1}{\sqrt{a_1}}, \frac{1}{\sqrt{a_{n+1}}}\right), \left(\frac{2}{\sqrt{a_2}}, \frac{1}{\sqrt{a_{2n}}}\right), \left(\frac{3}{\sqrt{a_3}}, \frac{1}{\sqrt{a_{2n-1}}}\right), \left(\frac{4}{\sqrt{a_4}}, \frac{1}{\sqrt{a_{2n-2}}}\right), \ldots,$$
$$\left(\frac{n-1}{\sqrt{a_{n-1}}}, \frac{1}{\sqrt{a_{n+3}}}\right), \left(\frac{n}{\sqrt{a_n}}, \frac{1}{\sqrt{a_{n+2}}}\right),$$

where $(a_n)_{n \geq 1}$ is a decreasing geometric progression, show that the area of this polygon is

$$A = \frac{3}{2\sqrt{a_1}} \left(\frac{1}{\sqrt{a_{2n}}} - \frac{1}{\sqrt{a_n}}\right)$$

32. *Proposed by Mihály Bencze, Braşov, Romania.* Let $f : [a, b] \to \mathbb{R}$ be a two times differentiable function such that f'' and f' are continuous. If $m = \min\limits_{x \in [a,b]} f''(x)$ and $M = \max\limits_{x \in [a,b]} f''(x)$, then prove that

$$\frac{m\,(b^2 - a^2)}{2} \leq bf'(b) - af'(a) - f(b) + f(a) \leq \frac{M\,(b^2 - a^2)}{2}$$

33. *Proposed by Ovidiu Furdui, Cluj, Romania.* Find the value of

$$\lim_{n \to \infty} \int_0^{\pi/2} \sqrt[n]{\sin^n x + \cos^n x}\, dx$$

34. *Proposed by Mihály Bencze, Braşov, Romania.* Solve the following equation

$$\underbrace{\sqrt{2 + \sqrt{2 + \ldots + \sqrt{2 + x}}}}_{\text{n-times}} + \underbrace{\sqrt{2 - \sqrt{2 + \ldots + \sqrt{2 + x}}}}_{\text{n-times}} = x\sqrt{2},$$

where $n \geq 3$.

35. *Proposed by Florin Stanescu, School Cioculescu Serban, Gaesti, jud. Dambovita, Romania.* Let $P(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$ be a polynomial with strictly positive real coefficients of degree $n \geq 3$, such that P' has only real zeros. If $0 \leq a < b$ show that

$$\frac{\int_a^b \frac{1}{P'(x)} dx}{\int_a^b \frac{1}{P''(x)} dx} \geq \frac{1}{b - a} \ln\left(\frac{P'(b)}{P'(a)}\right) \geq \frac{P'(b) - P'(a)}{P(b) - P(a)}$$

Solutions

No problem is ever permanently closed. We will be very pleased considering for publication new solutions or comments on the past problems.

22. *Proposed by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy.* Let x, y, z be positive real numbers. Prove that

$$\sum_{cyc} \frac{4x(x+2y+2z)}{(x+3y+3z)^2} \geq \sum_{cyc} \frac{(x+y)(3x+3y+4z)}{(2x+2y+3z)^2}$$

Solution by Albert Stadler, Switzerland. Obviously

$$\sum_{cyc} \frac{(x+y)(3x+3y+3z)}{(2x+2y+3z)^2} = \sum_{cyc} \frac{(y+z)(3y+3z+4x)}{(2y+2z+3x)^2}$$

The inequality then reads as

$$\sum_{cyc} \frac{4x(x+2y+2z)}{(x+3y+3z)^2} \geq \sum_{cyc} \frac{(y+z)(3y+3z+4x)}{(2y+2z+3x)^2}$$

By homogeneity, we can assume that $x+y+z=1$. So, we have to prove that

$$\sum_{cyc} \left(\frac{4x(2-x)}{(3-2x)^2} - \frac{(1-x)(3+x)}{(2+x)^2} \right) = \sum_{cyc} \frac{(1-3x)(3y+3z+4x)}{(2y+2z+3x)^2} \geq 0.$$

Let

$$f(x) = \frac{(1-3x)(4x^2+5x-27)}{(3-2x)^2(2+x)^2}, \qquad \text{and} \qquad g(x) = \frac{864}{343}\left(x - \frac{1}{3}\right).$$

We observe that $g(x)$ is the tangent to the graph of $f(x)$ at $x = 1/3$, because

$$f'\left(\frac{1}{3}\right) = \frac{-3\left(4\left(\frac{1}{3}\right)^2 + 5\left(\frac{1}{3}\right) - 27\right)}{\left(3-2\left(\frac{1}{3}\right)^2\right)\left(2+\frac{1}{3}\right)^2} = \frac{864}{343}$$

We claim that $f(x) \geq g(x)$, for $0 \leq x \leq 1$. Indeed

$$\frac{(1-3x)(4x^2+5x-27)}{(3-2x)^2(2+x)^2} - \frac{864}{343}\left(x - \frac{1}{3}\right) = \frac{(1-3x)^2(1107+1580x-512x^2-384x^3)}{343(3-2x)^2(2+x)^2},$$

and clearly, $1107 + 1580x - 512x^2 - 384x^3 > 0$, for $0 \leq x \leq 1$. So

$$\sum_{cyc} \frac{(1-3x)(3y+3z+4x)}{(2y+2z+3x)^2} = \sum_{cyc} f(x) \geq \sum_{cyc} g(x) = 0$$

Also solved by the proposer

23. *Proposed by Paolo Perfetti, department of Mathematics, Tor Vergata University, Rome, Italy.* Let f be a real, integrable function defined on $[0, 1]$ such that

$\int_0^1 f(x)dx = 0$ and $m = \min_{0\le x\le 1} f(x)$, $M = \max_{0\le x\le 1} f(x)$. Let us define $F(x) = \int_0^x f(y)dy$. Prove that

$$\int_0^1 F^2(x)dx \le \frac{-mM}{6(M-m)^2}(3M^2 - 8mM + 3m^2)$$

Solution by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria. First, we observe that

$$\frac{-mM}{6(M-m)^2}(3M^2 - 8mM + 3m^2) = -\frac{mM}{2} + \frac{m^2M^2}{3(M-m)^2} \ge \frac{m^2M^2}{3(M-m)^2}$$

and we will prove the following

Proposition. Let f be a nonzero real, integrable function defined on $[0,1]$ such that $\int_0^1 f(x)dx = 0$ and $m = \min_{0\le x\le 1} f(x)$, $M = \max_{0\le x\le 1} f(x)$, and let $F(x) = \int_0^x f(y)dy$, then $\displaystyle\int_0^1 F^2(x)dx \le \frac{m^2M^2}{3(M-m)^2}$, with equality if and only if f coincides for almost every x in $[0,1]$ with one of the functions f_0 or f_1 defined by

$$f_0(x) = \begin{cases} M & \text{if } x \in \left[0, \frac{-m}{M-m}\right) \\ \\ m & \text{if } x \in \left[\frac{-m}{M-m}, 1\right] \end{cases} \qquad f_1(x) = \begin{cases} m & \text{if } x \in \left[0, \frac{M}{M-m}\right) \\ \\ M & \text{if } x \in \left[\frac{M}{M-m}, 1\right] \end{cases}$$

