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# József Wildt International Mathematical Competition

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The solution of the problems W.1 - W.30 must be mailed before 30. October 2010, to Mihály Bencze, str. Hărmanului 6, 505600 Săcele - Négyfalu, Jud. Braşov, Romania, E-mail: benczemihaly@yahoo.com

**W.1.** If $\alpha \in \left(0, \frac{\pi}{2}\right)$, prove that $\ln\left(1+\frac{1}{x}\right) \geq \frac{1}{x+\alpha} + 2\ln\frac{1-\alpha}{\alpha} - \frac{1-2\alpha}{\alpha-\alpha^2}$ for any $x > 0$. The equality holds if and only if $x = \frac{\alpha^2}{1-2\alpha}$.

Ovidiu Pop

**W.2.** If $n \in N, n \geq 2$, prove that $2n\left[\sqrt{4-\frac{6}{n}+\frac{4}{n^2}}\right] = 2 + \left[\sqrt{4n^2-6n+4}\right]$, where $[\cdot]$ denote the integer part.

Ovidiu Pop

**W.3.** If $A, B \in M_2(R)$, the inequality $\det\left(A^2+B^2\right) \geq \det(AB-BA)$ holds.

Ovidiu Pop

**W.4.** Let $n \geq 2$ be a positive integer. Suppose that the monic polynomial with real coefficients $A(x) = x^n + \sum\limits_{k=0}^{n-1} a_k x^k$ has distinct nonzero real roots $x_k$, $1 \leq k \leq n$. Prove that $A^2(0)\sum\limits_{k=1}^{n}\frac{1}{x_k^2} = \left(A'(0) - A(0)A''(0)\right)$.

José Luis Díaz - Barrero

**W.5.** Compute the following sum: $\sum\limits_{i=1}^{\infty}\sum\limits_{j=1}^{\infty}\frac{(-1)^{i+j}}{i^2j^2}$.

José Luis Díaz - Barrero

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**W.6.** If $a, b, c > 0$ and $a + b + c = 1$, then

$$\sum \frac{a}{1+b^3} \geq \frac{3}{2}\left(\sqrt[3]{1+\sqrt{2}} - \sqrt[3]{1-\sqrt{2}}\right).$$

Sefket Arslanagic and Faruk Zejnulahi

**W.7.** Characterize all polynomials for which we have identity

$P(\sin x) + P(\cos x) = 1$, for all $x \in R$.

Sefket Arslanagic and Faruk Zejnulahi

**W.8.** In all acute triangle $ABC$ holds

1). $2\left(2 - \frac{r}{R}\right) \leq \sum A^2 \leq \pi\left(2 - \frac{r}{R}\right)$
2). $4\left(\frac{s^2+r^2-2Rr}{R^2} - 3\right) \leq \sum A^2B^2 \leq \pi^2\left(\frac{s^2+r^2-2Rr}{R^2} - 3\right)$
3). $8\left(\frac{3s^2+5r^2}{4R^2} - 3\right) \leq A^2B^2C^2 \leq \pi^3\left(\frac{3s^2+5r^2}{4R^2} - 3\right)$

Nicuşor Minculete and Mihály Bencze

**W.9.** Let $n$ be a positive integer. Show that there exist infinite many pairs $(x, y)$ of different positive rational numbers such that $x^{2n+1} + y^{2n}$ and $x^{2n} + y^{2n+1}$ are booth squares of rational numbers.

Pál Péter Dályay

**W.10.** Let $m$ and $n$ be positive integers, and let $S(n, m) = \sum_{k=1}^{\frac{n-1}{2}} \sec^{2m}\left(\frac{k\pi}{n+1}\right)$. If $n$ is odd and $n > 1$, show that $S(n, m)$ is rational.

Pál Péter Dályay

**W.11.** Let $f : [0, 1] \to R$ be derivable, with $f(0) = 0$, such that $|f'(x) - f'(0)| \leq x$, for every $x \in [0, 1]$. Prove that the sequence $a_n = \sum_{k=1}^{n} f\left(\frac{k}{n^2}\right)$ is convergent.

Cristinel Mortici

**W.12.** Let $P : R \to R$ be a polynomial map of degree $n$, $P(x) = \sum\limits_{k=0}^{n} a_k x^k$.
Suppose that $a_k \geq 0$ $(k = 0, 1, ..., n)$ and $\sum\limits_{k=0}^{n} a_k > 0$. Then the function $f(x) = \ln P(e^x)$, $x \in R$ is convex.

