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Math Competitions Corner

No. 1

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This section of the Journal offers readers an opportunity to solve interesting mathematical problems appeared previously in High School Mathematical Olympiads and University Competitions or used by trainers and contestants to prepare Math Competitions. Elegant solutions, generalizations of the problems posed and new proposals suitable to train students are always welcomed. Proposals should be accompanied by solutions. The origin of the problems appeared previously will be revealed when the solutions are published.

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Solutions to the problems stated in this issue should be posted before July 15, 2012

Problems

MC–1. *Find all functions* $f:\mathbb{N}\to[0,+\infty)$ *such that* $f(1000)=10$ *and*

$$f(n+1)=\sum_{k=0}^{n}\frac{1}{f^2(k)+f(k)f(k+1)+f^2(k+1)}$$

for all integer $n\geq 0$. *(Here,* $f^2(i)$ *means* $(f(i))^2$*.)*

MC–2. *Let $ABDC$ be a cyclic quadrilateral inscribed in a circle $\mathcal{C}$. Let M and N be the midpoints of the arcs AB and CD which do not contain C and A respectively. If MN meets side AB at P, then show that*

$$\frac{AP}{BP} = \frac{AC + AD}{BC + BD}$$

MC–3. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function on $\mathbb{R}$. A point x is called a shadow point of f if and only if there exists $y \in \mathbb{R}$, $y > x$ such that $f(y) > f(x)$. Suppose that all the points of the open interval $I = (a, b)$, $(a < b)$ are shadow points of f and a and b, are not shadow points of f. Prove that*

(i) $f(x) \leq f(b)$ *for all* $a < x < b$.

(ii) $f(a) = f(b)$.

MC–4. *Place n points on a circle and draw in all possible chord joining these points. If no three chords are concurrent, find (with proof) the number of disjoint regions created.*

MC–5. *Let $a, b, c \in (0, +\infty)$ such that $a + b + c = 1$. Prove that*

$$\frac{a}{a^3 + b^2c + c^2b} + \frac{b}{b^3 + c^2a + a^2c} + \frac{c}{c^3 + a^2b + b^2a} \leq 1 + \frac{8}{27abc}$$

MC–6. *Let m and n be distinct integer numbers. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(mx + ny) = mf(x) + nf(y)$ for all $x, y \in \mathbb{R}$.*

MC–7. *Let a, b, c be three positive real numbers such that $a + b + c + \sqrt{abc} = 2$. Prove that*

$$\frac{a}{(b-2)(c-2)} + \frac{b}{(c-2)(a-2)} + \frac{c}{(a-2)(b-2)} \geq \frac{3}{4}$$

MC–8. *Let a, b be positive real numbers and let $f : [a, b] \to \mathbb{R}$ be a continuous function. Prove that there exits $c \in (a, b)$ such that*

$$\frac{1}{2} f(c) = \left(\frac{1}{a^2 - c^2} + \frac{1}{b^2 - c^2} \right) \int_a^c c\, f(t)\, dt$$

MC–9. *Let* $A(z) = \sum_{k=0}^{n} a_k z^k$ *be a polynomial of degree* n *with complex coefficients having all its zeros in the disk* $\mathcal{C} = \{z \in \mathbb{C} : |z| \leq \sqrt{6}\}$. *Show that*

$$\Big|A(3z)\Big| \geq \left(\frac{3+\sqrt{6}}{2+\sqrt{6}}\right)^n \Big|A(2z)\Big|$$

for any complex number z *with* $|z| = 1$.

MC–10. *Compute*

$$\lim_{n\to\infty} \ln\left[\frac{1}{2^n}\prod_{k=1}^{n}\left(2+\frac{k}{n^2}\right)\right]$$

MC–11. *Let* a, b *be positive integers. Prove that*

$$\frac{\varphi(ab)}{\sqrt{\varphi^2(a^2)+\varphi^2(b^2)}} \leq \frac{\sqrt{2}}{2},$$

where $\varphi(n)$ *is the Euler's totient function.*

MC–12. *If* $x_i > 0$ *and* $\alpha_i \in \mathbb{R}$, *then*

$$V_n(x,\alpha) = \begin{vmatrix} x_1^{\alpha_1} & x_2^{\alpha_1} & \dots & x_n^{\alpha_1} \\ x_1^{\alpha_2} & x_2^{\alpha_2} & \dots & x_n^{\alpha_2} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{\alpha_n} & x_2^{\alpha_n} & \dots & x_n^{\alpha_n} \end{vmatrix}$$

is called a generalized Vandermonde determinant of order n. *Let* $0 < x_1 < x_2 < \ldots < x_n$, *and* $\alpha_1 < \alpha_2 < \ldots < \alpha_n$ *be real numbers. Prove that* $V_n(x,\alpha) > 0$.

MC–13. *Let* X *be a set of cardinal* n. *Determine*

$$\sum_{(A,B)\subseteq X\times X} card\,(A\cap B)$$

MC–14. *Compute*

$$\int_1^\infty \frac{dt}{2[t]+3[t]^2+[t]^3},$$

where $[x]$ *represents the integer part of* x.

MC–15. *Find all triplets* (x, y, z) *of real numbers such that*

$$\left.\begin{array}{r} x^2+\sqrt{y^2+12}=\sqrt{y^2+60}, \\ y^2+\sqrt{z^2+12}=\sqrt{z^2+60}, \\ z^2+\sqrt{x^2+12}=\sqrt{x^2+60}. \end{array}\right\}$$

MC–16. *Let* p *and* q *be two prime numbers and let* r *be a whole number. Find all possible values of* p, q, r *for which*

$$\frac{1}{p+1}+\frac{1}{q+1}-\frac{1}{(p+1)(q+1)}=\frac{1}{r}$$

MC–17. *Solve in* $\mathbb{R}$ *the following system of equations:*

$$\begin{array}{rcl} \sqrt{x}+\sqrt{y}+\sqrt{z} & = & 3 \\ x\sqrt{x}+y\sqrt{y}+z\sqrt{z} & = & 3 \\ x^2\sqrt{x}+y^2\sqrt{y}+z^2\sqrt{z} & = & 3 \end{array}$$

MC–18. *Find the biggest positive integer that cannot be written in the form* $5a + 503b$ *where* a *and* b *are nonnegative integer numbers.*

MC–19. *Let* $x_1, x_2, \cdots, x_n$ *be real numbers. Prove that,*

$$\left(\frac{1}{n}\sum_{k=1}^{n}\cosh x_k\right)^2 \geq 1+\left(\frac{1}{n}\sum_{k=1}^{n}\sinh x_k\right)^2$$

MC–20. *Let* a, b, c *be distinct nonzero complex numbers. Prove that*

$$\frac{a^2(1+b^2c^2)}{(a-b)(a-c)}+\frac{b^2(1+c^2a^2)}{(b-a)(b-c)}+\frac{c^2(1+a^2b^2)}{(c-a)(c-b)}$$

is an integer number and determine its value.