Proof. Since f is integrable, F is continuous on $[0,1]$. If $F = 0$, (i.e. $f = 0$ a.e.) there is nothing to be proved. So, in what follows we will suppose that $F \ne 0$. The continuity of F shows that the set $\mathcal{O} = \{x \in (0,1) : F(x) \ne 0\}$ is an open set. Moreover, since $F(0) = F(1) = 0$, we see that $F(t) = 0$ for every $t \in [0,1] \setminus \mathcal{O}$. The open set $\mathcal{O}$ is the union of at most denumerable family of disjoint open intervals. Thus there exist $\mathcal{N} \subset \mathbb{N}$ and a family $(I_n)_{n\in\mathcal{N}}$ of non-empty *disjoint* open subintervals of $(0,1)$ such that $\mathcal{O} = \cup_{n\in\mathcal{N}} I_n$. Suppose that $I_n = (a_n, b_n)$. Since a_n and b_n belong to $[0,1] \setminus \mathcal{O}$, we conclude that $F(a_n) = F(b_n) = 0$, while F keeps a constant sign on I_n. Let us consider two cases :

(a) $F(x) > 0$ for $x \in I_n$. From $m \le f \le M$ we conclude that, for $x \in I_n$, we have

$$F(x) = F(x) - F(a_n) = \int_{a_n}^x f(t)\,dt \le M(x - a_n)$$

and

$$F(x) = -(F(b_n) - F(x)) = \int_x^{b_n} (-f)(t)\,dt \le -m(b_n - x) = m(x - b_n)$$

So,

$$\forall\, x \in I_n, \qquad 0 < F(x) \le \min(M(x - a_n), m(x - b_n)),$$

and consequently

$$\begin{aligned}\int_{I_n} F^2(x)\,dx &\le \int_{a_n}^{b_n} \left(\min(M(x-a_n), m(x-b_n))\right)^2\,dx\\ &= \int_{a_n}^{a_n-m(b_n-a_n)/(M-m)} M^2(x-a_n)^2dx + \int_{b_n-M(b_n-a_n)/(M-m)}^{b_n} m^2(b_n-x)^2\,dx\\ &= M^2\int_0^{m(a_n-b_n)/(M-m)} x^2dx + m^2\int_0^{M(b_n-a_n)/(M-m)} x^2dx\\ &= \frac{m^2M^2}{3(M-m)^2}(b_n-a_n)^3 = \frac{m^2M^2}{3(M-m)^2}|I_n|^3\end{aligned}$$

with equality if and only if $F(x) = \min(M(x-a_n), m(x-b_n))$ for every $x \in I_n$. That is, if and only if, $f(x) = M$ for almost every $x \in \left[a_n, \frac{Ma_n-mb_n}{M-m}\right)$, and $f(x) = m$ for almost every $x \in \left[\frac{Ma_n-mb_n}{M-m}, b_n\right]$.

(b) $F(x) < 0$ for $x \in I_n$. From $m \le f \le M$ we conclude that, for $x \in I_n$, we have

$$F(x) = F(x) - F(a_n) = \int_{a_n}^{x} f(t)\,dt \ge m(x-a_n)$$

and

$$F(x) = -(F(b_n) - F(x)) = \int_x^{b_n} (-f)(t)\,dt \ge -M(b_n-x)$$

So,

$$\forall\, x \in I_n, \qquad 0 < -F(x) \le \min(-m(x-a_n), M(b_n-x)),$$

and consequently

$$\begin{aligned}\int_{I_n} F^2(x)\,dx &\le \int_{a_n}^{b_n} \left(\min(m(a_n-x), M(b_n-x))\right)^2\,dx\\ &= \int_{a_n}^{a_n+M(b_n-a_n)/(M-m)} m^2(x-a_n)^2dx + \int_{b_n+m(b_n-a_n)/(M-m)}^{b_n} M^2(b_n-x)^2\,dx\\ &= m^2\int_0^{M(b_n-a_n)/(M-m)} x^2dx + M^2\int_0^{-m(b_n-a_n)/(M-m)} x^2dx\\ &= \frac{m^2M^2}{3(M-m)^2}(b_n-a_n)^3 = \frac{m^2M^2}{3(M-m)^2}|I_n|^3,\end{aligned}$$

with equality if and only if $F(x) = \max(m(x-a_n), M(x-b_n))$ for every $x \in I_n$. That is, if and only if, $f(x) = m$ for almost every $x \in \left[a_n, \frac{Mb_n-ma_n}{M-m}\right)$, and $f(x) = M$ for almost every $x \in \left[\frac{Mb_n-ma_n}{M-m}, b_n\right]$.

So, in both cases we have

$$\int_{I_n} F^2(x)\,dx \le \frac{m^2M^2}{3(M-m)^2}|I_n|^3$$

and consequently

$$\begin{aligned}\int_0^1 F^2(x)\,dx &= \sum_{n\in\mathcal{N}}\int_{I_n} F^2(x)\,dx \le \frac{m^2M^2}{3(M-m)^2}\sum_{n\in\mathcal{N}}|I_n|^3\\ &\le \frac{m^2M^2}{3(M-m)^2}\left(\sum_{n\in\mathcal{N}}|I_n|\right)^3 = \frac{m^2M^2}{3(M-m)^2}|\mathcal{O}|^3\\ &\le \frac{m^2M^2}{3(M-m)^2},\end{aligned}$$

where we used the well-known inequality $\sum_{n\in\mathcal{N}}\lambda_n^3 \le \left(\sum_{n\in\mathcal{N}}\lambda_n\right)^3$. The desired inequality is, thus, proved. Moreover, the equality case can occur if and only if $\mathcal{O}=(0,1)$ and $f(x)=f_0(x)$ a.e. or $f(x)=f_2(x)$ a.e., where f_0 and f_1 are the functions defined in the statement of the proposition. This completes the proof.

Also solved by the proposer

24. *Proposed by D.M. Batinetu-Giurgiu, Bucharest and Neculai Stanciu, Buzau, Romania.* Let $(a_n)_{n\ge1}, (b_n)_{n\ge1}$ be sequences of positive real numbers such that $\lim\limits_{n\to\infty}\dfrac{a_{n+1}}{n^2\cdot a_n} = \lim\limits_{n\to\infty}\dfrac{b_{n+1}}{n^3\cdot b_n} = a > 0$. Compute

$$\lim_{n\to\infty}\left(\sqrt[n+1]{\frac{b_{n+1}}{a_{n+1}}} - \sqrt[n]{\frac{b_n}{a_n}}\right)$$

Solution 1 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy. From the definition of the limit of a sequence immediately follows that

$$\forall\,\varepsilon\;\exists\,n_0:\; n>n_0 \implies a-\varepsilon < \frac{a_{n+1}}{n^2a_n} < a+\varepsilon$$
$$\forall\,\varepsilon\;\exists\,n_0:\; n>n_0 \implies a-\varepsilon < \frac{b_{n+1}}{n^3b_n} < a+\varepsilon$$

Thus for any $k\ge1$, we get

$$\frac{((n_0+k-1)!)^3}{((n-1)!)^3}(a-\varepsilon)^k b_{n_0} \le b_{n_0+k} \le \frac{((n_0+k-1)!)^3}{((n-1)!)^3}(a+\varepsilon)^k b_{n_0}$$
$$\frac{((n_0+k-1)!)^2}{((n-1)!)^2}(a-\varepsilon)^k a_{n_0} \le a_{n_0+k} \le \frac{((n_0+k-1)!)^2}{((n-1)!)^2}(a+\varepsilon)^k a_{n_0}$$