Sorin Rădulescu and Marius Rădulescu

**W.13.** Find all positive integers $n$ such that $2^{3^n} + 3^{2^n} + 10$ is:

a). a perfect square;
b). a perfect cube.

Cristinel Mortici

**W.14.** Let $p \in N$ be a prime number and $i \in \{1, 2, ..., p-1\}$. Prove that the sum $\sum\limits_{p > k_i > ... > k_1 \geq 1} k_1 k_2 ... k_i$ is divisible by $p$.

Róbert Szász

**W.15.** Let $E(x, y) = x^4 + x^3 y - 8y^4$, be, where $(x, y) \in R^2$

1). Find all the real solutions of the equation $E(x, y) = 0$
2). Prove that exists an infinity of solutions $(x, y)$ for the anterior equation, which verify the condition: $x - y \equiv 0 \pmod 3$.
3). Find all the extremum points of the function $E(x, y)$, when $(x, y) \in R^2$.

Laurenţiu Modan

**W.16.** We consider the set $A = \{a_1, a_2, ..., a_n\}$ and $F = \{f \,|f : A \to A, f(a_1) = ... = f(a_k) = \alpha, \text{ where } a_i, \alpha \in A, \ (i \in \{1, 2, ..., k\})|\}$
1). Find $|F|$
2). Solve the equation $|F| = 18$, when $k$ is a prime and odd number.

Laurenţiu Modan

**W.17.** Let $ABCD$ be a convex quadrilateral with $B + D = 60^\circ$. suppose that the quadrilateral is Heron (having all sides integers, and area integer too). Prove that is a such quadrilateral, the perimeter is even sides; and the area is divisible by 3.

József Sándor

**W.18.** If $a, b > 1$, then
$\log_a(\log_b ab) + \log_b(\log_a ab) \le 1 + \min\left\{\log_a \frac{b}{b-1}, \log_b \frac{a}{a-1}\right\}$.

György Szöllősy

**W.19.** Let $F_n(x), L_n(x)$ the Fibonacci respective the Lucas polynomials, $F_0(x) = 0$, $L_0(x) = 2$, $F_1(x) = L_1(x) = 1$. Prove that
$F_m(x) L_n(x) + F_n(x) L_m(x) = 2F_{m+n}(x)$ for all $x \in R$, $n, m \in N$.

D.M. Bătineţu - Giurgiu

**W.20.** Find the integer solutions of the system: $\left\{\begin{array}{l} x^2 + 2yz - 36 < 0 \\ y^2 + 2zx + 16 = 0 \\ z^2 + 2xy + 16 = 0 \end{array}\right.$.

Michael Th. Rassias, Cambridge, UK

**W.21.** Let $ABC$ be a triangle, and $M \in Int(ABC)$ such that $AMB\measuredangle = BMC\measuredangle = CMA\measuredangle$. Prove that
$\left(MA^3 + MB^3 + MC^3\right)\left(\frac{1}{MA} + \frac{1}{MB} + \frac{1}{MC}\right) \ge 4\sqrt{3} Area\,[ABC]$.

Mihály Bencze

**W.22.** Let $ABCD$ be a convex quadrilateral such that $A = C = \frac{\pi}{2}$, and $AE, CF$ are the bisectors of angles $A$ and $C$, where $E, F \in (BD)$. Prove that
$\left(\frac{AE}{BE} + \frac{AE}{ED}\right)\left(\frac{CF}{BF} + \frac{CF}{FD}\right) \le 4\cos^2 \frac{B-D}{4}$.