The computations are straightforward. Now, we have

$$\sqrt[n_0+k+1]{\frac{b_{n_0+k+1}}{a_{n_0+k+1}}} = \exp\left\{\frac{1}{n_0+k+1}\ln\frac{b_{n_0+k+1}}{a_{n_0+k+1}}\right\}$$

and

$$\frac{(n_0+k)!}{(n_0-1)!}\left(\frac{a-\varepsilon}{a+\varepsilon}\right)^{k+1}\frac{b_{n_0}}{a_{n_0}} \le \frac{b_{n_0+k+1}}{a_{n_0+k+1}} \le \frac{(n_0+k)!}{(n_0-1)!}\left(\frac{a+\varepsilon}{a-\varepsilon}\right)^{k+1}\frac{b_{n_0}}{a_{n_0}}$$

Let us define $A = \frac{a+\varepsilon}{a-\varepsilon}$ and $B = \frac{b_{n_0}}{(n_0-1)!a_{n_0}}$. Using Stirling's formulae, namely $n! = (n/e)^n\sqrt{2\pi n}(1+o(1))$, we get

$$\ln \frac{b_{n_0+k+1}}{a_{n_0+k+1}} = (k+1)\ln A + \ln B +$$
$$+\left((n_0+k)\ln(n_0+k) - (n_0+k) + \frac{1}{2}\ln(2\pi) + \frac{1}{2}\ln(n_0+k) + \ln(1+o(1))\right)$$

Thus,

$$\frac{1}{n_0+k+1}\ln\frac{b_{n_0+k+1}}{a_{n_0+k+1}} = \frac{(k+1)\ln A + (n_0+k)\ln(n_0+k) - (n_0+k)}{n_0+k+1} + o(1) =$$
$$= \ln A + \ln k - 1 + o(1)$$

and then, using the fact that $e^x = 1 + o(1)$ when $x \to 0$), yields

$$\exp\left\{\frac{1}{n_0+k+1}\ln\frac{b_{n_0+k+1}}{a_{n_0+k+1}}\right\} = \frac{Ak}{e}(1+o(1))$$

Subtracting we obtain

$$\exp\left\{\frac{1}{n_0+k+1}\ln\frac{b_{n_0+k+1}}{a_{n_0+k+1}}\right\} - \exp\left\{\frac{1}{n_0+k}\ln\frac{b_{n_0+k}}{a_{n_0+k}}\right\} =$$
$$= \frac{A}{e}(1+o(1)) = \left(\frac{a+\varepsilon}{a-\varepsilon}\right)\frac{1}{e} + o(1)$$

Since ε is arbitrarily small, then the limit is $1/e$.

Solution 2 by Anastasios Kotronis, Athens, Greece. It is well known (see[1] p.46 for example) that z_n being a sequence of positive numbers, $\lim_{n\to+\infty}\frac{z_{n+1}}{z_n} = \ell \in \mathbb{R} \Rightarrow \lim_{n\to+\infty}(z_n)^{1/n} = \ell$. We set $z_n = \frac{b_n}{n^n a_n}$, so

$$\frac{z_{n+1}}{z_n} = \frac{b_{n+1}}{n^3 b_n}\left(\frac{a_{n+1}}{n^2 a_n}\right)^{-1}\left(1+\frac{1}{n}\right)^{-n}\frac{n}{n+1} \to e^{-1}$$

and $\lim_{n\to+\infty}(z_n)^{1/n} = e^{-1}$ on account of the preceding. Therefore,

$$\left(\frac{(n+1)(z_{n+1})^{1/(n+1)}}{n(z_n)^{1/n}}\right)^n = \frac{b_{n+1}}{n^3 b_n}\left(\frac{a_{n+1}}{n^2 a_n}\right)^{-1}(z_{n+1})^{-1/(n+1)}\frac{n}{n+1} \to e$$

Now

$$\sqrt[n+1]{\frac{b_{n+1}}{a_{n+1}}} - \sqrt[n]{\frac{b_n}{a_n}} = (z_n)^{1/n}\left(\frac{\frac{(n+1)(z_{n+1})^{1/(n+1)}}{n(z_n)^{1/n}} - 1}{\ln\left(\frac{(n+1)(z_{n+1})^{1/(n+1)}}{n(z_n)^{1/n}}\right)}\ln\left(\frac{(n+1)(z_{n+1})^{1/(n+1)}}{n(z_n)^{1/n}}\right)^n\right) \to e^{-1},$$

since

$$\lim_{n\to+\infty}\frac{\frac{(n+1)(z_{n+1})^{1/(n+1)}}{n(z_n)^{1/n}} - 1}{\ln\left(\frac{(n+1)(z_{n+1})^{1/(n+1)}}{n(z_n)^{1/n}}\right)} = \lim_{n\to+\infty}\frac{\exp\left(\ln\left(\frac{(n+1)(z_{n+1})^{1/(n+1)}}{n(z_n)^{1/n}}\right)\right) - 1}{\ln\left(\frac{(n+1)(z_{n+1})^{1/(n+1)}}{n(z_n)^{1/n}}\right)}$$
$$= \lim_{x\to 0}\frac{e^x - 1}{x} = 1$$

References

[1] W.J. Kaczor, M.T. Nowak Problems in Mathematical Analysis I, Real Numbers, Sequences and Series , A.M.S., 2000.

Also solved by Albert Stadler, Switzerland; Moubinool Omarjee, Paris, France; and the proposer.

25. *Proposed by José Luis Díaz-Barrero, BARCELONA TECH, Barcelona, Spain.* Let D, E, F be three points lying on the sides BC, AB, CA of ΔABC. Let M be a point lying on cevian AD. If E, M, F are collinear then show that

$$\left(\frac{BC\cdot MD}{MA}\right)\left(\frac{EA}{DC\cdot EB}+\frac{FA}{BD\cdot FC}\right)\geq 4$$

Solution 1 by Titu Zvonaru, Comanesti and Neculai Stanciu, George Emil Palade Secondary School, Buzau, Romania.
We denote

$$a=BC, \frac{BD}{DC}=x, \frac{AF}{FC}=y, \frac{AE}{EB}=z.$$

We have

$$BD=\frac{ax}{x+1}, DC=\frac{a}{x+1},$$

and by the relation (R_2) from Rec. Math, 2/2011, pp. 108, we obtain that

$$\frac{AM}{MD}=\frac{ayz}{\frac{ax}{x+1}\cdot z+\frac{a}{x+1}\cdot y}\Leftrightarrow \frac{AM}{MD}=\frac{yz(x+1)}{xz+y}$$

Hence, the given inequality becomes

$$\frac{a(xz+y)}{yz(x+1)}\left(\frac{z}{\frac{a}{x+1}}+\frac{y}{\frac{ax}{x+1}}\right)\geq 4\Leftrightarrow (xz+y)^2\geq 4xyz\Leftrightarrow (xz-y)^2\geq 0.$$

Last inequality trivially holds, and the proof is complete.

Solution 2 by the proposer. First, we write the inequality claimed as

$$\frac{1}{2}\left(\frac{BC\cdot MD}{MA}\right)\geq 2\left(\frac{EA}{DC\cdot EB}+\frac{FA}{BD\cdot FC}\right)^{-1}$$

On account of AM-HM inequality it will be suffice to prove that if E, M, F are collinear then holds

$$BC\cdot\frac{MD}{MA}=DC\cdot\frac{EB}{EA}+BD\cdot\frac{FC}{FA}$$

Indeed, assume that points E, M, F are collinear. It is easy to check that $\triangle EBB'\sim\triangle AEA'$, $\triangle CFC'\sim\triangle AFA'$, and $\triangle DMD'\sim\triangle AMA'$. Then, we have

$$\frac{EB}{EA}=\frac{BB'}{AA'},\quad \frac{FC}{FA}=\frac{CC'}{AA'},\quad \frac{MD}{MA}=\frac{DD'}{AA'}$$

and the statement becomes

$$DC\cdot\frac{BB'}{AA'}+BD\cdot\frac{CC'}{AA'}=BC\cdot\frac{DD'}{AA'}$$

or

$$DC\cdot BB'+BD\cdot CC'=DC\cdot DD'$$

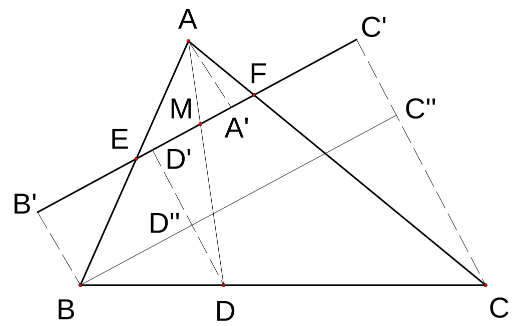

FIGURE 1. Problem 25

Now we distinguish two cases according to $BB' < CC'$ or $BB' > CC'$. Suppose that $BB' < CC'$. We draw the parallel to $B'C'$ that cuts line DD' at D'' and line CC' at C''. Since $\triangle BDB'' \sim \triangle BCC''$, then $\dfrac{DD''}{CC''} = \dfrac{BD}{BC}$ from which follows

$$DD'' = \frac{BD}{BC} CC'' = \frac{BD}{BC}(CC' - BB')$$

Thus,

$$DD' = DD'' + D''D' = \frac{BD}{BC}(CC' - BB') + BB' = \frac{BD \cdot CC'}{BC} + BB' \left(1 - \frac{BD}{BC}\right)$$
$$= \frac{BD \cdot CC'}{BC} + \frac{BB'}{BC}(BC - BD) = \frac{BD \cdot CC'}{BC} + \frac{CD \cdot BB'}{BC}$$

and the statement follows. Likewise, the same result is obtained when $BB' > CC'$. Finally, observe that equality holds when $\triangle ABC$ is equilateral and D, E, F are the vertices if its medial triangle.

Also solved by Codreanu Ioan-Viorel, Satulung, Maramures, Romania; Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria; and the proposer.

26. *Proposed by Enkel Hysnelaj, University of Technology, Sydney, Australia.* Determine all functions $f : \mathbb{R} - \{0, 1\} \to \mathbb{R}$, which satisfy the relation

$$f\left(\frac{x-1}{x}\right) + f\left(\frac{1}{1-x}\right) = ax^2 + bx + c,$$

where $a, b, c \in \mathbb{R}$.

Solution by Valmir Bucaj, Texas Lutheran University, Seguin, TX. Letting $y = \frac{1}{1-x}$, and substituting for x in the original equation we get

$$f\left(\frac{1}{1-y}\right) + f(y) = a\left(\frac{y-1}{y}\right)^2 + b\left(\frac{y-1}{y}\right) + c \tag{1}$$

Similarly, letting $y = \frac{x-1}{x}$, and substituting for x we get

$$f(y) + f\left(\frac{y-1}{y}\right) = a\left(\frac{1}{1-y}\right)^2 + b\left(\frac{1}{1-y}\right) + c \tag{2}$$

Adding (1) and (2) gives

$$2f(y)+f\left(\frac{y-1}{y}\right)+f\left(\frac{1}{1-y}\right)=a\left(\frac{y-1}{y}\right)^2+b\left(\frac{y-1}{y}\right)+a\left(\frac{1}{1-y}\right)^2+b\left(\frac{1}{1-y}\right)+2c$$

Since,

$$f\left(\frac{y-1}{y}\right)+f\left(\frac{1}{1-y}\right)=ay^2+by+c,$$

after substituting in the preceding, we get

$$2f(y)=a\left(\frac{y-1}{y}\right)^2+b\left(\frac{y-1}{y}\right)+a\left(\frac{1}{1-y}\right)^2+b\left(\frac{1}{1-y}\right)-ay^2-by+c$$

Finally,

$$f(y)=\frac{a}{2}\left[\left(\frac{y-1}{y}\right)^2+\left(\frac{1}{1-y}\right)^2-y^2\right]+\frac{b}{2}\left[\left(\frac{y-1}{y}\right)+\left(\frac{1}{1-y}\right)-y\right]+\frac{c}{2},$$

and the result follows by setting $y=x$.

Also solved by Albert Stadler, Switzerland; Moubinnol Omarjee, Paris, France; Adrian Naco, Albania; Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria; and the proposer

27. *Proposed by David R. Stone, Georgia Southern University, Statesboro, GA, USA.* With $\pi(x)=$ the number of primes $\leq x$, show that there exist constants a and b such that

$$e^{ax}<x^{\pi(x)}<e^{bx}$$

for x sufficiently large.

Solution by the proposer. Tchebychef proved (1850) that there exist constants a and b such that

$$a\frac{x}{lnx}<\pi(x)<b\frac{x}{lnx},$$

for x sufficiently large. Thus

$$x^{a\,\frac{x}{lnx}}<x^{\pi(x)}<x^{b\,\frac{x}{lnx}}$$

So

$$\left(x^{\frac{1}{lnx}}\right)^{ax}<x^{\pi(x)}<\left(x^{\frac{1}{lnx}}\right)^{bx}$$

Therefore

$$e^{ax}<x^{\pi(x)}<e^{bx},$$

which concludes the proof.

Comment by the Editors. Anastasios Kotronis, Athens, Greece, let us know that the problem solved above, apart from its first solution given by P. L. Chebyshev in Memoire sur les nombres premiers, *Journal de Math. Pures et Appl.* 17 (1852), 366-390, (which is reproduced on the analytic number theory books like Tom M. Apostol's *Introduction to analytic number theory* and K. Chandrasekharan's book with the same title),also it has been given an elementary solution using different methods, by M. Nair in the article On Chebyshev-type inequalities for primes, *Amer. Math. Monthly,* 89, no. 2, p.126-129, (which is reproduced in the book *Introduction to analytic and probabilistic number theory* by Gerald Tenenbaum).

Also solved by Albert Stadler, Switzerland

28 *Proposed by Florin Stanescu, School Cioculescu Serban, Gaesti, jud. Dambovita, Romania.* Let ABC be a triangle with semi-perimeter p. Prove that

$$\frac{a}{\sqrt{p-a}}+\frac{b}{\sqrt{p-b}}+\frac{c}{\sqrt{p-c}}\geq 2\sqrt{3p},$$

where $[AB]=c,[AC]=b,[BC]=a$.

Solution by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria. Let us define the positive real numbers x, y and z by

$$x=1-\frac{a}{p},\quad y=1-\frac{b}{p},\quad z=1-\frac{c}{p}$$

Clearly we have $x+y+z=1$ and

$$\frac{a}{\sqrt{p}\sqrt{p-a}}+\frac{b}{\sqrt{p}\sqrt{p-b}}+\frac{c}{\sqrt{p}\sqrt{p-c}}=f(x)+f(y)+f(z), \tag{3}$$

where

$$f:(0,1)\to\mathbb{R}, f(t)=\frac{1-t}{\sqrt{t}}=t^{-1/2}-t^{1/2}$$

The function f is convex since it is the sum of two convex functions, so

$$f(x)+f(y)+f(z)\geq 3f\left(\frac{x+y+z}{3}\right)=3f\left(\frac{1}{3}\right)=2\sqrt{3}, \tag{4}$$

and the desired inequality follows from (3) and (4).