Mihály Bencze

**W.23.** Prove that $\int\limits_0^{\frac{\pi}{2}} (5 + 3\cos 4x)^n \, dx \le \frac{4^n \pi (4n+1)!!}{(4n+2)!!}$.

Mihály Bencze

**W.24.** If $a_i, \lambda_i > 0, k_i \ge 1$ $(i = 1, 2, ..., n)$, $\sum\limits_{i=1}^{n} \lambda_i = 1$, $A = \sum\limits_{i=1}^{n} \lambda_i a_i$, then
$\lambda_1 \sqrt[k_1]{a_1 + \sqrt[k_2]{a_2 + ... + \sqrt[k_n]{a_n}}} + \lambda_2 \sqrt[k_1]{a_2 + \sqrt[k_2]{a_3 + ... + \sqrt[k_n]{a_1}}} + ...$
$+ \lambda_n \sqrt[k_1]{a_n + \sqrt[k_2]{a_1 + ... + \sqrt[k_n]{a_{n-1}}}} \le \sqrt[k_1]{A + \sqrt[k_2]{A + ... + \sqrt[k_n]{A}}}$.

Mihály Bencze

**W.25.** If $x \in R$ then
$\left(1+\sin^2 x\right)^{12}+\left(1+\cos^2 x\right)^{12}+\left(1+\sin^2 x\right)^4+\left(1+\cos^2 x\right)^4>$
$>\frac{5}{2}+\left(1+\sin^2 x\right)^9+\left(1+\cos^2 x\right)^9.$

Mihály Bencze

**W.26.** If $\alpha, x_k>0$ $(k=1,2,3,4,5,6,7,8)$, then
$\left(\left(\frac{1}{(\alpha+x_1)^2}+\frac{1}{(\alpha+x_2)^2}\right)^2+\left(\frac{1}{(\alpha+x_3)^2}+\frac{1}{(\alpha+x_4)^2}\right)^2\right)^2+$
$+\left(\left(\frac{1}{(\alpha+x_5)^2}+\frac{1}{(\alpha+x_6)^2}\right)^2+\left(\frac{1}{(\alpha+x_7)^2}+\frac{1}{(\alpha+x_8)^2}\right)^2\right)^2 \geq \frac{1}{\alpha^8+x_1x_2x_3x_4x_5x_6x_7x_8}.$

Mihály Bencze

**W.27.** If $p_k$ denote the $k^{th}$ prime $(p_1=2)$ and $S_n=\frac{1}{p_1p_2}+\frac{1}{p_2p_3}+...+\frac{1}{p_np_{n+1}}$ then exist $n_0 \in N^*$ such that for all $n \geq n_0$ $(n \in N)$ holds $\left[3^{k+1}S_n\right]=3^k-1$ for all $k \in N^*$, when $[\cdot]$ denote the integer part.

Mihály Bencze

**W.28.** If $a_i>1$ $(i=1,2,...,n)$ and $k \in \{3,4,...,n\}$, then
$\frac{\prod\limits_{cyclic}\left(\log_{a_2}a_1\log_{a_3}a_1...\log_{a_{k-1}}a_1+\log_{a_3}a_1\log_{a_4}a_1...\log_{a_k}a_1\right)^{k-2}}{\left(\prod\limits_{i=1}^{n}\log_{a_{i+1}}a_i\log_{a_{i+2}}a_i...\log_{a_{i+k-1}}a_i\right)^{k-3}} \geq$
$\geq (k-1)^{n(k-2)}\prod\limits_{cyclic}\log_{\frac{a_2+a_3+...+a_k}{k-1}}a_1.$

Mihály Bencze

**W.29.** Denote $a_k$ $(k=1,2,...,n)$ the sides of a convex polygon and
$f_m(x)=\sum\limits_{k=1}^{m}\frac{x^k}{k!}$. Prove that $\prod\limits_{cyclic}\left(1+f_m\left(\frac{a_2+...+a_n-a_n}{a_1}\right)\right)^{a_1} \leq e^{(n-3)\sum\limits_{k=1}^{n}a_k}$.

Mihály Bencze

**W.30.** In all triangle $ABC$ holds:
1). $\sum\limits_{\substack{k+p+r=n\\k,p,r\geq 0}}\frac{\cos(kA+pB+rC)}{k!p!r!}=$
$=\frac{1}{R^n}\left((R+r)^n-\binom{n}{2}(R+r)^{n-2}s^2+\binom{n}{4}(R+r)^{n-4}s^4-...\right)$
2). $\sum\limits_{\substack{k+p+r=n\\k,p,r\geq 0}}\frac{\sin(kA+pB+rC)}{k!p!r!}=\frac{1}{R^n}\left(\binom{n}{1}(R+r)^{n-1}s-\binom{n}{3}(R+r)^{n-3}s^3+...\right)$

Mihály Bencze