Also solved by Albert Stadler, Switzerland; Titu Zvonaru, Comanesti and Neculai Stanciu, George Emil Palade Secondary School, Buzau, Romania, Bruno Salgueiro Fanego, Viveiro, Spain; and the proposer.

MATHCONTEST SECTION

This section of the Journal offers readers an opportunity to solve interesting and elegant mathematical problems mainly appeared in Math Contest around the world and most appropriate for training Math Olympiads. Proposals are always welcomed. The source of the proposals will appear when the solutions be published.

Proposals

21. Let $a > -3/4$ be a real number. Show that

$$\sqrt[3]{\frac{a+1}{2}+\frac{a+3}{6}\sqrt{\frac{4a+3}{3}}}+\sqrt[3]{\frac{a+1}{2}-\frac{a+3}{6}\sqrt{\frac{4a+3}{3}}}$$

is an integer and determine its value.

22. Let a_1, a_2, a_3, a_4 be nonzero real numbers defined by $a_k = \dfrac{\sin(k\beta+\alpha)}{\sin k\beta}$, $(1 \le k \le 4), \alpha, \beta \in \mathbb{R}$. Calculate

$$\begin{vmatrix} 1+a_1^2+a_2^2 & a_1+a_2+1/a_4 & 1+a_2(a_1+a_3) \\ a_1+a_2+1/a_4 & 2+1/a_4^2 & a_2+a_3+1/a_4 \\ 1+a_2(a_1+a_3) & a_2+a_3+1/a_4 & 1+a_2^2+a_3^2 \end{vmatrix}$$

23. Let A be a set of positive integers. If the prime divisors of elements in A are among the prime numbers $p_1, p_2, \ldots, p_n$ and $|A| > 3 \cdot 2^n + 1$, then show that it contains one subset of four distinct elements whose product is the fourth power of an integer.

24. Find all triples (x, y, z) of real numbers such that

$$\left.\begin{array}{l} 12x-4z^2=25, \\ 24y-36x^2=1, \\ 20z-16y^2=9. \end{array}\right\}$$

25. Let a, b, c be the lengths of the sides of a triangle ABC with inradius r and cicumradius R. Prove that

$$\frac{r}{2r+R} \le \sqrt[3]{\left(\frac{a}{b+c+2a}\right)\left(\frac{b}{c+a+2b}\right)\left(\frac{c}{a+b+2c}\right)} \le \frac{1}{4}$$

Solutions

16. *A number of three digits is written as xyz in base* 7 *and as zxy in base* 9*. Find the number in base* 10*.*

(III Spanish Math Olympiad (1965-1966))

Solution by José Luis Díaz-Barrero, BARCELONA TECH, Barcelona, Spain. To write a number in the system of base 7 the only integers used are $0, 1, 2, 3, 4, 5, 6$, and to write it in base 9 only the integers form 0 to 8 are used. Consequently, $x, y, z \in \{0, 1, 2, 3, 4, 5, 6\}$. The number xyz in base 7 represents the number $x \cdot 7^2 + y \cdot 7 + z$ in base 10. On the other hand, the number zxy in base 9 represents the number $z \cdot 9^2 + y \cdot 9 + x$ in base 10. Therefore,

$$x \cdot 7^2 + y \cdot 7 + z = z \cdot 9^2 + y \cdot 9 + x$$

from which follows $8(3x - 5z) = y$. Since x, y, z are integers ranging form 0 to 6, then $y = 0$ and $3x = 5y$. From the preceding, we have $x = z = 0$ or $x = 5$ and $z = 3$. In the first case, we obtain the number 000 which does not have three digits; and from the second, we get the number 503 in base 7 and 305 in base 9. Both numbers in base 10 are the number 248 and this is the answer. □

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain.

17. *A regular convex polygon of $L+M+N$ sides must be colored using three colors: red, yellow and blue, in such a way that L sides must be red, M yellow and N blue. Give the necessary and sufficient conditions, using inequalities, to obtain a colored polygon with no two consecutive sides of the same color.*

(XI Spanish Math Olympiad (1973-1974))

Solution by José Luis Díaz-Barrero, BARCELONA TECH, Barcelona, Spain. Let $K = L + M + N$. We distinguish two cases: (a) K is even, then it must be

$$L \leq \frac{K}{2}, \qquad M \leq \frac{K}{2}, \qquad N \leq \frac{K}{2}$$

That is, $L + M \geq N, \quad L + N \geq M$ and $M + N \geq L$.

(b) If K is odd, then it must be

$$0 < L \leq \frac{K-1}{2}, \quad 0 < M \leq \frac{K-1}{2}, \quad 0 < N \leq \frac{K-1}{2}$$

That is, $L + M > N > 0$, $L + N > M > 0$ and $M + N > L > 0$. We claim that these conditions that are necessary are also sufficient. Indeed, WLOG we can assume that $L \geq M \geq N$, independently of the parity of K. We begin coloring in red sides first, third, fifth,... from the starting point in a circular sense until complete L red sides. Then, remains to be colored $L - 1$ non consecutive sides and $K - (2L - 1) = M + N - L + 1 \geq 1$ consecutive sides. Since $L \geq M$, then $M + N - L + 1 \leq N + 1$, therefore this set of consecutive sides can not be colored alternatively yellow-blue-yellow, etc., without painting two consecutive sides of the same color. Finally, the non consecutive $L - 1$ sides are colored yellow or blue until to complete the M yellow sides and the N blue sides, and our claim is proven.

□

Also solved by José Gibergans Báguena, BARCELONA TECH, Barcelona, Spain.

18. *Let $ABDC$ be a cyclic quadrilateral inscribed in a circle $\mathcal{C}$. Let M and N be the midpoints of the arcs AB and CD which do not contain C and A respectively. If MN meets side AB at P, then show that*

$$\frac{AP}{BP} = \frac{AC + AD}{BC + BD}$$

(IMAC 2011)

Solution by Ivan Geffner Fuenmayor, Technical University of Catalonia (BARCELONA TECH), Barcelona, Spain. Applying Ptolemy's theorem to the inscribed quadrilateral $ACND$, we have

$$AD \cdot CN + AC \cdot ND = AN \cdot CD$$

Since N is the midpoint of the arc CD, then we have $CN = ND = x$, and

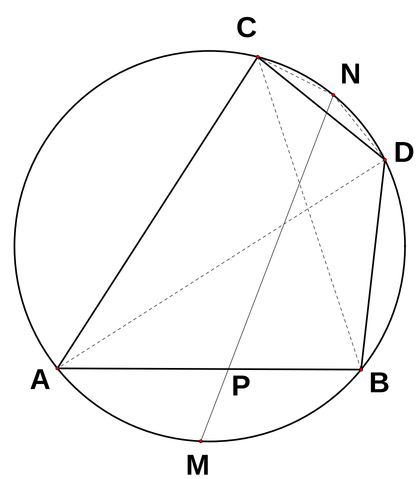

FIGURE 2. Problem 18

$AN \cdot CD = (AC + AD)x$. Likewise, considering the inscribed quadrilateral $CNDB$ we have $BN \cdot CD = (BD + BC)x$. Dividing the preceding expressions, yields

$$\frac{AN}{BN} = \frac{AC + AD}{BD + BC}$$

Applying the bisector angle theorem, we have

$$\frac{AN}{BN} = \frac{AP}{BP}$$

This completes the proof.

□

Also solved by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria; and José Gibergans Báguena, BARCELONA TECH, Barcelona, Spain.

19. *Place n points on a circle and draw in all possible chord joining these points. If no three chord are concurrent, find (with proof) the number of disjoint regions created.*

(IMAC-2011)

Solution 1 by José Gibergans Báguena, BARCELONA TECH, Barcelona, Spain. First, we prove that if a convex region crossed by L lines with P interior points of intersection, then the number of disjoint regions created is $R_L = L+P+1$. To prove the preceding claim, we argue by Mathematical Induction on L. Let $\mathcal{R}$ be an arbitrary convex region in the plane. For each $L \geq 0$, let $A(L)$ be the statement that for each $P \in \left\{1, 2, \ldots, \binom{L}{2}\right\}$, if L lines that cross $\mathcal{R}$, with P intersection points inside $\mathcal{R}$, then the number of disjoint regions created inside $\mathcal{R}$ is $R_L = L + P + 1$.

When no lines intersect $\mathcal{R}$, then $P = 0$, and so, $R_0 = 0+0+1 = 1$ and $A(0)$ holds. Fix some $K \geq 0$ and suppose that $A(K)$ holds for K lines and some $P \geq 0$ with $R_K = K + P + 1$ regions. Consider a collection $\mathcal{C}$ of $K + 1$ lines each crossing $\mathcal{R}$ (not just touching), choose some line $\ell \in \mathcal{C}$, and apply $A(K)$ to $\mathcal{C}\backslash\{\ell\}$ with some P intersection points inside $\mathcal{R}$ and $R_K = K + P + 1$ regions. Let S be the number of lines intersecting ℓ inside $\mathcal{R}$. Since one draws a $(K + 1)-$st line ℓ, starting outside $\mathcal{R}$, a new region is created when ℓ first crosses the border of $\mathcal{R}$, and whenever ℓ crosses a line inside of $\mathcal{R}$. Hence the number of new regions is $S + 1$. Hence, the number of regions determined by the $K + 1$ lines is, on account of $A(K)$,

$$R_{K+1} = R_K + S + 1 = (K + P + 1) + S + 1 = (K + 1) + (P + S) + 1,$$

where $P+S$ is the total number of intersection points inside $\mathcal{R}$. Therefore, $A(K+1)$ holds and by the PMI the claim is proven.

Finally, since the circle is convex and any intersection point is determined by a unique $4-$tuple of points, then there are $P = \binom{n}{4}$ intersection points and $L = \binom{n}{2}$ chords and the number of regions is $\mathcal{R} = \binom{n}{4} + \binom{n}{2} + 1.$

□

Solution 2 by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria. Let us denote by a_n the number of disjoint regions created. Clearly $a_1 = 1$, $a_2 = 2$ and $a_3 = 4$. Suppose that we have a_n regions obtained on placing the n points $A_1, \ldots, A_n$ in this order, and let us add the $n + 1^{st}$ point A_0 on the arc A_nA_1 that does not contain any other point. The chords A_0A_1 and A_0A_n add two regions. And for $1 < k < n$, there are $(k-1)(n-k)$ points of itersection of the chord A_0A_k with the other chords. Hence, the chord A_0A_k passes through $(k - 1)(n - k) + 1$ regions. Consequently, drawing the chord

A_0A_k adds $(k-1)(n-k)+1$ new regions. Thus

$$a_{n+1} = a_n + \sum_{k=1}^{n}((k-1)(n-k)+1)$$

But

$$\begin{aligned}\sum_{k=1}^{n}((k-1)(n-k)+1) &= -\sum_{k=1}^{n}k^2 + (n+1)\sum_{k=1}^{n}k + (1-n)n \\ &= -\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)^2}{2} - n(n-1) \\ &= \binom{n+2}{3} - 2\binom{n}{2}\end{aligned}$$

So,

$$\begin{aligned}a_n &= 1 + \sum_{k=1}^{n-1}\binom{k+2}{3} - 2\sum_{k=1}^{n-1}\binom{k}{2} \\ &= 1 + \sum_{k=1}^{n-1}\left(\binom{k+3}{4} - \binom{k+2}{4}\right) - 2\sum_{k=1}^{n-1}\left(\binom{k+1}{3} - \binom{k}{3}\right) \\ &= 1 + \binom{n+2}{4} - 2\binom{n}{3} = \binom{n}{4} + \binom{n}{2} + 1\end{aligned}$$

which is the required number of regions. □

Also solved by José Luis Díaz-Barrero, BARCELONA TECH, Barcelona, Spain.

20. *Let a, b, c be positive real numbers such that $a+b+c=1$. Prove that*

$$\sqrt[3]{\left(\frac{1+a}{b+c}\right)^{\frac{1-a}{bc}}\left(\frac{1+b}{c+a}\right)^{\frac{1-b}{ca}}\left(\frac{1+c}{a+b}\right)^{\frac{1-c}{ab}}} \geq 64$$

(József Wildt Competition 2009)

Solution by José Luis Díaz-Barrero, BARCELONA TECH, Barcelona, Spain. Consider the function $f:(0,1)\to\mathbb{R}$ defined by $f(x) = \frac{1}{x}\ln\left(\frac{1+x}{1-x}\right)$. Since $f(x) = 2\sum_{k=0}^{\infty}\frac{x^{2k}}{2k+1}$ for $|x|<1$, then $f'(x) = 4\sum_{k=0}^{\infty}\frac{kx^{2k-1}}{2k+1}$ $(|x|<1)$ and $f''(x) = 4\sum_{k=0}^{\infty}\frac{k(2k-1)x^{2k-2}}{2k+1}$ $(|x|<1)$. Therefore, $f'(x)>0$ and $f''(x)>0$ for all $x\in(0,1)$, and f is increasing and convex.

Applying Jensen's inequality, we have $f\left(\dfrac{a+b+c}{3}\right) \le \dfrac{f(a)+f(b)+f(c)}{3}$ or equivalently,

$$\frac{3}{a+b+c}\ln\left(\frac{3+(a+b+c)}{3-(a+b+c)}\right) \le \frac{1}{3}\left[\ln\left(\frac{1+a}{1-a}\right)^{1/a} + \ln\left(\frac{1+b}{1-b}\right)^{1/b} + \ln\left(\frac{1+c}{1-c}\right)^{1/c}\right]$$

Taking into account that $a+b+c=1$ and the properties of logarithms, we get

$$\sqrt[3]{\left(\frac{1+a}{1-a}\right)^{1/a}\left(\frac{1+b}{1-b}\right)^{1/b}\left(\frac{1+c}{1-c}\right)^{1/c}} \ge 8 \qquad (5)$$

WLOG we can assume that $a \ge b \ge c$. We have, $\dfrac{1}{a} \le \dfrac{1}{b} \le \dfrac{1}{c}$ and $g(a) \ge g(b) \ge g(c)$, where g is the increasing function defined by $g(x) = \ln\left(\dfrac{1+x}{1-x}\right)$. Applying rearrangement's inequality, we get

$$\frac{1}{b}\,g(a) + \frac{1}{c}\,g(b) + \frac{1}{a}\,g(c) \ge \frac{1}{a}\,g(a) + \frac{1}{b}\,g(b) + \frac{1}{c}\,g(c)$$

or

$$\left(\frac{1+a}{1-a}\right)^{1/b}\left(\frac{1+b}{1-b}\right)^{1/c}\left(\frac{1+c}{1-c}\right)^{1/a} \ge \left(\frac{1+a}{1-a}\right)^{1/a}\left(\frac{1+b}{1-b}\right)^{1/b}\left(\frac{1+c}{1-c}\right)^{1/c}$$

From the preceding and (5) we obtain

$$\sqrt[3]{\left(\frac{1+a}{b+c}\right)^{1/b}\left(\frac{1+b}{c+a}\right)^{1/c}\left(\frac{1+c}{a+b}\right)^{1/a}} = \sqrt[3]{\left(\frac{1+a}{1-a}\right)^{1/b}\left(\frac{1+b}{1-b}\right)^{1/c}\left(\frac{1+c}{1-c}\right)^{1/a}}$$
$$\ge \sqrt[3]{\left(\frac{1+a}{1-a}\right)^{1/a}\left(\frac{1+b}{1-b}\right)^{1/b}\left(\frac{1+c}{1-c}\right)^{1/c}} \ge 8$$

Likiwise, applying rearrangement's inequality again, we get

$$\frac{1}{c}\,g(a) + \frac{1}{a}\,g(b) + \frac{1}{b}\,g(c) \ge \frac{1}{a}\,g(a) + \frac{1}{b}\,g(b) + \frac{1}{c}\,g(c)$$

and

$$\sqrt[3]{\left(\frac{1+a}{b+c}\right)^{1/c}\left(\frac{1+b}{c+a}\right)^{1/a}\left(\frac{1+c}{a+b}\right)^{1/b}} = \sqrt[3]{\left(\frac{1+a}{1-a}\right)^{1/c}\left(\frac{1+b}{1-b}\right)^{1/a}\left(\frac{1+c}{1-c}\right)^{1/b}}$$
$$\ge \sqrt[3]{\left(\frac{1+a}{1-a}\right)^{1/a}\left(\frac{1+b}{1-b}\right)^{1/b}\left(\frac{1+c}{1-c}\right)^{1/c}} \ge 8$$

Multiplying up the preceding inequalities yields,

$$\sqrt[3]{\left(\frac{1+a}{b+c}\right)^{\frac{1}{b}+\frac{1}{c}}\left(\frac{1+b}{c+a}\right)^{\frac{1}{c}+\frac{1}{a}}\left(\frac{1+c}{a+b}\right)^{\frac{1}{a}+\frac{1}{b}}} \ge 64$$

from which the statement follows. Equality holds when $a=b=c=1/3$, and we are done. □

Also solved by José Gibergans Báguena, BARCELONA TECH, Barcelona, Spain.

MATHNOTES SECTION

On a Discrete Constrained Inequality

Mihály Bencze and José Luis Díaz-Barrero

Abstract. In this note a constrained inequality is generalized and some refinements and applications of it are also given.

1. Introduction

In [1] the following problem was posed: *Let a, b, c be positive real numbers such that $a+b+c=1$. Prove that*

$$(ab+bc+ca)\left(\frac{a}{b^2+b}+\frac{b}{c^2+c}+\frac{c}{a^2+a}\right)\geq\frac{3}{4} \tag{6}$$

A solution to the preceding proposal and some related results appeared in [2]. Our aim in this short paper is to generalize it and to give some of its applications.

2. Main Results

In the sequel some generalizations and refinements of (6) are given. We begin with

Theroem 1. *Let x and $a_k, b_k, (1\leq k\leq n)$ be positive real numbers. Then*

$$\begin{aligned}\left(\sum_{k=1}^n a_k\right)\left(\sum_{k=1}^n \frac{a_k}{(x+b_k)^2}\right) &\geq \left(\sum_{k=1}^n \frac{a_k}{x+b_k}\right)^2\\ &\geq \left(\sum_{k=1}^n a_k\right)^4\Bigg/\left(\sum_{k=1}^n a_k(x+b_k)\right)^2\end{aligned}$$

Proof. Setting $\vec{u}=\left(\frac{\sqrt{a_1}}{x+b_1},\frac{\sqrt{a_2}}{x+b_2},\ldots,\frac{\sqrt{a_n}}{x+b_n}\right)$ and $\vec{v}=\left(\sqrt{a_1},\sqrt{a_2},\ldots,\sqrt{a_n}\right)$ into CBS inequality, we have

$$\left(\sum_{k=1}^n \frac{a_k}{x+b_k}\right)^2=\left(\sum_{k=1}^n \frac{\sqrt{a_k}}{x+b_k}\sqrt{a_k}\right)^2\leq\left(\sum_{k=1}^n a_k\right)\left(\sum_{k=1}^n \frac{a_k}{(x+b_k)^2}\right)$$

and the LHS inequality is proven. To prove RHS inequality we set $\vec{u}=\left(\sqrt{\frac{a_1}{x+b_1}},\sqrt{\frac{a_2}{x+b_2}},\ldots,\sqrt{\frac{a_n}{x+b_n}}\right)$ and $\vec{v}=\left(\sqrt{a_1(x+b_1)},\sqrt{a_2(x+b_2)},\ldots,\sqrt{a_n(x+b_n)}\right)$ into CBS inequality again and we get

$$\left(\sum_{k=1}^n a_k\right)^2\leq\left(\sum_{k=1}^n \frac{a_k}{x+b_k}\right)\left(\sum_{k=1}^n a_k(x+b_k)\right)$$

from which the statement immediately follows. □

A constrained inequality that can be derived immediately from the preceding result is given in the following

Corllary 1. *Let* $a_k, b_k, (1 \leq k \leq n)$ *be positive real numbers such that* $\sum_{k=1}^{n} a_k = 1$. *Then holds:*

$$\left(\sum_{k=1}^{n} a_k(1+b_k)^2\right)\left[\sum_{k=1}^{n} \frac{a_k}{(1+b_k)^2} + \left(\sum_{k=1}^{n} \frac{a_k}{1+b_k}\right)^2\right] \geq 2$$

Proof. Setting $x = 1$ in Theorem 1, we get

$$\left(\sum_{k=1}^{n} a_k(1+b_k)^2\right)\left(\sum_{k=1}^{n} \frac{a_k}{(1+b_k)^2}\right) \geq 1$$

and

$$\left(\sum_{k=1}^{n} a_k(1+b_k)^2\right)\left(\sum_{k=1}^{n} \frac{a_k}{1+b_k}\right)^2 \geq 1$$

Adding up the preceding inequalities the statement follows.

□

Theroem 2. *Let* $0 \leq y < z$ *and* $a_k, b_k, (1 \leq k \leq n)$ *be positive real numbers. Then*

$$\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} \frac{a_k}{(y+b_k)(z+b_k)}\right)$$

$$\geq \left[\sum_{k=1}^{n} \frac{a_k^2}{(y+b_k)(z+b_k)} + \log \prod_{1\leq i<j\leq n} \left(\frac{(y+b_j)(z+b_i)}{(y+b_i)(z+b_j)}\right)^{\frac{2a_i a_j}{b_j - b_i}}\right]$$

$$\geq \left(\sum_{k=1}^{n} a_k\right)^4 \Big/ \left[\left(y\sum_{k=1}^{n} a_k + \sum_{k=1}^{n} a_k b_k\right)\left(z\sum_{k=1}^{n} a_k + \sum_{k=1}^{n} a_k b_k\right)\right]$$

Proof. From Theorem 1, we have

$$\int_y^z \left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} \frac{a_k}{(x+b_k)^2}\right) dx \geq \int_y^z \left(\sum_{k=1}^{n} \frac{a_k}{x+b_k}\right)^2 dx$$

$$\geq \int_y^z \left[\left(\sum_{k=1}^{n} a_k\right)^4 \Big/ \left(\sum_{k=1}^{n} a_k(x+b_k)\right)^2\right] dx.$$

After a little straightforward algebra the statement follows and the proof is complete.

□

Corllary 2. *Let* $y < z$ *and* $a_k, b_k, (1 \leq k \leq n)$ *be strictly positive real numbers. Then exists* $c \in (y, z)$ *such that*

$$\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} \frac{a_k}{(y+b_k)(z+b_k)}\right) \geq \left(\sum_{k=1}^{n} \frac{a_k}{c+b_k}\right)^2$$

$$\geq \left(\sum_{k=1}^{n} a_k\right)^4 \Big/ \left[\left(y\sum_{k=1}^{n} a_k + \sum_{k=1}^{n} a_k b_k\right)\left(z\sum_{k=1}^{n} a_k + \sum_{k=1}^{n} a_k b_k\right)\right]$$

Proof. Applying Lagrange's Mean Value Theorem to the function
$f(x) = \int_0^x \left(\sum_{k=1}^{n} \frac{a_k}{t+b_k}\right)^2 dt$ yields, $\int_y^z \left(\sum_{k=1}^{n} \frac{a_k}{t+b_k}\right)^2 dt = (z-y)\left(\sum_{k=1}^{n} \frac{a_k}{c+b_k}\right)^2$
Putting this in Theorem 2 the inequality claimed follows and this completes the proof. □

Applying again Theorem 2 with $y = 0$ and $z = 1$, we get

Corllary 3. *Let* $a_k, b_k, (1 \le k \le n)$ *be positive real numbers such that* $\sum_{k=1}^{n} a_k = 1$. *Then*

$$\begin{aligned}\sum_{k=1}^{n} \frac{a_k}{b_k(1+b_k)} &\ge \sum_{k=1}^{n} \frac{a_k^2}{b_k(1+b_k)} + \log \prod_{1\le i<j\le n} \left(\frac{b_j(1+b_i)}{b_i(1+b_j)}\right)^{\frac{2a_ia_j}{b_j-b_i}} \\ &\ge \frac{1}{\left(\sum_{k=1}^{n} a_kb_k\right)\left(1+\sum_{k=1}^{n} a_kb_k\right)}\end{aligned}$$

Corllary 4. *Let* a_k $(1 \le k \le n)$ *be positive real numbers such that* $\sum_{k=1}^{n} a_k = 1$. *Then*

$$\begin{aligned}\sum_{cyclic} \frac{a_1}{a_2(1+a_2)} \ge \sum_{cyclic} \frac{a_1^2}{a_2(1+a_2)} &+ \log \prod_{1\le i<j\le n} \left(\frac{a_{j+1}(1+a_{i+1})}{a_{i+1}(1+a_{j+1})}\right)^{\frac{2a_ia_j}{a_{j+1}-a_{i+1}}} \\ &\ge \frac{1}{\left(\sum_{cyclic} a_1a_2\right)\left(1+\sum_{cyclic} a_1a_2\right)}\end{aligned}$$

Proof. Setting $b_k = a_{k+1}$, $(1 \le k \le n)$ and $a_{n+1} = a_1$ into the preceding corollary the statement follows. □

Notice that this result is a generalization and refinement of the inequality posed in [1]. Indeed, for $n = 3$ we have

Corllary 5. *Let* a, b, c *be positive numbers of sum one. Prove that*

$$\begin{aligned}\frac{a}{b(1+b)} + \frac{b}{c(1+c)} + \frac{c}{a(1+a)} &\ge \frac{a^2}{b(1+b)} + \frac{b^2}{c(1+c)} + \frac{c^2}{a(1+a)} \\ &+ \log\left(\left(\frac{a(1+c)}{c(1+a)}\right)^{\frac{2bc}{a-c}} \left(\frac{b(1+a)}{a(1+b)}\right)^{\frac{2ca}{b-a}} \left(\frac{c(1+b)}{b(1+c)}\right)^{\frac{2ab}{c-b}}\right) \ge \frac{9}{4}.\end{aligned}$$

Proof. Taking into account that for all a, b, c positive numbers with sum one is $ab + bc + ca \le \frac{1}{3}(a+b+c)^2 \le \frac{1}{3}$ and corollary 4, we get

$$\begin{aligned}\frac{a}{b(1+b)} + \frac{b}{c(1+c)} + \frac{c}{a(1+a)} &\ge \frac{a^2}{b(1+b)} + \frac{b^2}{c(1+c)} + \frac{c^2}{a(1+a)} \\ &+ \log\left(\left(\frac{a(1+c)}{c(1+a)}\right)^{\frac{2bc}{a-c}} \left(\frac{b(1+a)}{a(1+b)}\right)^{\frac{2ca}{b-a}} \left(\frac{c(1+b)}{b(1+c)}\right)^{\frac{2ab}{c-b}}\right)\end{aligned}$$

$$\geq \frac{1}{(ab+bc+ca)(1+ab+bc+ca)} \geq \frac{9}{4}$$

□

Finally, combining the inequality posed in [1] by Dospinescu and the inequality presented in [2] by Janous, namely

$$(xy+yz+zx)\left(\frac{x}{1+y^2}+\frac{y}{1+z^2}+\frac{z}{1+x^2}\right) \leq \frac{3}{4} \quad (x+y+z=1),$$

two applications are given.

Problem 1. *Let a, b, c be positive real numbers. Prove that*

$$\sum_{cyclic} \frac{a}{b(a+2b+c)} \geq \frac{3(a+b+c)}{4(ab+bc+ca)} \geq \sum_{cyclic} \frac{a}{b^2+(a+b+c)^2}$$

Solution. Putting $x = \frac{a}{a+b+c}, y = \frac{b}{a+b+c}$ and $x = \frac{c}{a+b+c}$ into

$$\left(\sum_{cyclic} xy\right)\left(\sum_{cyclic} \frac{x}{y(1+y)}\right) \geq \frac{3}{4} \geq \left(\sum_{cyclic} xy\right)\left(\sum_{cyclic} \frac{x}{1+y^2}\right)$$

the statement follows. □

Setting in the expressions of x, y, z the elements of a triangle ABC and applying the previous procedure new inequalities for the triangle can be derived. For instance, using the sides a, b, c and the radii of ex-circles r_a, r_b, r_c, we have the following inequalities similar to the ones appeared in [3].

Problem 2. *Let ABC be a triangle. Prove that*

(1) $\displaystyle\sum_{cyclic} \frac{a}{b(2s+b)} \geq \frac{3s}{2(s^2+r^2+4rR} \geq \sum_{cyclic} \frac{a}{4s^2+b^2},$

(2) $\displaystyle\sum_{cyclic} \frac{r_a}{r_b(4R+r+r_b)} \geq \frac{3s}{4r(4R+r)} \geq \sum_{cyclic} \frac{r_a}{r_b^2+(4R+r)},$

where the notations are usual.

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